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On the Structure of Rough Approximations

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Abstract. We study rough approximations based on indiscernibility relations which are not necessarily reflexive, symmetric or transitive. For this, we define in a lattice-theoretical setting two maps which mimic the rough approximation operators and note that this setting is suitable also for other operators based on binary relations. Properties of the ordered sets of the upper and the lower approximations of the elements of an atomic Boolean lattice are studied.

1. Introduction

The basic ideas of rough set theory introduced by Pawlak [15] deal with situations in which the objects of a certain universe can be identified only within the limits determined by the knowledge represented by a given indiscernibility relation. The indiscernibility relation enables us to divide objects of the universe U into three disjoint sets with respect to any subset $X \subseteq U$:

(1) the objects, which surely are in X;

(2) the objects, which are surely not in X;

(3) the objects, which possibly are in X.

The objects in class 1 form the lower approximation of X, and the objects of type 1 and 3 form together its upper approximation. The boundary of X consists of the objects in class 3.

Usually indiscernibility relations are supposed to be equivalences, but in this work we do not restrict the properties of an indiscernibility relation. Namely, as we will see, it can be argued that neither reflexivity, symmetry nor transitivity are necessary properties of indiscernibility relations.

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We start our study by defining formally the upper and the lower approximations of an indiscernibility relation $\approx$ on U. For any $x \in U$, we denote

$$[x]_\approx = \{y \in U \mid x \approx y\}.$$

Thus, $[x]_\approx$ consists of the elements which cannot be discerned from x. For any subset X of U, let

$$X^\blacktriangledown = \{x \in U \mid [x]_\approx \subseteq X\} \quad \text{and} \tag{1.1}$$

$$X^\blacktriangle = \{x \in U \mid X \cap [x]_\approx \neq \emptyset\}. \tag{1.2}$$

The sets $X^\blacktriangledown$ and $X^\blacktriangle$ are called the *lower* and the *upper approximation* of X, respectively. The set $B(X) = X^\blacktriangle - X^\blacktriangledown$ is the *boundary* of X.

The above definitions mean that $x \in X^\blacktriangle$ if there is an element in X to which x is $\approx$-related. Similarly, $x \in X^\blacktriangledown$ if all the elements to which x is $\approx$-related are in X. Furthermore, $x \in B(X)$ if both in X and outside X there are elements which cannot be discerned from x. A set X is said to be *exact* if $X^\blacktriangledown = X^\blacktriangle$. It is obvious that the boundary of an exact set X is empty. This means that for any object $x \in U$, we can with certainty decide whether $x \in X$ by using the knowledge provided by $\approx$.

Two sets X and Y are said to be *equivalent*, denoted by $X \equiv Y$, if $X^\blacktriangledown = Y^\blacktriangledown$ and $X^\blacktriangle = Y^\blacktriangle$. The equivalence classes of $\equiv$ are called *rough sets*. The idea is that if subsets of U are observed within the limits given by the knowledge represented by $\approx$, then the sets in the same rough set look the same; $X \equiv Y$ means that exactly the same elements belong certainly to X and to Y, and that exactly the same elements belong possibly to X and to Y.

It seems that there is no natural representative for a rough set. However, this problem can be easily avoided by using Iwiński's [6] approach to rough sets based on the fact that each rough set $\mathcal{R}$ is uniquely determined by the pair $(X^\blacktriangledown, X^\blacktriangle)$, where X is any member of $\mathcal{R}$. Now there is a natural order relation on the set of all rough sets defined by

$$(X^\blacktriangledown, X^\blacktriangle) \leq (Y^\blacktriangledown, Y^\blacktriangle) \iff X^\blacktriangledown \subseteq Y^\blacktriangledown \text{ and } X^\blacktriangle \subseteq Y^\blacktriangle.$$

Example 1.1. Suppose that the object set $U = \{1, 2, 3, 4\}$ consists of four persons called 1, 2, 3, and 4, respectively. In Table 1 is presented some information concerning these persons.

	GENDER	AGE
1	Male	Old
2	Female	Young
3	Male	Middle-aged
4	Female	Young

Table 1. The gender and the age of persons 1, 2, 3, and 4

Let the relation $\approx$ be defined so that two persons are $\approx$-related if and only if their values for the attributes GENDER and AGE are the same. Then obviously $\approx$ is an equivalence which has the $\approx$-classes $\{1\}$, $\{3\}$, and $\{2, 4\}$.

Let us denote the subsets of U, which differ from $\emptyset$ and U, by sequences of their elements. For example, $\{1,2,3\}$ is written as 123. It is easy to see that the rough sets are $\{\emptyset\}$, $\{1\}$, $\{3\}$, $\{2,4\}$, $\{13\}$, $\{12,14\}$, $\{23,34\}$, $\{24\}$, $\{123,134\}$, $\{124\}$, $\{234\}$, and $\{U\}$. The corresponding ordered set is presented in Figure 1.

