ON THE INDEPENDENCE OF EQUATIONS IN THREE VARIABLES

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Abstract. We prove that an independent system of equations in three variables with a nonperiodic solution and at least two equations consists of balanced equations only. For that, we show that the intersection of two different entire systems contains only balanced equations, where an entire system is the set of all equations solved by a given morphism. Furthermore, we establish that two equations which have a common nonperiodic solution have the same set of periodic solutions or are not independent.

1. Introduction

Systems of word equations in three variables are investigated in this article. Consider for example the following system S of two equations

$$xyz = zyx \qquad \text{and} \qquad xyyz = zyyx$$

which has a solution α with $\alpha(x) = \alpha(z) = a$ and $\alpha(y) = b$. This solution is called nonperiodic since $\alpha(x)$, $\alpha(y)$, and $\alpha(z)$ are not powers of the same word. The system S is also independent since there exist the solutions β and γ, with $\beta(x) = a$ and $\beta(y) = b$ and $\beta(z) = aba$ and $\gamma(x) = a$ and $\gamma(y) = b$ and $\gamma(z) = abba$, which solve either one of the two equations but not the other. Both equations in this example are balanced, that is, the number of occurrences of each variable on the left and the right hand side is the same.

The main result of this article states that every independent system with at least two equations and a nonperiodic solution, consists of balanced equations only.

Let α be a nonperiodic solution. We call the set of all equations solved by α the entire system of equations generated by α. It is shown that the intersection of two different entire systems can only contain balanced equations. Furthermore, we establish that two equations which have a common nonperiodic solution have the same set of periodic solutions or are not independent. These two facts prove the result mentioned above.

Even though this result is interesting in its own right, it also provides more insight into the following question: Does there exist an independent system of three equations in three variables that has a nonperiodic solution? If it does, it must contain balanced equations only. This question was implicitely raised by Culik II and Karhumäki [3] first in 1983.

This introduction is followed by the preliminaries, Section 2, where the notations for this article are fixed. Section 3 introduces Spehner's [7] characterization of solutions of equations in three variables which is compared with an earlier result

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by Budkina and Markov [1] in Section 4. The intersection of Spehner's entire systems is investigated in Section 5 which provides the foundation for the proof of the main result in Section 6. This paper ends with concluding remarks in Section 7.

2. Preliminaries

In this section we fix the notations for this paper. We refer to [4, 5, 2] for more basic and general definitions.

Let $X = \{x, y, z\}$ be a fixed set of three *variables* and let X^+ denote the semigroup of all finite, nonempty words over X. Let ε denote the empty word. A word v is a *prefix* of u, denoted by $v \le u$, if there exists a word w such that $u = vw$. If $w \ne \varepsilon$, then v is a *proper prefix* of u, denoted by $v < u$. Accordingly, $v \preccurlyeq u$ and $v \prec u$ denote that v is a *suffix* and *proper suffix* of u, respectively, that is $u = wv$.

An *equation* in X is any pair (u, w) of words in X^+, usually written as $u = w$. An equation is called *balanced* if every variable has the same number of occurrences on each side. Let e be an equation $u = w$, then we abbreviate $v_1 \le u$ and $v_2 \le w$ by $(v_1, v_2) \le e$, and $v_1 \preccurlyeq u$ and $v_2 \preccurlyeq w$ by $(v_1, v_2) \preccurlyeq e$. We say that an equation e *starts* (or *ends*) with v_1 and v_2, if $(v_1, v_2) \le e$ (or $(v_1, v_2) \preccurlyeq e$, respectively) or $(v_2, v_1) \le e$ (or $(v_2, v_1) \preccurlyeq e$, respectively). Let $A = \{a, b\}$ be a fixed set of two *letters*. A *solution* of an equation $u = w$ is a morphism $\alpha \colon X^+ \to A^+$ such that $\alpha(u) = \alpha(w)$. Note, that for any solution in a finite set of letters a solution in A can be found, since any finitely generated semigroup can be embedded in A^+. An equation $u = w$ is called *reduced* if ε is the greatest common prefix and suffix of u and w. Note, that every equation can be transformed into its reduced form, that is by dropping common pre- and suffixes, without changing its set of solutions.

Two morphisms $\alpha \colon X^+ \to A^+$ and $\beta \colon Y^+ \to B^+$ are *isomorphic* if there exist two isomorphisms $\rho_1 \colon X^+ \to Y^+$ and $\rho_2 \colon A^+ \to B^+$ such that $\rho_2\alpha = \beta\rho_1$, that is, the following diagram commutes:

$$\begin{array}{ccc} X^+ & \xrightarrow{\alpha} & A^+ \\ {\scriptstyle\rho_1}\big\downarrow & & \big\downarrow{\scriptstyle\rho_2} \\ Y^+ & \xrightarrow{\beta} & B^+ \end{array}$$

We call α a *permutation* of β if $X = Y$ and $A = B$. A morphism is called *periodic* if it is isomorphic to a morphism in $\{a\}^+$, that is, there exists a word w such that $\alpha(x) \in \{w\}^+$ for all $x \in X$; otherwise it is called *nonperiodic*. Let $v \in X^+$ and α be as before, then $A_\alpha(v)$ and $B_\alpha(v)$ denote the number of occurrences of the letters a and b, respectively, in $\alpha(v)$. We might abbreviate $\alpha(v)$ with v, if the context is clear.

A *system* of equations, or system for short, is a nonempty set of equations. A *solution* of a system is a morphism that solves all equations in the system. Two systems are *equivalent* if they have exactly the same set of solutions. A system of equations is called *independent* if it is not equivalent to any of its proper subsets.

Let $\alpha \colon X^+ \to A^+$ be a morphism. Then the kernel

$$\ker(\alpha) = \{(u, w) \mid \alpha(u) = \alpha(w)\} = \alpha^{-1} \circ \alpha$$

of α is also called an *entire system* generated by α in the context of equations, see [7], denoted by K_α. Here, K_α consists of all equations for which α is a solution. A subset C of A^+ is called a *base* of C^+ if $C = C^+ \setminus (C^+)^2$. A subset C of A^+ is called *incontractable* if C is a base of C^+ and for any $D \subseteq A^+$, if C^+ and D^+ are isomorphic, then $\sum_{v\in C} |v| \leq \sum_{v\in D} |v|$. We call α an *incontractable morphism* if the base of $\alpha(X^+)$ is an incontractable subset of A^+.

Consider for example equation $xyx = zz$ which is solved by all of the following morphisms α_p defined by

$$x \mapsto (ab)^p a \qquad y \mapsto baab \qquad z \mapsto (ab)^{p+1} a$$

where $p \geq 0$. Now, α_0, that is $x \mapsto a$ and $y \mapsto baab$ and $z \mapsto aba$, is incontractable, whereas α_p, for any $p \geq 1$, is not incontractable. Note, that α_p is a principal solution (cf. [4]) for all $p \geq 0$.

For the rest of this paper we consider nontrivial reduced equations and systems of equations in X and solutions in A^+ only.

3. A Characterization of Incontractable Solutions

In this section we will give Spehner's characterization, see Proposition 2.5 in [7], of nonperiodic incontractable solutions of entire systems.

Theorem 1. *For every entire system S generated by a nonperiodic morphism, there exists a unique, up to permutation, incontractable nonperiodic morphism α such that $S = K_\alpha$.*

Let $\alpha\colon X^+ \to A^+$ be a nonperiodic incontractable solution, then α is of one of the following *types*. We define α by a triple $(\alpha(x), \alpha(y), \alpha(z))$, and we fix that $\gcd(p+1, q+1) = 1$ in the following.

(1) Let $p, q \geq 1$, then
$$\alpha\colon \left(a, b^{q+1}, b^{p+1}\right) .$$
(2) Let $p, q, k \geq 1$ and $i, j \geq 0$ and $i + j \leq k$, then
$$\alpha\colon \left(a, \left(ba^k\right)^q b, a^i \left(ba^k\right)^p ba^j\right) .$$
(3) Let $p > q \geq 1$ and $1 \leq i, j < k < i + j$, then
$$\alpha\colon \left(a, \left(ba^k\right)^q b, a^i \left(ba^k\right)^{p-q-1} ba^j\right) .$$
Note that $q + 1 \neq p - q$ since $\gcd(p + 1, q + 1) = 1$ by assumption.

(4) Let $q \geq 1$ and $1 \leq i, j \leq k < i + j$ and $n \geq 0$ and $k_t \geq 0$, for all $1 \leq t \leq n$, then
$$\alpha\colon \left(a, \left(vba^k\right)^q vb, a^i ba^j\right) \quad \text{where} \quad v = ba^{k_1+i+j} ba^{k_2+i+j} \cdots ba^{k_n+i+j} .$$
Spehner gives $n \geq 1$ in [7], but this is a misprint. Indeed, we have thet equation $xyxyx = zyz$ implies an entire system with an incontractable solution where $y \mapsto bab$ and $z \mapsto aba$ which is not in any of Spehner's types.

(5) Let $p, q, i, j \geq 1$ and $m \geq 0$, then
$$\alpha\colon \left(a, \left(ba^{i+j+m}\right)^q ba^j, a^i b \left(a^{i+j+m} b\right)^p\right) .$$
In Spehner's paper the second component in $\alpha(X)$ is $b\left(a^{i+j} b\right)^q a^j$ which we think is a typo.

(6) Let $q \geq 1$ and $i, i', j, j' \geq 0$ and $ii' = jj' = 0$ and $n \geq 0$ and $k_t \geq 0$, for all $1 \leq t \leq n$, then

$$\alpha\colon \left(a, \left(a^{i'} b a^{k_1+i+j} b a^{k_2+i+j} \cdots b a^{k_n+i+j} b a^{j'}\right)^q, a^i b a^j\right) .$$

As for type 4, Spehner gives $n \geq 1$ in [7], but this is a misprint. Indeed, the equation $x = z$ implies an entire system with an incontractable solution where $y \mapsto b$ and $z \mapsto a$ which would not be in any of Spehner's types. However, this example could belong to case 1, if $p, q \geq 0$ there. But then the entire system implied by $xy = zx$ is not contained in any type here since $y \mapsto ba$ and $z \mapsto ab$. So, assuming $n \geq 0$ in types 4 and 6 closes the gap leaving type 1 as given in [7].

Note, that if $i' + j' < i + j$ then $q = 1$ otherwise α does not satisfy any nontrivial equation.

(7) In this type

$$\alpha\colon\ (a, b, \varepsilon)\ .$$

We observe that no nonerasing morphism is isomorphic to a morphism of this type. Since we consider only nonerasing solutions here, type 7 will not be further investigated.

(8) Let $p, q, r \geq 1$ and $i, j \geq 0$, then

$$\alpha\colon \left(a, \left(ba^{i+j+r}\right)^q ba^{j+r}, a^{i+r} b \left(a^{i+j+r} b\right)^p\right) .$$

The following theorem completes Spehner's characterization.

Theorem 2. *Any incontractable solution β is equal to a permutation of α in one of the eight previous types.*

Furthermore, we observe the following fact.

Lemma 3. *If α is an incontractable solution then $\alpha(x') = a$, up to renaming of a, for some $x' \in X$.*

4. A Comparison of Spehner's and Budkina & Markov's Characterization

Let a semigroup that is isomorphic to a subsemigroup with k generators of a free semigroup be called *F-semigroup* with k generators. All F-semigroups with 3 generators have been completely characterized by Budkina and Markov in 1973 [1] and by Spehner in 1976 [6]. We use Spehner's presentation on the Semigroups conference in 1986 [7] to base our result on. Unfortunately, we had to correct some details of Proposition 2.5 in [7] as mentioned in the previous section. In order to justify these corrections we will compare our version of Spehner's Proposition 2.5 with Budkina and Markov's Theorem 1 in [1].

Budkina and Markov establish that any F-semigroup S with three generators is of one of the following types.

(i) S is isomorphic to a subsemigroup of $\{a\}^+$.

(ii) S is isomorphic to the free product of an F-semigroup with two generators and an infinite monogenic semigroup, that is,

$$\{a, b, c\}^+ \qquad \text{or} \qquad \{a^q, b, a^p\}^+$$

with $p, q \geq 1$ and $\gcd(p, q) = 1$.

(iii) S is isomorphic to

$$\left\{a, ba^{k_1}b\cdots ba^{k_n}ba^{j'}, a^ib\right\}^+ \quad \text{or} \quad \left\{a, a^{i'}ba^{k_1}b\cdots ba^{k_n}b, ba^j\right\}^+$$

where $i,j \geq 1$ and $i',j',n \geq 0$ and $k_t \geq i$ and $k_t \geq j$, for all $1 \leq t \leq n$, respectively.

(iv) S is isomorphic to

$$\left\{a, ba^{k_1}b\cdots ba^{k_n}b, a^iba^j\right\}^+$$

where $i,j \geq 1$ and $n \geq 0$ and $k_t \geq i+j$, for all $1 \leq t \leq n$.

(v) S is isomorphic to

$$\left\{a, a^{i'}\left(ba^k\right)^q ba^{j'}, a^i\left(ba^k\right)^p ba^j\right\}^+$$

where $k,p,q \geq 1$ and $0 \leq i,i',j,j' \leq r$ and $\gcd(p+1,q+1) = 1$ and $ii' = jj' = 0$.

(vi) S is isomorphic to

$$\left\{a, \left(ba^k\right)^q b, a^iba^j\right\}^+$$

where $q \geq 1$ and $1 \leq i,j \leq k$ and $k < i+j$.

(vii) S is isomorphic to

$$\left\{a, \left(ba^{k_1}b\cdots ba^{k_n}ba^k\right)^q ba^{k_1}b\cdots ba^{k_n}b, a^iba^j\right\}^+$$

where $q \geq 1$ and $1 \leq i,j \leq k$ and $k < i+j$ and $n \geq 0$ and $k_t \geq i+j$, for all $1 \leq t \leq n$.

We observe the following correspondence between Spehner's and Budkina & Markov's types.

Budkina & Markov	**Spehner**
(ii)	(1) and (6), where $i+j=0$
(iii)	(6) where $i+j>0$ and $ij=0$
(iv)	(6) where $ij>0$
(v)	(2), (3), where $p>q+1$, (5), (8)
(vi)	(3), where $p=q+1$, (4), where $n=0$
(vii)	(4), where $n>0$

Note, that if α is of type (6) and $i=j=0$, then $(\alpha(X))^+$ is isomorphic to $\{a,b\}^+$, note also, that $\{a\}^+$ and $\{a,b,c\}^+$ are not of rank 2.

5. About the Intersection of Entire Systems

In this section we show that entire systems with nonperiodic solutions intersect with balanced equations only. We investigate Spehner's types of entire systems for that purpose. Recall that we require solutions to be nonerasing. So, type (7) will not be considered here.

For the rest of this section, let $\alpha,\beta\colon X^+ \to A^+$ be two nonperiodic incontractable morphisms which are not identical up to renaming of a and b. Let us fix an equation $u=w$, denoted by e, and let α and β both satisfy e. Assume that e is not balanced.

We can assume that $\alpha(x) = a$ without restriction of generality. Let τ_α and τ_β denote the type of α and β, respectively. In general, let us subscript variables with α and β to indicate the solution they belong to.

The numbers of occurences of letters a and b will be frequently made use of in the proofs of this section. Therefore, observe that

$$|u|_x + A_\alpha(y)|u|_y + A_\alpha(z)|u|_z = |w|_x + A_\alpha(y)|w|_y + A_\alpha(z)|w|_z$$

and

$$B_\alpha(y)|u|_y + B_\alpha(z)|u|_z = B_\alpha(y)|w|_y + B_\alpha(z)|w|_z$$

which implies

$$\frac{A_\alpha(z)B_\alpha(y) - A_\alpha(y)B_\alpha(z)}{B_\alpha(y)} = \frac{|w|_x - |u|_x}{|u|_z - |w|_z} \ .$$

Case: Assume $\alpha(x) = \beta(x) = a$. Then we have

$$B_\beta(y)|u|_y + B_\beta(z)|u|_z = B_\beta(y)|w|_y + B_\beta(z)|w|_z$$

which implies

$$\frac{B_\alpha(y)}{B_\alpha(z)} = \frac{|w|_z - |u|_z}{|u|_y - |w|_y} = \frac{B_\beta(y)}{B_\beta(z)} \ . \tag{1}$$

Case: Assume $\alpha(x) = \beta(y) = a$. Then we have

$$B_\beta(x)|u|_x + B_\beta(z)|u|_z = B_\beta(x)|w|_x + B_\beta(z)|w|_z$$

which implies

$$\frac{B_\beta(z)}{B_\beta(x)} = \frac{|w|_x - |u|_x}{|u|_z - |w|_z}$$

and

$$A_\alpha(z) - A_\alpha(y)\frac{B_\alpha(z)}{B_\alpha(y)} = \frac{B_\beta(z)}{B_\beta(x)} \tag{2}$$

and

$$A_\beta(z) - A_\beta(x)\frac{B_\beta(z)}{B_\beta(x)} = \frac{B_\alpha(z)}{B_\alpha(y)} \ . \tag{3}$$

Case: Assume $\alpha(x) = \beta(z) = a$. Then we obtain in the same way

$$A_\alpha(y) - A_\alpha(z)\frac{B_\alpha(y)}{B_\alpha(z)} = \frac{B_\beta(y)}{B_\beta(x)} \tag{4}$$

and

$$A_\beta(y) - A_\beta(x)\frac{B_\beta(y)}{B_\beta(x)} = \frac{B_\alpha(y)}{B_\alpha(z)} \ . \tag{5}$$

The first lemma states that we can assume α and β to be of a certain shape.

Lemma 4. *If all equations $v_1 = v_2$ in $K_\alpha \cap K_\beta$ such that*

$$\alpha(x_1) \leq \alpha(x_2) \iff \beta(x_2) \leq \beta(x_1)$$

where $x_1 \leq v_1$ and $x_2 \leq v_2$ and $x_1, x_2 \in X$ are balanced, then all equations in $K_\alpha \cap K_\beta$ are balanced.

Proof. Assume $v_1 = v_2$ is any equation in $K_\alpha \cap K_\beta$.

Case: $\alpha(x_1) = \alpha(x_2)$ or $\beta(x_1) = \beta(x_2)$. Note that if $\alpha(x_1) = \alpha(x_2)$ and $\beta(x_1) = \beta(x_2)$, then $v_1 = v_2$ is balanced by assumption.

Let $\alpha(x_1) = \alpha(x_2)$ and $\beta(x_1) < \beta(x_2)$, the other case is symmetric. Assume that $v_1 = v_2$ is not balanced. Since α and β are incontractable, we have $\alpha(x) = a$ and $\alpha(y) = \alpha(z) = b$ up to permutation of letters and variables as can be easily seen from the types of incontractable solutions in Section 3. Now, $|v_1|_x = |v_2|_x$, and we have also $\beta(x) = a$, otherwise the shape of β implies that either $|v_1|_y = |v_2|_y$ or $|v_1|_z = |v_2|_z$, since $B_\beta(v_1) = B_\beta(v_2)$, and $v_1 = v_2$ is balanced, which is a contradiction. But now, equation (1) gives

$$\frac{B_\alpha(y)}{B_\alpha(z)} = \frac{B_\beta(y)}{B_\beta(z)} = 1 \ ;$$

a contradiction again since we have necessarily $\gcd(B_\beta(y), B_\beta(z)) = 1$ and also $\max\{B_\beta(y), B_\beta(z)\} > 1$, otherwise β is a permutation of α.

Case: $\alpha(x_1) < \alpha(x_2)$ and $\beta(x_1) < \beta(x_2)$. Let $\alpha' \colon X^+ \to A^+$ and $\beta' \colon X^+ \to A^+$ such that

$$\alpha'(x_2) = \alpha(x_1)^{-1}\alpha(x_2) \qquad \text{and} \qquad \beta'(x_2) = \beta(x_1)^{-1}\beta(x_2)$$

and $\alpha'(x_3) = \alpha(x_3)$ and $\beta'(x_3) = \beta(x_3)$ for all $x_3 \in X$ such that $x_3 \neq x_2$, and let $\sigma \colon X^+ \to X^+$ such that

$$\sigma(x_2) = x_1 x_2$$

and $\sigma(x_3) = x_3$ for all $x_3 \in X$ such that $x_3 \neq x_2$. Let $v_1' = v_2'$ be the reduced equation $\sigma(v_1) = \sigma(v_2)$.

Now, α' and β' are both solutions for $v_1' = v_2'$ and

$$v_1' = v_2' \ \in \ K_{\alpha'} \cap K_{\beta'}$$

and it is easy to see by the shape of σ that $v_1 = v_2$ is balanced, if, and only if, $v_1' = v_2'$ is balanced.

Let α'' and β'' be incontractable morphisms such that $K_{\alpha'} = K_{\alpha''}$ and also $K_{\beta'} = K_{\beta''}$. Then α'' is of smaller size than α and β'' is of smaller size than β. Now, either $v_1' = v_2'$ is balanced or we can repeat this construction. In the former case we have that $v_1 = v_2$ is balanced. By repetition of the above construction, we get a sequence of equations where the size of the corresponding incontractable solutions gets strictly smaller. Therefore, the sequence of equations must end with a balanced equation which implies that also $v_1 = v_2$ is balanced. □

Remark 5. Since, by Lemma 4, only equations $v_1 = v_2$ in $K_\alpha \cap K_\beta$ such that

$$\alpha(x_1) \le \alpha(x_2) \iff \beta(x_2) \le \beta(x_1)$$

where $x_1 \le v_1$ and $x_2 \le v_2$ and $x_1, x_2 \in X$ need to be shown to be balanced, the case where $\alpha(x) = \beta(x) = a$ and $u = w$ begins or ends with x does not need to be investigated in the following.

In the next two subsections we investigate the intersection of entire systems of different type first and of the same type then.

5.1. **Entire Systems of Different Type.** The entire systems K_α and K_β are assumed to be of different type throughout this subsection.

Lemma 6. *If $\tau_\alpha = 1$ or $\tau_\beta = 1$ then $K_\alpha \cap K_\beta$ contains only balanced equations.*

Proof. If $\tau_\alpha = 1$ then $|u|_x = |w|_x$ which implies $\beta(x) = a$, for, otherwise $|u|_y = |w|_y$ or $|u|_z = |w|_z$, by counting the number of occurences of b in $\beta(y)$ and $\beta(z)$, and $u = w$ is balanced. Now, $\tau_\beta \in \{2, 6\}$ since $u = w$ starts and ends with y and z. But, α implies $(yx_1, zx_2) \le e$ where $x_1, x_2 \in \{y, z\}$ which implies that $\tau_\beta \ne 2$ since $k_\beta \ge 1$. But, also $\tau_\beta \ne 6$ since equation (1) gives

$$\frac{q_\alpha + 1}{p_\alpha + 1} = B_\beta(y) \qquad \text{or} \qquad \frac{q_\alpha + 1}{p_\alpha + 1} = \frac{1}{B_\beta(z)} \ ;$$

a contradiction since $\gcd(p_\alpha + 1, q_\alpha + 1) = 1$. □

We will not consider type (1) for the rest of this section anymore.

Lemma 7. *If $(y, z) \le e$ and $\alpha(x) = \beta(x) = a$ then $K_\alpha \cap K_\beta$ contains only balanced equations.*

Proof. Since $\alpha(x) = a$, we have that $b \le \alpha(y)$ and $b \le \alpha(z)$ and α must be of type 2 or 6, and since $\beta(x) = a$ we have $b \le \beta(y)$ and $b \le \beta(z)$ and β must be of type 2 or 6. Equation (1) gives that $B_\alpha(y)/B_\alpha(z)$ is an integer if α is of type 6, and if α is of type 2 then $B_\alpha(y)/B_\alpha(z)$ is not an integer since $\gcd(p_\alpha+1, q_\alpha+1) = 1$. The same holds for $B_\beta(y)/B_\beta(z)$. So, α and β must be of the same type; a contradiction. □

Lemma 8. *If $(x, y) \le e$ and $\alpha(x) = \beta(y) = a$ then $K_\alpha \cap K_\beta$ contains only balanced equations.*

Proof. We have $a \le \alpha(y)$ and $a \le \beta(x)$ and $b \le \alpha(z)$ and $b \le \beta(z)$. It follows that $\tau_\alpha = 6$ and α is of the shape

$$\begin{aligned} x &\mapsto a \\ y &\mapsto \left(a^{i'_\alpha} b a^{k_{1,\alpha}+j_\alpha} b \cdots b a^{k_{n,\alpha}+j_\alpha} b a^{j'_\alpha}\right)^{q_\alpha} \\ z &\mapsto b a^{j_\alpha} \end{aligned}$$

where $q_\alpha, i'_\alpha \ge 1$ and $j_\alpha, j'_\alpha \ge 0$ and $j_\alpha j'_\alpha = 0$ and $k_{t,\alpha} \ge 0$, for all $1 \le t \le n_\alpha$ and $n_\alpha \ge 0$, and

$$(x^{i'_\alpha} z, y) \le e \ . \tag{6}$$

Case: Assume $\tau_\beta = 2$. Then β is of the shape

$$\begin{aligned} x &\mapsto a^{i_\beta} \left(b a^{k_\beta}\right)^{p_\beta} b a^{j_\beta} \\ y &\mapsto a \\ z &\mapsto \left(b a^{k_\beta}\right)^{q_\beta} b \end{aligned}$$

where $p_\beta, q_\beta, k_\beta, i_\beta \ge 1$ and $j_\beta \ge 0$ and $i_\beta + j_\beta \le k_\beta$. We have that solution β implies $(xy^{k_\beta - j_\beta} z, y) \le e$ and we have $k_\beta = j_\beta$, by equation (6), a contradiction, or $(xy^{k_\beta - i_\beta - j_\beta} x, y) \le e$ and $k_\beta = i_\beta + j_\beta$ and equation (3) gives

$$q_\alpha(n_\alpha + 1) = \frac{1}{k_\beta} = 1$$

and $k_\beta = 1$ and $j_\beta = 0$. Now, β implies $(x, yz) \preccurlyeq e$ and α implies $(x, xz) \preccurlyeq e$ or $(x, zz) \preccurlyeq e$ since $j'_\alpha = 0$; a contradiction.

Case: Assume $\tau_\beta = 3$. Then β is of the shape

$$\begin{aligned} x &\mapsto a^{i_\beta} \left(ba^{k_\beta}\right)^{p_\beta - q_\beta - 1} ba^{j_\beta} \\ y &\mapsto a \\ z &\mapsto \left(ba^{k_\beta}\right)^{q_\beta} b \end{aligned}$$

where $p_\beta > q_\beta \geq 1$ and $1 \leq i_\beta, j_\beta < k_\beta < i_\beta + j_\beta$. Solution β implies $(xy, y) \leq e$ which contradicts α, see equation (6).