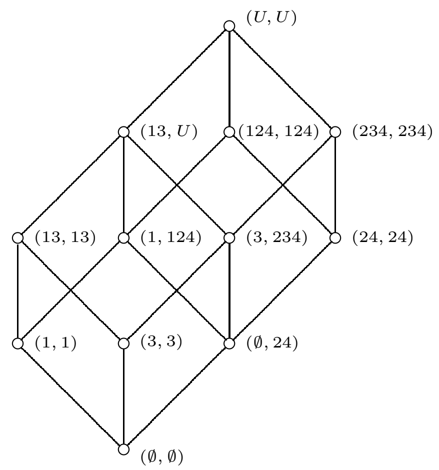

Figure 1. The ordered set of rough sets

For an arbitrary binary relation we can now give the following definition.

Definition 1.2. A binary relation $\approx$ on a nonempty set U is said to be

1. *reflexive*, if $x \approx x$ for all $x \in U$;
2. *symmetric*, if $x \approx y$ implies $y \approx x$ for all $x, y \in U$;
3. *transitive*, if $x \approx y$ and $y \approx z$ imply $x \approx z$ for all $x, y, z \in U$;
4. a *quasi-ordering*, if it is reflexive and transitive;
5. a *tolerance relation*, if it is reflexive and symmetric;
6. an *equivalence relation*, if it is reflexive, symmetric, and transitive.

It is commonly assumed that indiscernibility relations are equivalences. However, in the literature one can find studies in which rough approximation operators are determined by tolerances (see e.g. [8, 16, 17]). In [18] Slowinski and Vanderpooten studied approximation operators defined by a reflexive binary relation and in [19] Yao and Lin considered approximations based on arbitrary binary relations. Furthermore, Orłowska [14] has examined operators determined by frames of information relations reflecting distinguishability or indistinguishability of objects of an information system. Note also that in [11] Kortelainen studied so-called compositional modifiers based on quasi-orderings, which are quite similar to operators (1.1) and (1.2).

Next we will argue that there exist indiscernibility relations which are not reflexive, symmetric, or transitive.

Reflexivity. It may seem reasonable to assume that every object is indiscernible from itself. But in some occasions this is not true, since it is possible that our information is so imprecise. For example, we may discern persons by comparing photographs taken of them. But it may happen that we are unable to recognize that a same person appears in two different photographs.

Symmetry. Usually it is supposed that indiscernibility relations are symmetric, which means that if we cannot discern x from y, then we cannot discern y from x either. But indiscernibility relations may be directional. For example, if a person x speaks English and Finnish, and a person y speaks English, Finnish, and German, then x cannot distinguish y from himself by the property "knowledge of languages" since y can communicate with x in any language that x speaks. On the other hand, y can distinguish x from himself by asking a simple question in German, for example.

Transitivity. Transitivity is the least obvious of the three properties usually associated with indiscernibility relations. For example, if we define an indiscernibility relation on a set of human beings in such a way that two person are indiscernible with respect to the property "age" if their time of birth differs by less than two hours, then there may exist three persons x, y, and z, such that x is born an hour before y and y is born $1\frac{1}{2}$ hours before z. Hence, x is indiscernible from y and y is indiscernible from z, but x and z are not indiscernible.

This work is structured as follows. The following section is devoted to basic notations and general conventions concerning ordered sets. In Section 3 we introduce generalizations of lower and upper approximations in a lattice-theoretical setting and study their properties. We note also that this paper is an extended version of the presentation [9] made by the author at the Third International Conference on Rough Sets and Current Trends in Computing (RSCTC 2002).

2. Preliminaries

We assume that the reader is familiar with the usual lattice-theoretical notation and conventions, which can be found in [1, 3], for example.

First we recall some definitions concerning properties of maps. Let $\mathcal{P} = (P, \leq)$ be an ordered set. A map $f\colon P \to P$ is said to be *extensive*, if $x \leq f(x)$ for all $x \in P$. The map f is *order-preserving* if $x \leq y$ implies $f(x) \leq f(y)$. Moreover, f is *idempotent* if $f(f(x)) = f(x)$ for all $x \in P$.

A map $c\colon P \to P$ is said to be a *closure operator* on $\mathcal{P}$, if c is extensive, order-preserving, and idempotent. An element $x \in P$ is *c-closed* if $c(x) = x$. Furthermore, if $i\colon P \to P$ is a closure operator on $\mathcal{P}^\partial = (P, \geq)$, then i is an *interior operator* on $\mathcal{P}$.