Case: Assume $\tau_\beta = 4$. Then β is of the shape

$$\begin{aligned} x &\mapsto a^{i_\beta} ba^{j_\beta} \\ y &\mapsto a \\ z &\mapsto \left(ba^{k_{1,\beta}+i_\beta+j_\beta} b \cdots ba^{k_{n_\beta,\beta}+i_\beta+j_\beta} ba^{k_\beta}\right)^{q_\beta} ba^{k_{1,\beta}+i_\beta+j_\beta} b \cdots ba^{k_{n_\beta,\beta}+i_\beta+j_\beta} b \end{aligned}$$

where $q_\beta \geq 1$ and $1 \leq i_\beta, j_\beta < k_\beta < i_\beta + j_\beta$ and $k_{t,\beta} \geq 0$, for all $1 \leq t \leq n_\beta$ and $n_\beta \geq 0$. Now, $(x^{i'_\alpha} z, y^{i_\beta} z) \leq e$ and by the shape of $\beta(z)$ necessarily $i'_\alpha \geq n_\beta + 1$. If $i'_\alpha = n_\beta + 1$ then $k_\beta = j_\beta$, and if $i'_\alpha > n_\beta + 1$ then $k_\beta = i_\beta + j_\beta$; a contradiction in both cases.

Case: Assume $\tau_\beta = 5$ or $\tau_\beta = 8$. Then $a \leq \beta(x)$ and $b \preccurlyeq \beta(x)$ and $a = \beta(y)$ and $b \leq \beta(z)$ and $a \preccurlyeq \beta(z)$ and e ends in y and z which implies $j_\alpha = j'_\alpha = 0$ and equation (2) gives

$$\frac{B_\beta(z)}{B_\beta(x)} = -\frac{A_\alpha(y)}{B_\alpha(y)} \ ;$$

a contradiction. □

Lemma 9. *If $(x, z) \leq e$ and $\alpha(x) = \beta(y) = a$ then $K_\alpha \cap K_\beta$ contains only balanced equations.*

Proof. We have $a \leq \alpha(z)$ and $b \leq \alpha(y)$ and $b \leq \beta(x)$ and $b \leq \beta(z)$, and it follows that $\tau_\beta \in \{2, 6\}$.

Case: Assume $\tau_\beta = 2$. Then β is of the shape

$$\begin{aligned} x &\mapsto \left(ba^{k_\beta}\right)^{q_\beta} b \\ y &\mapsto a \\ z &\mapsto \left(ba^{k_\beta}\right)^{p_\beta} ba^{j_\beta} \end{aligned} \tag{7}$$

or

$$\begin{aligned} x &\mapsto \left(ba^{k_\beta}\right)^{p_\beta} ba^{j_\beta} \\ y &\mapsto a \\ z &\mapsto \left(ba^{k_\beta}\right)^{q_\beta} b \end{aligned} \tag{8}$$

where $0 \leq j_\beta \leq k_\beta$.

Subcase: Assume $j_\beta = 0$. Then both shapes of τ_β are symmetric and we consider only case (7). Note, equations solved by β end with x and z which gives $\tau_\alpha \in \{3, 4, 6\}$. If $\tau_\alpha = 3$ then $k_\alpha = j_\alpha$ since β implies $(x, zy) \leq e$ but this contradicts the definition of type 3 where $k_\alpha > j_\alpha$. If $\tau_\alpha = 4$ or $\tau_\alpha = 6$ then equation (3) gives

$$B_\alpha(y) = \frac{q_\beta + 1}{k_\beta \left(p_\beta - q_\beta\right)} \ ;$$

a contradiction since $\gcd(p_\beta+1, q_\beta+1) = 1$.

Subcase: Assume $j_\beta > 0$ and β is of shape (7). Then e ends with y and z, and $\tau_\alpha = 6$. Solution β implies now $(xy, zy^{k_\beta - j_\beta}) \le e$ and $k_\beta = j_\beta$ or $\alpha(y) = b$ by the shape of α. If $k_\beta = j_\beta$ then equation (3) gives

$$B_\alpha(y) = \frac{q_\beta + 1}{k_\beta\,(p_\beta + 1)} \;;$$

a contradiction since $\gcd(p_\beta+1, q_\beta+1)$. If $\alpha(y) = b$ then equation (2) gives

$$\frac{q_\beta + 1}{p_\beta + 1} = i_\alpha \;;$$

a contradiction since $\gcd(p_\beta+1, q_\beta+1) = 1$.

Subcase: Assume $j_\beta > 0$ and β is of shape (8). Then e ends with x and y, and $\tau_\alpha \in \{5, 6, 8\}$.

If $\tau_\alpha = 5$ or $\tau_\alpha = 8$ then α implies $(x, zx) \le e$ and β implies $(x, zy) \le e$; a contradiction.

If $\tau_\alpha = 6$ then β implies $(xy^{k_\beta - j_\beta}, zy) \le e$ and $k_\beta = j_\beta$ by the shape of α. Now, equation (3) gives

$$-k_\beta = \frac{1}{q_\alpha(n_\alpha + 1)} \;;$$

a contradiction.

Case: Assume $\tau_\beta = 6$. Then β is of the shape

$$\begin{aligned} x &\mapsto ba^{j_\beta} \\ y &\mapsto a \\ z &\mapsto \left(ba^{k_{1,\beta}+j_\beta} b \cdots ba^{k_{n_\beta,\beta}+j_\beta} ba^{j'_\beta}\right)^{q_\beta} \end{aligned} \tag{9}$$

or

$$\begin{aligned} x &\mapsto \left(ba^{k_{1,\beta}+j_\beta} b \cdots ba^{k_{n_\beta,\beta}+j_\beta} ba^{j'_\beta}\right)^{q_\beta} \\ y &\mapsto a \\ z &\mapsto ba^{j_\beta} \end{aligned} \tag{10}$$

where $q_\beta, i'_\beta \ge 1$ and $j_\beta, j'_\beta \ge 0$ and $j_\beta j'_\beta = 0$ and $k_{t,\beta} \ge 0$, for all $1 \le t \le n_\beta$ and $n_\beta \ge 0$.

Subcase: Assume β is of shape (9). Then equation (3) gives

$$\frac{B_\alpha(z)}{B_\alpha(y)} = A_\beta(z) - A_\beta(x) B_\beta(z)$$

which implies $B_\alpha(y) = 1$, but then $\tau_\alpha = 6$; a contradiction.

Subcase: Assume β is of shape (10). Then equation (3) gives

$$\frac{B_\alpha(z)}{B_\alpha(y)} = j_\beta - \frac{A_\beta(x)}{B_\beta(x)}$$

where $j_\beta \geq 1$ and $j'_\beta = 0$ and e ends with y and z which implies $\tau_\alpha = 2$ with $i_\alpha \geq 1$ and $j_\alpha = 0$. So, α is of the shape

$$\begin{aligned} x &\mapsto a \\ y &\mapsto \left(ba^{k_\alpha}\right)^{q_\alpha} b \\ z &\mapsto a^{i_\alpha} \left(ba^{k_\alpha}\right)^{p_\alpha} b \end{aligned}$$

where $i_\alpha \leq k_\alpha$. If $i_\alpha = k_\alpha$ then equation (2) gives

$$\frac{1}{q_\beta + 1} = k_\alpha \frac{p_\alpha + 1}{q_\alpha + 1}$$

a contradiction since $\gcd(p_\alpha + 1, q_\alpha + 1) = 1$. If $i_\alpha < k_\alpha$ then α implies that $(x^{k_\alpha} y, x^{k_\alpha - i_\alpha} z) \preccurlyeq e$ which gives $q_\beta = j_\beta = 1$ and $n_\beta = 0$ But now, equation (3) gives

$$\frac{p_\alpha + 1}{q_\alpha + 1} = 1 \ ;$$

a contradiction since $\gcd(p_\alpha + 1, q_\alpha + 1) = 1$. □

Lemma 10. *If $(y, z) \leq e$ and $\alpha(x) = \beta(y) = a$ then $K_\alpha \cap K_\beta$ contains only balanced equations.*

Proof. Since $\alpha(x) = a$ we have $b \leq \alpha(y)$ and $b \leq \alpha(z)$ and α must be of type 2 or 6, and since $\beta(y) = a$ we have $a \leq \beta(z)$ and $b \leq \beta(x)$.

Case: Assume $\tau_\alpha = 2$. Then $i_\alpha = 0$ and α is of the shape

$$\begin{aligned} x &\mapsto a \\ y &\mapsto \left(ba^{k_\alpha}\right)^{q_\alpha} b \\ z &\mapsto \left(ba^{k_\alpha}\right)^{p_\alpha} ba^{j_\alpha} \end{aligned}$$

where $p_\alpha, q_\alpha, k_\alpha \geq 1$ and $j_\alpha \geq 0$ and $j_\alpha \leq k_\alpha$. Now, α implies

$$(yx^{k_\alpha}, zx^{k_\alpha - j_\alpha}) \leq e \tag{11}$$

and

$$(x^{k_\alpha - j_\alpha} y x^{j_\alpha}, x^{k_\alpha - j_\alpha} z) \preccurlyeq e \ . \tag{12}$$

Subcase: Assume $\tau_\alpha = 2$ and $\tau_\beta = 3$. Then β is of the form

$$\begin{aligned} x &\mapsto \left(ba^{k_\beta}\right)^{q_\beta} b \\ y &\mapsto a \\ z &\mapsto a^{i_\beta} \left(ba^{k_\beta}\right)^{p_\beta - q_\beta - 1} ba^{j_\beta} \end{aligned}$$

where $1 \leq i_\beta, j_\beta < k_\beta < i_\beta + j_\beta$ and equations solved by β end with y and z, which implies $j_\alpha = 0$. But now, by (11) and (12), we have $i_\beta = j_\beta = 1$ and $1 < k_\beta < 2$; a contradiction.

Subcase: Assume $\tau_\alpha = 2$ and $\tau_\beta = 4$. Then β is of the form

$$\begin{aligned} x &\mapsto \left(ba^{k_{1,\beta} + i_\beta + j_\beta} b \cdots ba^{k_{n_\beta,\beta} + i_\beta + j_\beta} ba^{k_\beta}\right)^{q_\beta} ba^{k_{1,\beta} + i_\beta + j_\beta} b \cdots ba^{k_{n_\beta,\beta} + i_\beta + j_\beta} b \\ y &\mapsto a \\ z &\mapsto a^{i_\beta} ba^{j_\beta} \end{aligned}$$

where $q_\beta \geq 1$ and $1 \leq i_\beta, j_\beta \leq k_\beta < i_\beta + j_\beta$ and $n_\beta \geq 0$ and equations solved by β end with y and z, which implies $j_\alpha = 0$. From (11) and (12) follows that $i_\beta = j_\beta = k_\beta = 1$ and $n_\beta = 0$ and also $k_\alpha = 1$. Equation (2) gives

$$B_\beta(x) = \frac{q_\alpha + 1}{p_\alpha - q_\alpha} ;$$

a contradiction since $\gcd(q_\alpha + 1, p_\alpha + 1) = 1$.

Subcase: Assume $\tau_\alpha = 2$ and $\tau_\beta = 5$ or $\tau_\beta = 8$. Then an equation solved by β ends in x and y, and hence, cannot be solved by α.

Subcase: Assume $\tau_\alpha = 2$ and $\tau_\beta = 6$ and β is of the form

$$\begin{aligned} x &\mapsto \left(ba^{k_{1,\beta}+i_\beta+j_\beta} b \cdots ba^{k_{n_\beta,\beta}+i_\beta+j_\beta} ba^{j'_\beta}\right)^{q_\beta} \\ y &\mapsto a \\ z &\mapsto a^{i_\beta} ba^{j_\beta} \end{aligned}$$

where $i_\beta \geq 1$ and $j_\beta, j'_\beta \geq 0$ and $q_\beta \geq 1$ and $j_\beta j'_\beta = 0$. Note, that actually $j'_\beta = 0$ since equations solved by α end with y and z or x and z. If $k_\alpha = j_\alpha$ then equation (2) gives

$$B_\beta(x) = \frac{q_\alpha + 1}{k_\alpha(p_\alpha + 1)} ;$$

a contradiction since $\gcd(q_\alpha + 1, p_\alpha + 1) = 1$. Hence, $k_\alpha > j_\alpha$ and α implies $(yx, zx) \leq e$, but for equations solved by β we have $(y, zz) \leq e$ or $(y, zy) \leq e$; a contradiction.

Subcase: Assume $\tau_\alpha = 2$ and $\tau_\beta = 6$ and β is of the form

$$\begin{aligned} x &\mapsto ba^{j_\beta} \\ y &\mapsto a \\ z &\mapsto \left(a^{i'_\beta} ba^{k_{1,\beta}+j_\beta} b \cdots ba^{k_{n_\beta,\beta}+j_\beta} ba^{j'_\beta}\right)^{q_\beta} \end{aligned}$$

where $i' \geq 1$ and $j_\beta, j'_\beta \geq 0$ and $q_\beta \geq 1$ and $j_\beta j'_\beta = 0$. Note, that actually $j_\beta = 0$ since equations solved by α end with y and z or x and z. Equation (3) gives

$$\frac{p_\alpha + 1}{q_\alpha + 1} = A_\beta(z) ;$$

a contradiction since $\gcd(p_\alpha + 1, q_\alpha + 1) = 1$.

Case: Assume $\tau_\alpha = 6$. Then $i_\alpha = i'_\alpha = 0$ and α is of the shape

$$\begin{aligned} x &\mapsto a \\ y &\mapsto \left(ba^{k_{1,\alpha}+j_\alpha} b \cdots ba^{k_{n_\alpha,\alpha}+j_\alpha} ba^{j'_\alpha}\right)^{q_\alpha} \\ z &\mapsto ba^{j_\alpha} \end{aligned}$$

where $q_\alpha \geq 1$ and $j_\alpha, j'_\alpha \geq 0$ and $j_\alpha j'_\alpha = 0$ and $k_{t,\alpha} \geq 0$, for all $1 \leq t \leq n_\alpha$ and $n_\alpha \geq 0$. We observe for later considerations that if

$$(y^s x, z^t y) \leq e \tag{13}$$

with $s, t \geq 1$, and if $q_\alpha > 1$ or $s > 1$ then we have $\left(ba^{j_\alpha}\right)^{n_\alpha} ba^{j'_\alpha} b \leq \alpha(u)$ and $\left(ba^{j_\alpha}\right)^{n_\alpha+1} b \leq \alpha(w)$ and $j_\alpha = j'_\alpha = 0$ and $k_{t,\alpha} = 0$, for all $1 \leq t \leq n_\alpha$. But, then $|u|_x = |w|_x$ and since $\beta(y) = a$ we have also $|u|_z = |w|_z$ and e is balanced; a contradiction. Hence, $q_\alpha = s = 1$ and then $\left(ba^{j_\alpha}\right)^{n_\alpha} ba^{j'_\alpha} a \leq \alpha(u)$ and

$\left(ba^{j_\alpha}\right)^{n_\alpha+1} b \le \alpha(w)$ and $j_\alpha > 0$ and $j'_\alpha = 0$ and equations solved by α end with x and z.

Subcase: Assume $\tau_\alpha = 6$ and $\tau_\beta = 2$. Then β is of the form

$$\begin{aligned} x &\mapsto \left(ba^{k_\beta}\right)^{q_\beta} b \\ y &\mapsto a \\ z &\mapsto a^{i_\beta} \left(ba^{k_\beta}\right)^{p_\beta} ba^{j_\beta} \end{aligned}$$

where $p_\beta, q_\beta, k_\beta, i_\beta \ge 1$ and $j_\beta \ge 0$ and $i_\beta + j_\beta \le k_\beta$. Solution β implies that $(y^{i_\beta}x, zy^{k_\beta - i_\beta - j_\beta}z) \le e$ or $(y^{i_\beta}x, zy^{k_\beta - j_\beta}x) \le e$. If $k_\beta = i_\beta + j_\beta$ then equation (3) gives

$$B_\alpha(y) = \frac{q_\beta + 1}{k_\beta(p_\beta + 1)} \ ;$$

a contradiction since $\gcd(p_\beta+1, q_\beta+1) = 1$. Therefore, $(y^{i_\beta}x, zy) \le e$ which implies that e ends with x and z by equation (13), and hence, $j_\beta = 0$ and $i_\beta = 1$, since $s = 1$ for equation (13). Now, either $(yx, xz) \preccurlyeq e$ or $(yx, yz) \preccurlyeq e$. In the previous case equation (3) implies

$$B_\alpha(y) = \frac{q_\beta + 1}{p_\beta + 1} \ ;$$

a contradiction since $\gcd(p_\beta + 1, q_\beta + 1) = 1$. In the latter case we have then $\left(ba^{j_\alpha}\right)^{n_\alpha} ba \preccurlyeq \alpha(u)$ and $\left(ba^{j_\alpha}\right)^{n_\alpha} bba^{j_\alpha} \preccurlyeq \alpha(w)$ and $n_\alpha = 0$, since $j_\alpha > 0$, and we have $B_\alpha(y) = 1$. Equation (3) gives

$$k_\beta \frac{p_\beta - q_\beta}{q_\beta + 1} + 1 = 1 \ ;$$

a contradiction.

Subcase: Assume $\tau_\alpha = 6$ and $\tau_\beta = 3$. Then β is of the form

$$\begin{aligned} x &\mapsto \left(ba^{k_\beta}\right)^{q_\beta} b \\ y &\mapsto a \\ z &\mapsto a^{i_\beta} \left(ba^{k_\beta}\right)^{p_\beta - q_\beta - 1} ba^{j_\beta} \end{aligned}$$

where $i_\beta, j_\beta \ge 1$ and $k_\beta < i_\beta + j_\beta$ and $p_\beta > q_\beta \ge 1$. Equation (3) gives

$$\frac{1}{B_\alpha(y)} = k_\beta \left(\frac{p_\beta - q_\beta}{q_\beta + 1} - 1\right) + i_\beta + j_\beta$$

and $p_\beta < q_\beta$ since $i_\beta + j_\beta > k_\beta$; a contradiction.

Subcase: Assume $\tau_\alpha = 6$ and $\tau_\beta = 4$. Then β is of the form

$$\begin{aligned} x &\mapsto \left(ba^{k_{1,\beta}+i_\beta+j_\beta} b \cdots ba^{k_{n_\beta,\beta}+i_\beta+j_\beta} ba^{k_\beta}\right)^{q_\beta} ba^{k_{1,\beta}+i_\beta+j_\beta} b \cdots ba^{k_{n_\beta,\beta}+i_\beta+j_\beta} b \\ y &\mapsto a \\ z &\mapsto a^{i_\beta} ba^{j_\beta} \end{aligned}$$

where $q_\beta \ge 1$ and $1 \le i_\beta, j_\beta \le k_\beta < i_\beta + j_\beta$ and $n_\beta \ge 0$. Equations solved by β end with y and z which implies $j_\alpha = j'_\alpha = 0$. Equation (2) gives

$$-\frac{A_\alpha(y)}{B_\alpha(y)} = \frac{1}{B_\beta(x)} \ ;$$

a contradiction.

Subcase: Assume $\tau_\alpha = 6$ and $\tau_\beta = 5$ or $\tau_\beta = 8$. Then β is of the form

$$\begin{aligned} x &\mapsto b\left(a^{i_\beta+j_\beta+m_\beta}b\right)^{q_\beta} a^{j_\beta} \\ y &\mapsto a \\ z &\mapsto a^{i_\beta} b \left(a^{i_\beta+j_\beta+m_\beta}b\right)^{p_\beta} \end{aligned} \qquad \text{or} \qquad \begin{aligned} x &\mapsto b\left(a^{i_\beta+j_\beta+r_\beta}b\right)^{q_\beta} a^{j_\beta+r_\beta} \\ y &\mapsto a \\ z &\mapsto a^{i_\beta+r_\beta} b \left(a^{i_\beta+j_\beta+r_\beta}b\right)^{p_\beta} \end{aligned}$$

where $p_\beta, q_\beta, i_\beta, j_\beta, r_\beta \geq 1$ and $m_\beta \geq 0$. Solution β implies $(yx, y) \preccurlyeq e$ which contradicts solution α. □

Lemma 11. *If $(y,z) \leq e$ and $\alpha(x) = \beta(z) = a$ then $K_\alpha \cap K_\beta$ contains only balanced equations.*

Proof. We have $b \leq \alpha(y)$ and $b \leq \alpha(z)$ and $b \leq \beta(x)$ and $a \leq \beta(y)$. It follows that $\tau_\alpha \in \{2, 6\}$.

Case: Assume $\tau_\alpha = 2$. Then $i_\alpha = 0$ and α is of the shape

$$\begin{aligned} x &\mapsto a \\ y &\mapsto \left(ba^{k_\alpha}\right)^{q_\alpha} b \\ z &\mapsto \left(ba^{k_\alpha}\right)^{p_\alpha} ba^{j_\alpha} \end{aligned}$$

where $p_\alpha, q_\alpha, k_\alpha \geq 1$ and $j_\alpha \geq 0$ and $k_\alpha \geq j_\alpha$. Now, α implies

$$(yx^{k_\alpha}, z) \leq e \,. \tag{14}$$

Subcase: Assume $\tau_\alpha = 2$ and $\tau_\beta = 3$. Then $k_\beta = j_\beta$ since equations solved by β begin with $(yz^{k_\beta - j_\beta}, z) \leq e$; a contradiction by the definition of type (3).

Subcase: Assume $\tau_\alpha = 2$ and $\tau_\beta = 4$. Then β is of the form

$$\begin{aligned} x &\mapsto \left(ba^{k_{1,\beta}+i_\beta+j_\beta}b \cdots ba^{k_{n_\beta,\beta}+i_\beta+j_\beta} ba^{k_\beta}\right)^{q_\beta} ba^{k_{1,\beta}+i_\beta+j_\beta}b \cdots ba^{k_{n_\beta,\beta}+i_\beta+j_\beta} b \\ y &\mapsto a^{i_\beta} ba^{j_\beta} \\ z &\mapsto a \end{aligned}$$

where $p_\beta > q_\beta \geq 1$ and $1 \leq i_\beta, j_\beta \leq k_\beta < i_\beta + j_\beta$ and $n_\beta \geq 0$ and $k_{t,\beta} \geq 0$, for all $1 \leq t \leq n_\beta$. Equation (14) gives by the shape of β that $(yx^{k_\alpha}, z^{i_\beta}x) \leq e$ which implies either $j_\beta = k_{i_\beta,\beta} + i_\beta + j_\beta$, if $i_\beta \leq n_\beta$, or $k_\beta = j_\beta$; a contradiction in both cases.

Subcase: Assume $\tau_\alpha = 2$ and $\tau_\beta = 5$ or $\tau_\beta = 8$. Then β is of the form

$$\begin{aligned} x &\mapsto \left(ba^{i_\beta+j_\beta+m_\beta}\right)^{q_\beta} ba^{j_\beta} \\ y &\mapsto a^{i_\beta} b \left(a^{i_\beta+j_\beta+m_\beta}b\right)^{p_\beta} \\ z &\mapsto a \end{aligned} \qquad \text{or} \qquad \begin{aligned} x &\mapsto \left(ba^{i_\beta+j_\beta+r_\beta}\right)^{q_\beta} ba^{j_\beta+r_\beta} \\ y &\mapsto a^{i_\beta+r_\beta} b \left(a^{i_\beta+j_\beta+r_\beta}b\right)^{p_\beta} \\ z &\mapsto a \end{aligned}$$

where $p_\beta, q_\beta, i_\beta, j_\beta, r_\beta \geq 1$ and $m_\beta \geq 0$, and β implies $(yz, z) \leq e$ which contradicts (14).

Subcase: Assume $\tau_\alpha = 2$ and $\tau_\beta = 6$. Then β is of the form

$$\begin{aligned} x &\mapsto \left(ba^{k_{1,\beta}+i_\beta+j_\beta}b \cdots ba^{k_{n_\beta,\beta}+i_\beta+j_\beta} ba^{j'_\beta}\right)^{q_\beta} \\ y &\mapsto a^{i_\beta} ba^{j_\beta} \\ z &\mapsto a \end{aligned} \tag{15}$$

or

$$\begin{aligned} x &\mapsto ba^{j_\beta} \\ y &\mapsto \left(a^{i'_\beta} ba^{k_{1,\beta}+j_\beta} b \cdots ba^{k_{n_\beta,\beta}+j_\beta} ba^{j'_\beta}\right)^{q_\beta} \\ z &\mapsto a \end{aligned} \tag{16}$$

where $q_\beta \geq 1$ and $i_\beta, i'_\beta, j_\beta, j'_\beta, n_\beta \geq 0$ and $i_\beta i'_\beta = j_\beta j'_\beta = 0$ and $k_{t,\beta} \geq 0$, for all $1 \leq t \leq n_\beta$. If β is of shape (15) then (14) implies $(yx, z^{i_\beta} x) \leq e$. We get either $j_\beta = k_{1,\beta} + i_\beta + j_\beta$, a contradiction, or $n_\beta = j_\beta = j'_\beta = 0$ and equation (5) gives

$$\frac{q_\alpha + 1}{p_\alpha + 1} = i_\beta \ ;$$

a contradiction since $\gcd(p_\alpha + 1, q_\alpha + 1) = 1$. If β is of shape (16) then equation (5) gives

$$\frac{q_\alpha + 1}{p_\alpha + 1} = A_\beta(y) - j_\beta B_\beta(y) \ ;$$

a contradiction since $\gcd(p_\alpha + 1, q_\alpha + 1) = 1$.

Case: Assume $\tau_\alpha = 6$. Then α is of the shape

$$\begin{aligned} x &\mapsto a \\ y &\mapsto \left(ba^{k_{1,\alpha}+j_\alpha} b \cdots ba^{k_{n_\alpha,\alpha}+j_\alpha} ba^{j'_\alpha}\right)^{q_\alpha} \\ z &\mapsto ba^{j_\alpha} \end{aligned}$$

where $j_\alpha, j'_\alpha, n_\alpha \geq 0$ and $j_\alpha j'_\alpha = 0$ and $k_{t,\alpha} \geq 0$, for all $1 \leq t \leq n_\alpha$.

Subcase: Assume $\tau_\alpha = 6$ and $\tau_\beta \in \{2, 3, 5, 8\}$. Then equation (4) gives

$$\frac{q_\beta + 1}{p_\beta + 1} = A_\alpha(y) - j_\alpha B_\alpha(y) \ ;$$

a contradiction since $\gcd(p_\beta + 1, q_\beta + 1) = 1$.