Let $\mathcal{P} = (P, \leq)$ and $\mathcal{Q} = (Q, \leq)$ be ordered sets. A map $f\colon P \to Q$ is an *order-embedding*, if for any $a, b \in P$, $a \leq b$ in $\mathcal{P}$ if and only if $f(a) \leq f(b)$ in $\mathcal{Q}$. Note that an order-embedding is always an injection. An order-embedding f onto Q is called an *order-isomorphism* between $\mathcal{P}$ and $\mathcal{Q}$. When there exists an order-isomorphism between $\mathcal{P}$ and $\mathcal{Q}$, we say that $\mathcal{P}$ and $\mathcal{Q}$ are *order-isomorphic* and write $\mathcal{P} \cong \mathcal{Q}$. If $\mathcal{P} = (P, \leq)$ and $\mathcal{Q}^\partial = (Q, \geq)$ are order-isomorphic, then $\mathcal{P}$ and $\mathcal{Q}$ are said to be *dually order-isomorphic*.

Next we define dual Galois connections. It is known [8] that the pair of maps which assigns to every set its upper and lower approximations forms a dual Galois connection when the corresponding indiscernibility relation is symmetric. In the next section we will show that an analogous result holds also in our generalized setting.

Definition 2.1. Let $\mathcal{P} = (P, \leq)$ be an ordered set. A pair $(^{\blacktriangleright}, ^{\blacktriangleleft})$ of maps $^{\blacktriangleright}\colon P \to P$ and $^{\blacktriangleleft}\colon P \to P$ (which we refer to as the *right map* and the *left map*, respectively) is called a *dual Galois connection* on $\mathcal{P}$ if $^{\blacktriangleright}$ and $^{\blacktriangleleft}$ are order-preserving and $p^{\blacktriangleleft\blacktriangleright} \leq p \leq p^{\blacktriangleright\blacktriangleleft}$ for all $p \in P$.

The following proposition presents some basic properties of dual Galois connections, which follow from the properties of Galois connections (see [3], for example).

Proposition 2.2. Let $(^{\blacktriangleright}, ^{\blacktriangleleft})$ be a dual Galois connection on a complete lattice $\mathcal{P} = (P, \leq)$.

(a) For all $p \in P$, $p^{\blacktriangleright\blacktriangleleft\blacktriangleright} = p^{\blacktriangleright}$ and $p^{\blacktriangleleft\blacktriangleright\blacktriangleleft} = p^{\blacktriangleleft}$.

(b) The map $c\colon P \to P$, $p \mapsto p^{\blacktriangleright\blacktriangleleft}$ is a closure operator on $\mathcal{P}$ and the map $k\colon P \to P$, $p \mapsto p^{\blacktriangleleft\blacktriangleright}$ is an interior operator on $\mathcal{P}$.

(c) If c and k are the mappings defined in (b), then restricted to the sets of c-closed elements P_c and k-closed elements P_k, respectively, $^{\blacktriangleright}$ and $^{\blacktriangleleft}$ yield a pair $^{\blacktriangleright}\colon P_c \to P_k$, $^{\blacktriangleleft}\colon P_k \to P_c$ of mutually inverse order-isomorphisms between the complete lattices $(P_c, \leq)$ and $(P_k, \leq)$.

Before we consider Boolean lattices, we present the following lemma. Note that condition (a) can be found in [3], for example. Here $\wp(P)$ denotes the *power set* of P, that is, the set of all subsets of P.

Lemma 2.3. Let $(P, \leq)$ be a complete lattice, $S, T \subseteq P$, and $\{X_i \mid i \in I\} \subseteq \wp(P)$.

(a) If $S \subseteq T$, then $\bigvee S \leq \bigvee T$.

(b) $\bigvee \left(\bigcup\{X_i \mid i \in I\}\right) = \bigvee\{\bigvee X_i \mid i \in I\}$.

Proof: (b) For all $i \in I$, $X_i \subseteq \bigcup\{X_i \mid i \in I\}$. This implies by (a) that $\bigvee X_i \leq \bigvee(\bigcup\{X_i \mid i \in I\})$ for all $i \in I$. Let x be an upper bound of $\{\bigvee X_i \mid i \in I\}$. Then for all $i \in I$ and $y_i \in X_i$, $y_i \leq \bigvee X_i \leq x$. This means that $z \leq x$ for all $z \in \bigcup\{X_i \mid i \in I\}$. Thus, $\bigvee \left(\bigcup\{X_i \mid i \in I\}\right) \leq x$, which completes the proof. □

Next we recall some definitions concerning Boolean lattices. They are bounded distributive lattices with a complementation operation.

Definition 2.4. A lattice $\mathcal{L} = (L, \leq)$ is called a *Boolean lattice*, if

(a) L is distributive,

(b) L has a least element 0 and a greatest element 1, and

(c) each $a \in L$ has a *complement* $a' \in L$ such that $a \vee a' = 1$ and $a \wedge a' = 0$.