Subcase: Assume $\tau_\alpha = 6$ and $\tau_\beta = 4$. Then β is of the form

$$\begin{aligned} x &\mapsto \left(ba^{k_{1,\beta}+i_\beta+j_\beta} b \cdots ba^{k_{n_\beta,\beta}+i_\beta+j_\beta} ba^{k_\beta}\right)^{q_\beta} ba^{k_{1,\beta}+i_\beta+j_\beta} b \cdots ba^{k_{n_\beta,\beta}+i_\beta+j_\beta} b \\ y &\mapsto a^{i_\beta} ba^{j_\beta} \\ z &\mapsto a \end{aligned}$$

where $q_\beta \geq 1$ and $1 \leq i_\beta, j_\beta \leq k_\beta < i_\beta + j_\beta$ and $n_\beta \geq 0$ and $k_{t,\beta} \geq 0$, for all $1 \leq t \leq n_\beta$. Equations solved by β end with y and z, hence, $j_\alpha = j'_\alpha = 0$ and equation (4) gives

$$\frac{q_\beta + 1}{p_\beta + 1} = A_\alpha(y) \ ;$$

a contradiction since $\gcd(p_\beta + 1, q_\beta + 1) = 1$. □

Lemma 12. *If $(x, y) \leq e$ and $\alpha(x) = \beta(z) = a$ then $K_\alpha \cap K_\beta$ contains only balanced equations.*

Proof. We have $a \leq \alpha(y)$ and $b \leq \alpha(z)$ and $b \leq \beta(x)$ and $b \leq \beta(y)$. It follows that $\tau_\alpha = 6$. Equation (4) gives

$$\frac{B_\beta(y)}{B_\beta(z)} = A_\alpha(y) - A_\alpha(z) B_\alpha(y)$$

which implies $\tau_\beta = 4$ and either $a \le \beta(x)$ or $a \le \beta(y)$ since $\beta(z) = a$; a contradiction. ◻

Lemma 13. *If $(x, z) \le e$ and $\alpha(x) = \beta(z) = a$ then $K_\alpha \cap K_\beta$ contains only balanced equations.*

Proof. We have $a \le \alpha(z)$ and $a \le \beta(x)$ and $b \le \alpha(y)$ and $b \le \beta(y)$.

Case: Assume $\tau_\alpha = 2$. Then α is of the shape

$$\begin{aligned} x &\mapsto a \\ y &\mapsto \left(ba^{k_\alpha}\right)^{q_\alpha} b \\ z &\mapsto a^{i_\alpha} \left(ba^{k_\alpha}\right)^{p_\alpha} ba^{j_\alpha} \end{aligned}$$

where $i_\alpha \le 1$ and $j_\alpha \le 0$ and $i_\alpha + j_\alpha \le k_\alpha$. Solution α implies

$$(x^{i_\alpha} y, zx^{k_\alpha - i_\alpha} y) \le e \tag{17}$$

or

$$(x^{i_\alpha} y, zx^{k_\alpha - i_\alpha - j_\alpha} z) \le e\ . \tag{18}$$

Subcase: Assume $\tau_\alpha = 2$ and $\tau_\beta = 3$. Then β is of the form

$$\begin{aligned} x &\mapsto a^{i_\beta} \left(ba^{k_\beta}\right)^{p_\beta} ba^{j_\beta} \\ y &\mapsto \left(ba^{k_\beta}\right)^{q_\beta} b \\ z &\mapsto a \end{aligned}$$

where $1 \le i_\beta, j_\beta < k_\beta < i_\beta + j_\beta$. Solution β implies $(xz^{k_\beta - j_\beta}, z) \le e$ which contradicts (17) and (18).

Subcase: Assume $\tau_\alpha = 2$ and $\tau_\beta = 4$. Then β is of the form

$$\begin{aligned} x &\mapsto a^{i_\beta} ba^{j_\beta} \\ y &\mapsto \left(ba^{k_{1,\beta}+i_\beta+j_\beta} b \cdots ba^{k_{n_\beta,\beta}+i_\beta+j_\beta} ba^{k_\beta}\right)^{q_\beta} ba^{k_{1,\beta}+i_\beta+j_\beta} b \cdots ba^{k_{n_\beta,\beta}+i_\beta+j_\beta} b \\ z &\mapsto a \end{aligned}$$

where $q_\beta \ge 1$ and $1 \le i_\beta, j_\beta \le k_\beta < i_\beta + j_\beta$ and $n_\beta \ge 0$ and $k_{t,\beta} \ge 0$, for all $1 \le t \le n_\beta$. Solution β implies $(x, z^{i_\beta} y) \le e$ and $k_\alpha = i_\alpha$ and $j_\alpha = 0$. But, equations solved by α end with y and z, and equations solved by β end with x and z; a contradiction.

Subcase: Assume $\tau_\alpha = 2$ and $\tau_\beta = 5$ or $\tau_\beta = 8$. Then β is of the form

$$\begin{aligned} x &\mapsto a^{i_\beta} b \left(a^{i_\beta+j_\beta+m_\beta} b\right)^{p_\beta} \\ y &\mapsto \left(ba^{i_\beta+j_\beta+m_\beta}\right)^{q_\beta} ba^{j_\beta} \\ z &\mapsto a \end{aligned} \qquad \text{or} \qquad \begin{aligned} x &\mapsto a^{i_\beta+r_\beta} b \left(a^{i_\beta+j_\beta+r_\beta} b\right)^{p_\beta} \\ y &\mapsto \left(ba^{i_\beta+j_\beta+r_\beta}\right)^{q_\beta} ba^{j_\beta+r_\beta} \\ z &\mapsto a \end{aligned}$$

where $p_\beta, q_\beta, i_\beta, j_\beta, r_\beta \ge 1$ and $m_\beta \ge 0$. Solution β implies $(zy, z) \preccurlyeq e$ and $j_\alpha = 0$, and α implies $(xy, z) \preccurlyeq e$; a contradiction.

Subcase: Assume $\tau_\alpha = 2$ and $\tau_\beta = 6$. Then β is of the form

$$\begin{aligned} x &\mapsto a^{i_\beta} ba^{j_\beta} \\ y &\mapsto \left(ba^{k_{1,\beta}+i_\beta+j_\beta} b \cdots ba^{k_{n_\beta,\beta}+i_\beta+j_\beta} ba^{j'_\beta}\right)^{q_\beta} \\ z &\mapsto a \end{aligned}$$

or

$$
\begin{aligned}
x &\mapsto \left(ab^{k_{1,\beta}+j_\beta}a \cdots ab^{k_{n_\beta,\beta}+j_\beta}ab^{j'_\beta}\right)^{q_\beta} \\
y &\mapsto b \\
z &\mapsto a
\end{aligned}
$$

where $q_\beta, i_\beta \geq 1$ and $j_\beta, j'_\beta \geq 0$ and $j_\beta j'_\beta = 0$ and $n_\beta \geq 0$ and $k_{t,\beta} \geq 0$, for all $1 \leq t \leq n_\beta$. In both cases, solution β implies $(x, z^{i_\beta}y) \leq e$ which gives either $k_\alpha = i_\alpha$ or $k_\alpha = i_\alpha + j_\alpha$ by (17) and (18), respectively. Equation (4) gives

$$
B_\beta(y) = -k_\alpha \qquad \text{or} \qquad \frac{1}{B_\beta(y)} = -k_\alpha
$$

respectively; a contradiction.

Case: Assume $\tau_\alpha = 3$. Then α is of the shape

$$
\begin{aligned}
x &\mapsto a \\
y &\mapsto \left(ba^{k_\alpha}\right)^{q_\alpha} b \\
z &\mapsto a^{i_\alpha}\left(ba^{k_\alpha}\right)^{p_\alpha - q_\alpha - 1} ba^{j_\alpha}
\end{aligned}
$$

where $p_\alpha > q_\alpha \geq 1$ and $1 \leq i_\alpha, j_\alpha < k_\alpha < i_\alpha + j_\alpha$.

Subcase: Assume $\tau_\alpha = 3$ and $\tau_\beta = 6$ and β is of the form

$$
\begin{aligned}
x &\mapsto a^{q_\beta} \\
y &\mapsto b \\
z &\mapsto a
\end{aligned}
$$

where $q_\beta \geq 1$. Then $|u|_y = |w|_y$ and the shape of α implies $|u|_z = |w|_z$ and $u = w$ is balanced.

Subcase: Assume $\tau_\alpha = 3$ and β is not of the previous shape. Then α implies $(x, zx) \leq e$ and β does not solve e; a contradiction.

Case: Assume $\tau_\alpha = 4$. Then α is of the shape

$$
\begin{aligned}
x &\mapsto a \\
y &\mapsto \left(ba^{k_{1,\alpha}+i_\alpha+j_\alpha}b \cdots ba^{k_{n_\alpha,\alpha}+i_\alpha+j_\alpha}ba^{k_\alpha}\right)^{q_\alpha} ba^{k_{1,\alpha}+i_\alpha+j_\alpha}b \cdots ba^{k_{n_\alpha,\alpha}+i_\alpha+j_\alpha}b \\
z &\mapsto a^{i_\alpha}ba^{j_\alpha}
\end{aligned}
$$

where $q_\alpha \geq 1$ and $1 \leq i_\alpha, j_\alpha \leq k_\alpha < i_\alpha + j_\alpha$ and $n_\alpha \geq 0$ and $k_{t,\alpha} \geq 0$, for all $1 \leq t \leq n_\alpha$. Solution α implies $(x^{i_\alpha}y, zx^{k_{1,\alpha}}z \cdots zx^{k_{n_\alpha,\alpha}}zx^{k_\alpha - j_\alpha}y) \leq e$.

Subcase: Assume $\tau_\alpha = 4$ and $\tau_\beta \in \{2, 3, 5, 8\}$. Then equation (4) gives

$$
A_\alpha(y) - A_\alpha(z)B_\alpha(y) = \frac{q_\beta + 1}{p_\beta + 1}\ ;
$$

a contradiction since $\gcd(p_\beta + 1, q_\beta + 1) = 1$.

Subcase: Assume $\tau_\alpha = 4$ and $\tau_\beta = 6$. Then β is of the form

$$
\begin{aligned}
x &\mapsto a^{i_\beta}ba^{j_\beta} \\
y &\mapsto \left(ba^{k_{1,\beta}+i_\beta+j_\beta}b \cdots ba^{k_{n_\beta,\beta}+i_\beta+j_\beta}ba^{j'_\beta}\right)^{q_\beta} \\
z &\mapsto a
\end{aligned}
$$

or

$$\begin{aligned} x &\mapsto \left(ab^{k_{1,\beta}+j_\beta}a\cdots ab^{k_{n_\beta,\beta}+j_\beta}ab^{j'_\beta}\right)^{q_\beta} \\ y &\mapsto b \\ z &\mapsto a \end{aligned}$$

where $q_\beta, i_\beta \geq 1$ and $j_\beta, j'_\beta \geq 0$ and $j_\beta j'_\beta = 0$ and $n_\beta \geq 0$ and $k_{t,\beta} \geq 0$, for all $1 \leq t \leq n_\beta$. In both cases, β implies $(x, z^{i_\beta}y) \leq e$ and $k_{t,\alpha} = 0$, for all $1 \leq t \leq n_\alpha$, and $k_\alpha = j_\alpha$ and equation (4) gives

$$-i_\alpha(q_\alpha + 1) - j_\alpha = \frac{B_\beta(y)}{B_\beta(x)} \ ;$$

a contradiction.

Case: Assume $\tau_\alpha = 5$. Then α is of the shape

$$\begin{aligned} x &\mapsto a \\ y &\mapsto \left(ba^{i_\alpha+j_\alpha+m_\alpha}\right)^{q_\alpha} ba^{j_\alpha} \\ z &\mapsto a^{i_\alpha}\left(ba^{i_\alpha+j_\alpha+m_\alpha}\right)^{p_\alpha} b \end{aligned}$$

where $p_\alpha, q_\alpha, i_\alpha, j_\alpha \geq 1$ and $m_\alpha \geq 0$.

Subcase: Assume $\tau_\alpha = 5$ and $\tau_\beta = 2$. Then α implies $(x, zx) \leq e$ and β implies $(x, z^{i_\beta}y) \leq e$ where $i_\beta \geq 1$; a contradiction.

Subcase: Assume $\tau_\alpha = 5$ and $\tau_\beta \in \{3, 4, 8\}$. Then equations solved by α end with x and y, but equations solved by β end with x and z or y and z; a contradiction.

Subcase: Assume $\tau_\alpha = 5$ and $\tau_\beta = 6$. Then β is of the form

$$\begin{aligned} x &\mapsto a^{i_\beta}ba^{j_\beta} \\ y &\mapsto \left(ba^{k_{1,\beta}+i_\beta+j_\beta}b\cdots ba^{k_{n_\beta,\beta}+i_\beta+j_\beta}ba^{j'_\beta}\right)^{q_\beta} \\ z &\mapsto a \end{aligned}$$

or

$$\begin{aligned} x &\mapsto \left(ab^{k_{1,\beta}+j_\beta}a\cdots ab^{k_{n_\beta,\beta}+j_\beta}ab^{j'_\beta}\right)^{q_\beta} \\ y &\mapsto b \\ z &\mapsto a \end{aligned}$$

where $q_\beta, i_\beta \geq 1$ and $j_\beta, j'_\beta \geq 0$ and $j_\beta j'_\beta = 0$ and $n_\beta \geq 0$ and $k_{t,\beta} \geq 0$, with $1 \leq t_\beta \leq n_\beta$. Equation (5) gives

$$\frac{q_\alpha + 1}{p_\alpha + 1} = A_\beta(y) - A_\beta(x)B_\beta(y) \qquad \text{and} \qquad \frac{q_\alpha + 1}{p_\alpha + 1} = -\frac{A_\beta(x)}{B_\beta(x)}$$

respectively; a contradiction since $\gcd(p_\alpha + 1, q_\alpha + 1) = 1$.

Case: Assume $\tau_\alpha = 6$. Then α is of the shape

$$\begin{aligned} x &\mapsto a \\ y &\mapsto \left(a^{i'_\alpha}ba^{k_{1,\alpha}+i_\alpha+j_\alpha}b\cdots ba^{k_{n_\alpha,\alpha}+i_\alpha+j_\alpha}ba^{j'_\alpha}\right)^{q_\alpha} \\ z &\mapsto a^{i_\alpha}ba^{j_\alpha} \end{aligned}$$

$q_\alpha \geq 1$ and $i_\alpha, i'_\alpha, j_\alpha, j'_\alpha \geq 0$ and $i_\alpha i'_\alpha = j_\alpha j'_\alpha = 0$ and $n_\alpha \geq 0$ and $k_{t,\alpha} \geq 0$, for all $1 \leq t \leq n_\alpha$.

Subcase: Assume $\tau_\alpha = 6$ and $\tau_\beta \in \{2, 3, 5, 8\}$. Then equation (4) gives

$$\frac{q_\beta + 1}{p_\beta + 1} = A_\alpha(y) - A_\alpha(z)B_\alpha(y) \ ;$$

a contradiction since $\gcd(p_\beta + 1, q_\beta + 1) = 1$.

Subcase: Assume $\tau_\alpha = 6$ and $\tau_\beta = 4$. Then β is of the shape

$$\begin{aligned} x &\mapsto a^{i_\beta} b a^{j_\beta} \\ y &\mapsto \left(ba^{k_{1,\beta}+i_\beta+j_\beta} b \cdots ba^{k_{n_\beta,\beta}+i_\beta+j_\beta} ba^{k_\beta}\right)^{q_\beta} ba^{k_{1,\beta}+i_\beta+j_\beta} b \cdots ba^{k_{n_\beta,\beta}+i_\beta+j_\beta} b \\ z &\mapsto a \end{aligned}$$

where $q_\beta \geq 1$ and $1 \leq i_\beta, j_\beta \leq k_\beta < i_\beta + j_\beta$ and $n_\beta \geq 0$ and $k_{t,\beta} \geq 0$, for all $1 \leq t \leq n_\beta$. Now, $i'_\alpha = j'_\alpha = 0$ and $(x^{i_\alpha} y, z^{i_\beta} y) \leq e$ and $k_{t,\beta} = 0$, for all $1 \leq t \leq n_\beta$, and $i_\alpha = n_\beta + 1$. This implies $k_\beta = j_\beta$ and $k_{t,\alpha} = 0$, for all $1 \leq t \leq n_\alpha$ and equation (4) gives

$$(q_\beta + 1)(n_\beta + 1) = -q_\alpha(i_\alpha + j_\alpha) \ ;$$

a contradiction.

Case: Assume $\tau_\alpha = 8$. Then we use arguments similar to case $\tau_\alpha = 5$. □

The previous lemmas and Remark 5 imply the following conclusion.

Proposition 14. *The intersection of two different entire systems of different type contains only balanced equations.*

5.2. **Entire Systems of Equal Type.** Let τ be the type where α and β are both taken from, and let $\rho \colon A^+ \to A^+$ be an isomorphism such that $\rho(a) = b$ and $\rho(b) = a$.

Lemma 15. *If $\alpha(x) = \beta(x) = a$ then $K_\alpha \cap K_\beta$ contains only balanced equations.*

Proof. We have for type 1 that

$$\frac{p_\alpha + 1}{q_\alpha + 1} = \frac{|w|_x - |u|_x}{|u|_z - |w|_z} = \frac{p_\beta + 1}{q_\beta + 1}$$

and $p_\alpha = p_\beta$ and $q_\alpha = q_\beta$ otherwise $\gcd(p+1, q+1) > 1$ for at least one of the two solutions, but now α is a permutation of β; a contradiction.

For all other types Remark 5 implies that equations solved by α and β begin and end with y and z, and we have only to consider types 2 and 6 further.

Case: Assume $\tau = 2$. Then $i_\alpha = j_\alpha = i_\beta = j_\beta = 0$ and equation (1) gives

$$\frac{q_\alpha + 1}{p_\alpha + 1} = \frac{q_\beta + 1}{p_\beta + 1}$$

which implies $p_\alpha = p_\beta$ and $q_\alpha = q_\beta$ otherwise $\gcd(p+1, q+1) > 1$ for at least one of the two solutions. Equations solved by α and β imply

$$(yx^{k_\alpha} x_1, zx^{k_\alpha} x_2) \leq e \qquad \text{and} \qquad (yx^{k_\beta} x_1, zx^{k_\beta} x_2) \leq e$$

respectively, where $x_1, x_2 \in \{y, z\}$ which gives $k_\alpha = k_\beta$ and α is a permutation of β; a contradiction.

Case: Assume $\tau = 6$. Then $i_\alpha = i'_\alpha = j_\alpha = j'_\alpha = i_\beta = i'_\beta = j_\beta = j'_\beta = 0$ and equation (1) gives

$$q_\alpha(n_\alpha + 1) = q_\beta(n_\beta + 1) \qquad \text{or} \qquad q_\alpha(n_\alpha + 1) = \frac{1}{q_\beta(n_\beta + 1)} \; .$$

In the latter case we get $q_\alpha = q_\beta = 1$ and $n_\alpha = n_\beta = 0$ and α is a permutation of β; a contradiction. In the former case, we have $\alpha(z) = \beta(z) = b$ and $(y, vy) \leq e$ where $v \in \{x, z\}^+$. Now, $y = v^s v'$, with $s \geq 0$ and $v' \leq v$, which implies together with $B_\alpha(y) = B_\beta(y)$ that $\alpha(y)$ and $\beta(y)$ are equally defined, and hence, α is a permutation of β; a contradiction. □

Lemma 16. *If $\alpha(x) = \beta(y) = a$ and $\tau = 2$ then $K_\alpha \cap K_\beta$ contains only balanced equations.*

Proof. Let α be of the shape

$$\begin{aligned} x &\mapsto a \\ y &\mapsto \left(ba^{k_\alpha}\right)^{q_\alpha} b \\ z &\mapsto a^{i_\alpha} \left(ba^{k_\alpha}\right)^{p_\alpha} ba^{j_\alpha} \end{aligned}$$

where $p_\alpha, q_\alpha, k_\alpha \geq 1$ and $i_\alpha, j_\alpha \geq 0$ and $i_\alpha + j_\alpha \leq k_\alpha$. Now, β is of the shape

$$\begin{aligned} x &\mapsto a^{i_\beta} \left(ba^{k_\beta}\right)^{p_\beta} ba^{j_\beta} \\ y &\mapsto a \\ z &\mapsto \left(ba^{k_\beta}\right)^{q_\beta} b \end{aligned} \tag{19}$$

or

$$\begin{aligned} x &\mapsto \left(ba^{k_\beta}\right)^{q_\beta} b \\ y &\mapsto a \\ z &\mapsto a^{i_\beta} \left(ba^{k_\beta}\right)^{p_\beta} ba^{j_\beta} \end{aligned} \tag{20}$$

where $p_\beta, q_\beta, k_\beta \geq 1$ and $i_\beta, j_\beta \geq 0$ and $i_\beta + j_\beta \leq k_\beta$.

Case: Assume β is of shape (19). Then $i_\alpha, j_\alpha \geq 1$ and $i_\beta = j_\beta = 0$ since equations solved by α and β begin and end with x and z. By Lemma (4) we have $p_\beta > q_\beta$. Now, α implies $(x, zx^{k_\alpha - j_\alpha} y) \leq e$ or $(x, zx^{k_\alpha - i_\alpha - j_\alpha} z) \leq e$ and β implies $(x, xy) \leq e$ which gives that $k_\alpha = j_\alpha$; a contradiction since now $k_\alpha < i_\alpha + j_\alpha \leq k_\alpha$.

Case: Assume β is of shape (20). If e begins with x and z then $i_\alpha \geq 1$ and $i_\beta = 0$ and $q_\beta > p_\beta$ by Lemma (4) and α implies $(x, zx^{k_\alpha - j_\alpha} y) \leq e$ or $(x, zx^{k_\alpha - i_\alpha - j_\alpha} z) \leq e$ and β implies$(x, zy) \leq e$ which gives that $k_\alpha = j_\alpha$; a contradiction since we require $k_\alpha < i_\alpha + j_\alpha \leq k_\alpha$. If e begins with y and z then $i_\beta \geq 1$ and $q_\alpha > p_\alpha$ by Lemma (4) and $i_\alpha = 0$ and α implies $(yx^{k_\alpha}, zx^{k_\alpha - j_\alpha} y) \leq e$ or $(yx^{k_\alpha}, zx^{k_\alpha - j_\alpha} z) \leq e$ and solution β implies $(y^{i_\beta} xy^{k_\beta - i_\beta}, zy^{k_\beta - j_\beta} x) \leq e$ or $(y^{i_\beta} xy^{k_\beta - i_\beta}, zy^{k_\beta - i_\beta - j_\beta} z) \leq e$ and in any case $k_\alpha = j_\alpha = 1$, since $k_\beta - i_\beta \geq 1$. Now, equation (2) gives

$$\frac{p_\alpha + 1}{q_\alpha + 1} = \frac{p_\beta + 1}{q_\beta + 1}$$

but, equation (3) gives

$$k_\beta p_\beta + i_\beta + j_\beta - k_\beta q_\beta \frac{p_\beta + 1}{q_\beta + 1} = \frac{p_\beta + 1}{q_\beta + 1} \; ;$$

a contradiction. □

Lemma 17. *If $\alpha(x) = \beta(y) = a$ and $\tau \in \{3, 4, 5, 8\}$ then $K_\alpha \cap K_\beta$ contains only balanced equations.*

Proof. Equations solved by α begin with x and z whereas equations solved by β begin with x and y or y and z; a contradiction. □

Lemma 18. *If $(x, y) \leq e$ and $\alpha(x) = \beta(y) = a$ and $\tau = 6$ then $K_\alpha \cap K_\beta$ contains only balanced equations.*

Proof. Let α be of the shape

$$\begin{aligned} x &\mapsto a \\ y &\mapsto \left(a^{i'_\alpha} b a^{k_{1,\alpha}+j_\alpha} b \cdots b a^{k_{n_\alpha,\alpha}+j_\alpha} b a^{j'_\alpha}\right)^{q_\alpha} \\ z &\mapsto b a^{j_\alpha} \end{aligned}$$

where $q_\alpha, i'_\alpha \geq 1$ and $j_\alpha, j'_\alpha \geq 0$ and $j_\alpha j'_\alpha = 0$ and $n_\alpha \geq 0$ and $k_{t,\alpha} \geq 0$, for all $1 \leq t \leq n_\alpha$. Actually, $j_\alpha \geq 1$ and $j'_\alpha = 0$, otherwise equation (2) gives

$$-\frac{A_\alpha(y)}{B_\alpha(y)} = \frac{B_\beta(z)}{B_\beta(x)} ;$$

a contradiction. So, $u = w$ ends in x and z. Now, β is of the shape

$$\begin{aligned} x &\mapsto a^{i_\beta} b \\ y &\mapsto a \\ z &\mapsto \left(b a^{k_{1,\beta}+i_\beta} b \cdots b a^{k_{n_\beta,\beta}+i_\beta} b\right)^{q_\beta} \end{aligned} \tag{21}$$

or

$$\begin{aligned} x &\mapsto \left(a^{i'_\beta} b a^{k_{1,\beta}} b \cdots b a^{k_{n_\beta,\beta}} b\right)^{q_\beta} \\ y &\mapsto a \\ z &\mapsto b \end{aligned} \tag{22}$$

where $q_\beta, i_\beta, i'_\beta \geq 1$ and $n_\beta \geq 0$ and $k_{t,\beta} \geq 0$, for all $1 \leq t \leq n_\beta$.

Case: Assume β is of shape (21). Then equation (3) gives

$$\frac{1}{q_\alpha(n_\alpha + 1)} = q_\beta \left(\sum_{1 \leq t \leq n_\beta} k_{t,\beta} - i_\beta \right) = 1$$

which implies $q_\alpha = q_\beta = 1$ and $n_\alpha = 0$, and equation (2) gives

$$j_\alpha - i'_\alpha = n_\beta + 1$$

which implies $j_\alpha > i'_\alpha \geq 1$. Now, α implies $(x, yx) \leq e$ and β implies $(x, y^{i_\beta} z) \leq e$; a contradiction.

Case: Assume β is of shape (22). Equation (3) gives

$$\frac{1}{B_\alpha(y)} = -\frac{A_\beta(x)}{B_\beta(x)} ;$$

a contradiction. □

Lemma 19. *If $(x,z) \le e$ and $\alpha(x) = \beta(y) = a$ and $\tau = 6$ then $K_\alpha \cap K_\beta$ contains only balanced equations.*

Proof. Let α be of the shape

$$\begin{aligned} x &\mapsto a \\ y &\mapsto \left(ba^{k_{1,\alpha}+i_\alpha+j_\alpha} b \cdots ba^{k_{n_\alpha,\alpha}+i_\alpha+j_\alpha} ba^{j'_\alpha}\right)^{q_\alpha} \\ z &\mapsto a^{i_\alpha} ba^{j_\alpha} \end{aligned}$$

where $q_\alpha, i_\alpha \ge 1$ and $j_\alpha, j'_\alpha \ge 0$ and $j_\alpha j'_\alpha = 0$ and $n_\alpha \ge 0$ and $k_{t,\alpha} \ge 0$, for all $1 \le t \le n_\alpha$. Now, β is of the shape

$$\begin{aligned} x &\mapsto ba^{j_\beta} \\ y &\mapsto a \\ z &\mapsto \left(ba^{k_{1,\beta}+j_\beta} b \cdots ba^{k_{n_\beta,\beta}+j_\beta} ba^{j'_\beta}\right)^{q_\beta} \end{aligned} \tag{23}$$

or

$$\begin{aligned} x &\mapsto \left(ba^{k_{1,\beta}+j_\beta} b \cdots ba^{k_{n_\beta,\beta}+j_\beta} ba^{j'_\beta}\right)^{q_\beta} \\ y &\mapsto a \\ z &\mapsto ba^{j_\beta} \end{aligned} \tag{24}$$

where $q_\beta \ge 1$ and $j_\beta, j'_\beta \ge 0$ and $j_\beta j'_\beta = 0$ and $n_\beta \ge 0$ and $k_{t,\beta} \ge 0$, for all $1 \le t \le n_\beta$.

Case: Assume β is of shape (23). Then we have $\beta(z) < \beta(x)$ by Lemma (4), and hence, $q_\beta = 1$ and $j'_\beta = n_\beta = 0$ and equation (3) gives

$$\frac{1}{q_\alpha(n_\alpha+1)} = -j_\beta \;;$$

a contradiction.

Case: Assume β is of shape (24). If $j_\beta = 0$ then equation (3) gives

$$\frac{1}{q_\alpha(n_\alpha+1)} = -\frac{A_\beta(x)}{B_\beta(x)} \;;$$

a contradiction. So, $j_\beta \ge 1$ and $j_\alpha = j'_\alpha = j'_\beta = 0$ since e ends in y and z. Now, equation (2) gives

$$\frac{i_\alpha - \sum_{1\le t\le n_\alpha} k_{t,\alpha}}{n_\alpha+1} = \frac{1}{q_\beta(n_\beta+1)}$$

which implies $n_\alpha + 1 \ge n_\beta + 1$ and equation (3) gives

$$\frac{1}{q_\alpha(n_\alpha+1)} = \frac{j_\beta - \sum_{1\le t\le n_\beta} k_{t,\beta}}{n_\beta+1}$$

which implies $n_\alpha + 1 \le n_\beta + 1$, and hence, $n_\alpha = n_\beta$ and $q_\alpha = q_\beta = 1$ and

$$i_\alpha - \sum_{1\le t\le n} k_{t,\alpha} = j_\beta - \sum_{1\le t\le n} k_{t,\beta} = 1 \tag{25}$$

where $n = n_\alpha = n_\beta$. Solution α implies $(x^{i_\alpha}y, z) \leq e$ and β implies $(xy, z) \leq e$ which gives $i_\alpha = 1$, and α also implies $(xy, z) \preccurlyeq e$ and β implies $(xy^{j_\beta}, z) \preccurlyeq e$ which gives $j_\beta = 1$. From equation (25) follows

$$\begin{aligned} \alpha\colon x &\mapsto a \\ y &\mapsto (ba)^n b \\ z &\mapsto ab \end{aligned} \qquad \text{and} \qquad \begin{aligned} \beta\colon x &\mapsto (ba)^n b \\ y &\mapsto a \\ z &\mapsto ba \end{aligned}$$

By Proposition 2.5 in [7] we get the same generic equation for both entire systems K_α and K_β generated by α and β, respectively, namely

$$xy = z^{n+1}$$

and hence, K_α and K_β are not different; a contradiction. □

Lemma 20. *If $(y, z) \leq e$ and $\alpha(x) = \beta(y) = a$ and $\tau = 6$ then $K_\alpha \cap K_\beta$ contains only balanced equations.*

Proof. Let α be of the shape

$$\begin{aligned} x &\mapsto a \\ y &\mapsto \left(ba^{k_{1,\alpha}+j_\alpha}b \cdots ba^{k_{n_\alpha,\alpha}+j_\alpha}ba^{j'_\alpha}\right)^{q_\alpha} \\ z &\mapsto ba^{j_\alpha} \end{aligned}$$

where $q_\alpha \geq 1$ and $j_\alpha, j'_\alpha \geq 0$ and $j_\alpha j'_\alpha = 0$ and $n_\alpha \geq 0$ and $k_{t,\alpha} \geq 0$, for all $1 \leq t \leq n_\alpha$. Actually, $j_\alpha \geq 1$ and $j'_\alpha = 0$ since otherwise equation (2) gives

$$-\frac{A_\alpha(y)}{B_\alpha(y)} = \frac{B_\beta(z)}{B_\beta(x)} \ ;$$

a contradiction. So, e ends in x and z. Now, β is of the shape

$$\begin{aligned} x &\mapsto b \\ y &\mapsto a \\ z &\mapsto \left(a^{i'_\beta}ba^{k_{1,\beta}}b \cdots ba^{k_{n_\beta,\beta}}b\right)^{q_\beta} \end{aligned} \tag{26}$$

or

$$\begin{aligned} x &\mapsto \left(ba^{k_{1,\beta}+i_\beta}b \cdots ba^{k_{n_\beta,\beta}+i_\beta}b\right)^{q_\beta} \\ y &\mapsto a \\ z &\mapsto a^{i_\beta}b \end{aligned} \tag{27}$$

where $q_\beta, i_\beta, i'_\beta, j'_\beta \geq 1$ and $n_\beta \geq 0$ and $k_{t,\beta} \geq 0$, for all $1 \leq t \leq n_\beta$.

Case: Assume β is of shape (26). Then equation (3) gives

$$\frac{1}{q_\alpha(n_\alpha + 1)} = q_\beta \left(i'_\beta + \sum_{1 \leq t \leq n_\beta} k_{t,\beta} \right) = 1$$

which implies $q_\alpha = q_\beta = i'_\beta = 1$ and $n_\alpha = k_{t,\beta} = 0$, for all $1 \leq t \leq n_\beta$. Now, α implies $(yx^{j_\alpha}, z) \preccurlyeq e$ and β implies $(yx^{n_\beta+1}, z) \preccurlyeq e$ which gives $j_\alpha = n_\beta + 1$, and we have $\alpha = \rho \circ \beta$; a contradiction.

Case: Assume β is of shape (27). Then equation (2) gives

$$\frac{j_\alpha - \sum_{1\le t\le n_\alpha} k_{t,\alpha}}{n_\alpha + 1} = \frac{1}{q_\beta(n_\beta+1)} = 1$$

which implies $n_\alpha + 1 \ge n_\beta + 1$, and equation (3) gives

$$\frac{1}{q_\alpha(n_\alpha+1)} = \frac{i_\beta - \sum_{1\le t\le n_\beta} k_{t,\beta}}{n_\beta + 1} = 1$$

which implies $n_\beta + 1 \ge n_\alpha + 1$, and hence, $n_\alpha = n_\beta$ and $q_\alpha = q_\beta = 1$ and

$$j_\alpha - \sum_{1\le t\le n} k_{t,\alpha} = i_\beta - \sum_{1\le t\le n} k_{t,\beta} = 1 \tag{28}$$

where $n = n_\alpha = n_\beta$. Solution α implies $(yx, z) \le e$ and β implies $(y^{i_\beta}x, z) \le e$ which gives $i_\beta = 1$, and α also implies $(yx^{j_\alpha}, z) \preccurlyeq e$ and β implies $(yx, z) \preccurlyeq e$ which implies $j_\alpha = 1$. From equation (28) follows

$$\begin{aligned} \alpha\colon x &\mapsto a \\ y &\mapsto (ba)^n b \\ z &\mapsto ba \end{aligned} \qquad \text{and} \qquad \begin{aligned} \beta\colon x &\mapsto (ba)^n b \\ y &\mapsto a \\ z &\mapsto ab \end{aligned}$$

By Proposition 2.5 in [7] we get the same generic equation for both entire systems K_α and K_β generated by α and β, respectively, namely

$$yx = z^{n+1}$$

and hence, K_α and K_β are not different; a contradiction. □

Lemma 21. *If $\alpha(x) = \beta(z) = a$ and $\tau = 2$ then $K_\alpha \cap K_\beta$ contains only balanced equations.*

Proof. Let α be of the shape

$$\begin{aligned} x &\mapsto a \\ y &\mapsto \left(ba^{k_\alpha}\right)^{q_\alpha} b \\ z &\mapsto a^{i_\alpha} \left(ba^{k_\alpha}\right)^{p_\alpha} ba^{j_\alpha} \end{aligned}$$

where $p_\alpha, q_\alpha, k_\alpha \ge 1$ and $i_\alpha, j_\alpha \ge 0$ and $i_\alpha + j_\alpha \le k_\alpha$. Now, β is of the shape

$$\begin{aligned} x &\mapsto a^{i_\beta} \left(ba^{k_\beta}\right)^{p_\beta} ba^{j_\beta} \\ y &\mapsto \left(ba^{k_\beta}\right)^{q_\beta} b \\ z &\mapsto a \end{aligned} \tag{29}$$

or

$$\begin{aligned} x &\mapsto \left(ba^{k_\beta}\right)^{q_\beta} b \\ y &\mapsto a^{i_\beta} \left(ba^{k_\beta}\right)^{p_\beta} ba^{j_\beta} \\ z &\mapsto a \end{aligned} \tag{30}$$

where $p_\beta, q_\beta, k_\beta, i_\beta, j_\beta \ge 1$ and $i_\beta + j_\beta \le k_\beta$.

Case: Assume β is of shape (29). Then $i_\alpha, j_\alpha \geq 1$ since e must begin and end with x and z. Now, solution α implies $(x, zx^{k_\alpha - j_\alpha}y) \leq e$ or $(x, zx^{k_\alpha - i_\alpha - j_\alpha}z) \leq e$ and solution β implies $(x, z^{i_\beta}y) \leq e$ which gives $k_\alpha = i_\alpha + j_\alpha$. Equation (4) gives

$$-k_\alpha = \frac{q_\beta + 1}{p_\beta + 1} \ ;$$

a contradiction.

Case: Assume β is of shape (30). Then $i_\alpha = j_\alpha = 0$ and α implies $(yx, z) \leq e$ and β implies $(yz^{k_\beta - i_\beta - j_\beta}y, z) \leq e$ or $(yz^{k_\beta - j_\beta}x, z) \leq e$ which gives $k_\beta = j_\beta$; a contradiction. □

Lemma 22. *If $\alpha(x) = \beta(z) = a$ and $\tau = 3$ then $K_\alpha \cap K_\beta$ contains only balanced equations.*

Proof. Let α be of the shape

$$\begin{aligned} x &\mapsto a \\ y &\mapsto \left(ba^{k_\alpha}\right)^{q_\alpha} b \\ z &\mapsto a^{i_\alpha} \left(ba^{k_\alpha}\right)^{p_\alpha - q_\alpha - 1} ba^{j_\alpha} \end{aligned}$$

where $p_\alpha > q_\alpha \geq 1$ and $1 \leq i_\alpha, j_\alpha < k_\alpha < i_\alpha + j_\alpha$. Now, β must be of the shape

$$\begin{aligned} x &\mapsto a^{i_\beta} \left(ba^{k_\beta}\right)^{p_\beta - q_\beta - 1} ba^{j_\beta} \\ y &\mapsto \left(ba^{k_\beta}\right)^{q_\beta} b \\ z &\mapsto a \end{aligned}$$

where $p_\beta > q_\beta \geq 1$ and $1 \leq i_\beta, j_\beta < k_\beta < i_\beta + j_\beta$. Solution α implies $(x, zx) \leq u$ and β implies $(x, z^{i_\beta}y) \leq u$; a contradiction. □

Lemma 23. *If $\alpha(x) = \beta(z) = a$ and $\tau = 4$ then $K_\alpha \cap K_\beta$ contains only balanced equations.*

Proof. Let α be of the shape

$$\begin{aligned} x &\mapsto a \\ y &\mapsto \left(ba^{k_{1,\alpha}+i_\alpha+j_\alpha}b \cdots ba^{k_{n_\alpha,\alpha}+i_\alpha+j_\alpha}ba^{k_\alpha}\right)^{q_\alpha} ba^{k_{1,\alpha}+i_\alpha+j_\alpha}b \cdots ba^{k_{n_\alpha,\alpha}+i_\alpha+j_\alpha}b \\ z &\mapsto a^{i_\alpha}ba^{j_\alpha} \end{aligned}$$

where $q_\alpha \geq 1$ and $1 \leq i_\alpha, j_\alpha \leq k_\alpha < i_\alpha + j_\alpha$ and $n_\alpha \geq 0$ and $k_{t,\alpha} \geq 0$, for all $1 \leq t \leq n_\alpha$. Now, β must be of the shape

$$\begin{aligned} x &\mapsto a^{i_\beta}ba^{j_\beta} \\ y &\mapsto \left(ba^{k_{1,\beta}+i_\beta+j_\beta}b \cdots ba^{k_{n_\beta,\beta}+i_\beta+j_\beta}ba^{k_\beta}\right)^{q_\beta} ba^{k_{1,\beta}+i_\beta+j_\beta}b \cdots ba^{k_{n_\beta,\beta}+i_\beta+j_\beta}b \\ z &\mapsto a \end{aligned}$$

where $q_\beta \geq 1$ and $1 \leq i_\beta, j_\beta \leq k_\beta < i_\beta + j_\beta$ and $n_\beta \geq 0$ and $k_{t,\beta} \geq 0$, for all $1 \leq t \leq n_\beta$. Solution α implies $(x^{i_\alpha}y, zx^{k_{1,\alpha}}z \cdots zx^{k_{n_\alpha,\alpha}}zx^{k_\alpha - j_\alpha}y) \leq e$ and β implies $(xz^{k_{1,\beta}}x \cdots xz^{k_{n_\beta,\beta}}xz^{k_\beta - j_\beta}y, z^{i_\beta}y) \leq e$ which gives that $k_{t,\zeta} = 0$, for all $1 \leq t \leq n_\zeta$ and $\zeta \in \{\alpha, \beta\}$, and $k_\alpha = j_\alpha$ and $k_\beta = j_\beta$. Equation (4) gives

$$-(q_\alpha + 1)i_\alpha - j_\alpha = (q_\beta + 1)(n_\beta + 1) \ ;$$

a contradiction. □

Lemma 24. *If* $\alpha(x) = \beta(z) = a$ *and* $\tau \in \{5, 8\}$ *then* $K_\alpha \cap K_\beta$ *contains only balanced equations.*

Proof. Equations solved by α end in x and y whereas equations solved by β end in x and z or y and z; a contradiction. □

Lemma 25. *If* $(x, y) \leq e$ *and* $\alpha(x) = \beta(z) = a$ *and* $\tau = 6$ *then* $K_\alpha \cap K_\beta$ *contains only balanced equations.*

Proof. Let α be of the shape

$$\begin{aligned} x &\mapsto a \\ y &\mapsto \left(a^{i'_\alpha} b a^{k_{1,\alpha}+j_\alpha} b \cdots b a^{k_{n_\alpha,\alpha}+j_\alpha} b a^{j'_\alpha}\right)^{q_\alpha} \\ z &\mapsto b a^{j_\alpha} \end{aligned}$$

where $q_\alpha, i'_\alpha \geq 1$ and $j_\alpha, j'_\alpha \geq 0$ and $j_\alpha j'_\alpha = 0$ and $n_\alpha \geq 0$ and $k_{t,\alpha} \geq 0$, for all $1 \leq t \leq n_\alpha$. Now, β is of the shape

$$\begin{aligned} x &\mapsto b a^{j_\beta} \\ y &\mapsto \left(b a^{k_{1,\beta}+j_\beta} b \cdots b a^{k_{n_\beta,\beta}+j_\beta} b a^{j'_\beta}\right)^{q_\beta} \\ z &\mapsto a \end{aligned} \tag{31}$$

or

$$\begin{aligned} x &\mapsto \left(b a^{k_{1,\beta}+j_\beta} b \cdots b a^{k_{n_\beta,\beta}+j_\beta} b a^{j'_\beta}\right)^{q_\beta} \\ y &\mapsto b a^{j_\beta} \\ z &\mapsto a \end{aligned} \tag{32}$$

where $q_\beta \geq 1$ and $j_\beta, j'_\beta \geq 0$ and $j_\beta j'_\beta = 0$ and $n_\beta \geq 0$ and $k_{t,\beta} \geq 0$, for all $1 \leq t \leq n_\beta$.

Case: Assume β is of shape (31). Then from $\alpha(x) < \alpha(y)$ follows by Lemma (4) that $\beta(y) < \beta(x)$, and hence, $j_\beta \geq 1$ and $j'_\beta = 0$ and $p_\beta = 1$ and $n_\beta = 0$, and so, $\beta(y) = b$. But now, equation (5) gives

$$-j_\beta = q_\alpha(n_\alpha + 1) \ ;$$

a contradiction.

Case: Assume β is of shape (32). Then equation (4) gives

$$q_\alpha \left(i'_\alpha + j'_\alpha - j_\alpha + \sum_{1 \leq t \leq n_\alpha} k_{t,\alpha} \right) = \frac{1}{q_\beta(n_\beta + 1)} \tag{33}$$

which implies $q_\alpha = q_\beta = 1$ and $n_\beta = 0$, and equation (5) gives

$$j_\beta - j'_\beta = n_\alpha + 1$$

and we have $j_\beta \geq 1$ and $j'_\beta = 0$ and e ends with y and z. That implies $j_\alpha = j'_\alpha = 0$ and since equation (33) gives

$$i'_\alpha + \sum_{1 \leq t \leq n_\alpha} k_{t,\alpha} = 1$$

and $i'_\alpha \geq 1$, we have that $\alpha(y) = a^{i'_\alpha} b^{n_\alpha+1}$. Solution α implies $(x^{i_\alpha} z, y) \leq e$ and β implies $(xz, y) \leq e$ which gives $i_\alpha = 1$. Solution α implies $(y, xz^{n_\alpha+1}) \preccurlyeq e$ and β

implies $(y, xz^{j_\beta}) \preccurlyeq e$ which gives that $n_\alpha + 1 = j_\beta$, and now, $\alpha = \rho \circ \beta$; a contradiction. $\square$

Lemma 26. *If $(x, z) \leq e$ and $\alpha(x) = \beta(z) = a$ and $\tau = 6$ then $K_\alpha \cap K_\beta$ contains only balanced equations.*

Proof. Let α be of the shape

$$\begin{aligned} x &\mapsto a \\ y &\mapsto \left(ba^{k_{1,\alpha}+i_\alpha+j_\alpha}b \cdots ba^{k_{n_\alpha,\alpha}+i_\alpha+j_\alpha}ba^{j'_\alpha}\right)^{q_\alpha} \\ z &\mapsto a^{i_\alpha}ba^{j_\alpha} \end{aligned}$$

where $q_\alpha, i_\alpha \geq 1$ and $j_\alpha, j'_\alpha \geq 0$ and $j_\alpha j'_\alpha = 0$ and $n_\alpha \geq 0$ and $k_{t,\alpha} \geq 0$, for all $1 \leq t \leq n_\alpha$. Now, β is of the shape

$$\begin{aligned} x &\mapsto a^{i_\beta}ba^{j_\beta} \\ y &\mapsto \left(ba^{k_{1,\beta}+i_\beta+j_\beta}b \cdots ba^{k_{n_\beta,\beta}+i_\beta+j_\beta}ba^{j'_\beta}\right)^{q_\beta} \\ z &\mapsto a \end{aligned} \tag{34}$$

or

$$\begin{aligned} x &\mapsto \left(a^{i'_\beta}ba^{k_{1,\beta}+j_\beta}b \cdots ba^{k_{n_\beta,\beta}+j_\beta}ba^{j'_\beta}\right)^{q_\beta} \\ y &\mapsto ba^{j_\beta} \\ z &\mapsto a \end{aligned} \tag{35}$$

where $q_\beta, i_\beta, i'_\beta \geq 1$ and $j_\beta, j'_\beta \geq 0$ and $j_\beta j'_\beta = 0$ and $n_\beta \geq 0$ and $k_{t,\beta} \geq 0$, for all $1 \leq t \leq n_\beta$. Note, that if β is of shape (34) or (35) then

$$(x^{i_\alpha}y, z^\ell y) \leq e \tag{36}$$

where $\ell \in \{i_\beta, i'_\beta\}$.

Case: Assume β is of shape (34). From (36) follows that $k_{t,\zeta} = 0$, for all $1 \leq t \leq n_\zeta$ and $\zeta \in \{\alpha, \beta\}$, and equations (4) and (5) give

$$q_\beta(n_\beta + 1) = q_\alpha(j'_\alpha - i_\alpha - j_\alpha) \qquad \text{and} \qquad q_\alpha(n_\alpha + 1) = q_\beta(j'_\beta - i_\beta - j_\beta)$$

respectively. This implies $j'_\alpha, j'_\beta > 1$ and $j_\alpha = j_\beta = 0$. Now, e must end in x and y by α and in y and z by β; a contradiction.

Case: Assume β is of shape (35). From (36) follows that $k_{t,\alpha} = 0$, for all $1 \leq t \leq n_\alpha$, and equation (4) gives

$$\frac{1}{q_\beta(n_\beta + 1)} = q_\alpha(j'_\alpha - i_\alpha - j_\alpha) = 1$$

and $q_\alpha = q_\beta = 1$ and $j'_\alpha \geq 1$ and $j_\alpha = n_\beta = 0$ and also $j_\beta \geq 1$ and $j'_\beta = 0$, since e ends in y and z. Equation (5) gives

$$n_\alpha + 1 = j_\beta - i'_\beta$$

which implies $j_\beta > i'_\beta$, and we get from (36) that $(a^{i'_\beta}bb, a^{i'_\beta}ba^{j_\beta}) \leq (\beta(u), \beta(w))$ or $(a^{i'_\beta}ba^{i'_\beta}b, a^{i'_\beta}ba^{j_\beta}) \leq (\beta(u), \beta(w))$ which implies $j_\beta \leq i'_\beta$; a contradiction. $\square$

Lemma 27. *If $(y, z) \leq e$ and $\alpha(x) = \beta(z) = a$ and $\tau = 6$ then $K_\alpha \cap K_\beta$ contains only balanced equations.*

Proof. Let α be of the shape

$$\begin{aligned} x &\mapsto a \\ y &\mapsto \left(ba^{k_{1,\alpha}+j_\alpha}b\cdots ba^{k_{n_\alpha,\alpha}+j_\alpha}ba^{j'_\alpha}\right)^{q_\alpha} \\ z &\mapsto ba^{j_\alpha} \end{aligned}$$

where $q_\alpha \geq 1$ and $j_\alpha, j'_\alpha \geq 0$ and $j_\alpha j'_\alpha = 0$ and $n_\alpha \geq 0$ and $k_{t,\alpha} \geq 0$, for all $1 \leq t \leq n_\alpha$. Now, β is of the shape

$$\begin{aligned} x &\mapsto ba^{j_\beta} \\ y &\mapsto \left(a^{i'_\beta}ba^{k_{1,\beta}+j_\beta}b\cdots ba^{k_{n_\beta,\beta}+j_\beta}ba^{j'_\beta}\right)^{q_\beta} \\ z &\mapsto a \end{aligned}$$

or

$$\begin{aligned} x &\mapsto \left(ba^{k_{1,\beta}+i_\beta+j_\beta}b\cdots ba^{k_{n_\beta,\beta}+i_\beta+j_\beta}ba^{j'_\beta}\right)^{q_\beta} \\ y &\mapsto a^{i_\beta}ba^{j_\beta} \\ z &\mapsto a \end{aligned}$$

where $q_\beta, i_\beta, i'_\beta \geq 1$ and $j_\beta, j'_\beta \geq 0$ and $j_\beta j'_\beta = 0$ and $n_\beta \geq 0$ and $k_{t,\beta} \geq 0$, for all $1 \leq t \leq n_\beta$. We have for any shape of β that $\beta(z) < \beta(y)$, and hence, $\alpha(y) < \alpha(z)$ by Lemma (4). So, $q_\alpha = 1$ and $j'_\alpha = n_\alpha = 0$ and equation (4) gives

$$-j_\alpha = \frac{B_\beta(y)}{B_\beta(x)} \;;$$

a contradiction. □

The previous lemmas imply the following conclusion.

Proposition 28. *The intersection of two different entire systems of the same type contains only balanced equations.*

Proposition 29 follows directly from Proposition 14 and 28.

Proposition 29. *The intersection of two different entire systems contains only balanced equations.*

6. About Systems of Equations

This section contains the main result of this article. It states that an independent system with at least two equations and a nonperiodic solution consists of balanced equations only. Before this statement is proved, we observe the following propositions.

Proposition 30. *If two unbalanced equations e_1 and e_2 have a common nonperiodic solution, then they have the same set of periodic solutions.*

Proof. Let α be a nonperiodic solution of e_1 and e_2. Let $e_i = (u_i, w_i)$, and let $\delta_{x'}(e_i) = |u_i|_{x'} - |w_i|_{x'}$ where $x' \in X$ and $i \in \{1, 2\}$. Assume that α is incontractable and $\alpha(x) = a$ without restriction of generality. Now,

$$B_\alpha(y)\delta_y(e_i) + B_\alpha(z)\delta_z(e_i) = 0$$

for $i \in \{1, 2\}$, and

$$-\frac{B_\alpha(y)}{B_\alpha(z)} = \frac{\delta_z(e_1)}{\delta_y(e_1)} = \frac{\delta_z(e_2)}{\delta_y(e_2)}$$

and we have $s\delta_y(e_1) = t\delta_y(e_2)$ and $s\delta_z(e_1) = t\delta_z(e_2)$ with $s, t \neq 0$. It is easy to see that also $s\delta_x(e_1) = t\delta_x(e_2)$. Clearly, e_1 and e_2 have the same set of periodic solutions. □

Proposition 31. *If an unbalanced equation e_1 and a balanced equation e_2 have a common nonperiodic solution, then every solution of e_1 is a solution of e_2.*

Proof. It is clear that any periodic solution of e_1 is a solution of e_2 since every periodic solution is a solution for a balanced equation. Let α be a nonperiodic solution of e_1 and e_2. Assume that there exists a nonperiodic solution β of e_1 that is not a solution of e_2. Let α and β be incontractable without restriction of generality. Now, α and β generate two different enire systems since $e_2 \in K_\alpha$ and $e_2 \notin K_\beta$. But, $e_1 \in K_\alpha \cap K_\beta$ which implies that e_1 is balanced by Proposition 29; a contradiction. □

The main result of this article follows immediately.

Theorem 32. *If a system of equations has a nonperiodic solution and contains an unbalanced equation then it is not independent or it is a singleton.*

Proof. Let S be a system of at least two equations that has a nonperiodic solution α and contains at least one unbalanced equation e_1. Assume that S is independent, then it contains no balanced equation by Proposition 31. Let e_2 be an unbalanced equation in S different from e_1. Since S is independent, there exists a solution β that solves e_1 but does not solve e_2. From Proposition 30 follows that β is nonperiodic. We can assume that both α and β are incontractable without restriction of generality. Furthermore, α and β generate two different entire systems since $e_2 \in K_\alpha$ and $e_2 \notin K_\beta$. But, $e_1 \in K_\alpha \cap K_\beta$ which implies that e_1 is balanced by Proposition 29; a contradiction. □

Corollary 33. *An independent system with at least two equations and a nonperiodic solution consists for balanced equations only.*

7. Conclusions

We have shown that the intersection of the kernel of two different nonperiodic solutions in three variables contains only balanced equations, Proposition 29. From that result and the fact that the independence of systems in three variables depends on their nonperiodic solutions only, Proposition 30 and 31, follows that independent systems of equations in three variables that have a nonperiodic solution and contain more than one equation consist of balanced equations only, Theorem 32 and Corollary 33.

This result is a further step towards an answer of the question whether or not an independent system of three equations in three variables with a nonperiodic solution exists. We have established here that length arguments do not help to answer that question.

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