In the following lemma is some properties of Boolean lattices.

Lemma 2.5. Let $\mathcal{B} = (B, \leq)$ be a Boolean lattice. Then for all $a, b \in B$,

(a) $0' = 1$ and $1' = 0$,

(b) $a'' = a$,

(c) $(a \vee b)' = a' \wedge b'$ and $(a \wedge b)' = a' \vee b'$, and

(d) $a \wedge b' = 0$ if and only if $a \leq b$.

The following definition introduces the dual of a map. In Section 3 we show that the pair of maps which assigns the lower and upper approximations to each element are mutually dual.

Definition 2.6. Let $\mathcal{B} = (B, \leq)$ be a Boolean lattice. Two maps $f\colon B \to B$ and $g\colon B \to B$ are the *duals* of each other if

$$f(x') = g(x)' \text{ and } g(x') = f(x)'$$

for all $x \in B$.

The next lemma shows that the dual of a closure operator is an interior operator (see e.g. [13]).

Lemma 2.7. Let $\mathcal{B} = (B, \leq)$ be a Boolean lattice and let $f\colon B \to B$ be a closure operator on $\mathcal{B}$. If $g\colon B \to B$ is the dual of f, then g is an interior operator on $\mathcal{B}$.

By the following lemma, which can be found in [1], the infinite distributive laws hold in a complete Boolean lattice.

Lemma 2.8. Let $\mathcal{B} = (B, \leq)$ be a complete Boolean lattice. Then for all $\{x_i \mid i \in I\} \subseteq B$ and $y \in B$,

$$y \wedge (\textstyle\bigvee_{i \in I} x_i) = \textstyle\bigvee_{i \in I} (y \wedge x_i)$$

and

$$y \vee (\textstyle\bigwedge_{i \in I} x_i) = \textstyle\bigwedge_{i \in I} (y \vee x_i).$$

We end this section by introducing atomic lattices. Let $(P, \leq)$ be an ordered set and $x, y \in P$. We say that x *is covered by* y (or that y *covers* x), and write $x \prec y$, if $x < y$ and there is no element z in P with $x < z < y$.

Definition 2.9. Let $\mathcal{L} = (L, \leq)$ be a lattice with a least element 0. Then $a \in L$ is called an *atom* if $0 \prec a$. The set of atoms of $\mathcal{L}$ is denoted by $\mathcal{A}(\mathcal{L})$. The lattice $\mathcal{L}$ is *atomic* if every element x of L is the supremum of the atoms below it, that is, $x = \bigvee \{a \in \mathcal{A}(\mathcal{L}) \mid a \leq x\}$.

It is obvious that in a lattice $\mathcal{L} = (L, \leq)$ with a least element 0,

$$a \wedge x \neq 0 \iff a \leq x$$

for all $a \in \mathcal{A}(\mathcal{L})$ and $x \in L$. This implies that $a \wedge b = 0$ for all $a, b \in \mathcal{A}(\mathcal{L})$ such that $a \neq b$. Furthermore, if $\mathcal{L}$ is atomic, then for all $x \neq 0$ there exists an atom $a \in \mathcal{A}(\mathcal{L})$ such that $a \leq x$. Namely, if $\{a \in \mathcal{A}(\mathcal{L}) \mid a \leq x\} = \emptyset$, then $x = \bigvee \{a \in \mathcal{A}(\mathcal{L}) \mid a \leq x\} = \bigvee \emptyset = 0$.

3. Generalizations of approximations

This section is devoted to the study of the properties of approximations in a more general setting of complete atomic Boolean lattices. Note that researches have introduced many generalizations of approximation operators. For example, Yao [20] studied approximation operators defined in finite Boolean lattices, Düntsch, Orłowska, and Wang [4] presented the operators as a certain kind of interior and closure operators on a bounded distributive lattice, and Cattaneo [2] defined the approximations by the means of inner and outer definable elements of a bounded ordered set.

Definition 3.1. Let $\mathcal{B} = (B, \leq)$ be a complete atomic Boolean lattice. A map $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ is

(a) *extensive*, if $x \leq \varphi(x)$ for all $x \in \mathcal{A}(\mathcal{B})$;

(b) *symmetric*, if $x \leq \varphi(y)$ implies $y \leq \varphi(x)$ for all $x, y \in \mathcal{A}(\mathcal{B})$;

(c) *closed*, if $x \leq \varphi(y)$ implies $\varphi(x) \leq \varphi(y)$ for all $x, y \in \mathcal{A}(\mathcal{B})$.

Let $\approx$ be a binary relation on a set U. The ordered set $(\wp(U), \subseteq)$ is a complete atomic Boolean lattice. Since the atoms $\{x\}$ $(x \in U)$ of $(\wp(U), \subseteq)$ can be identified with the elements of U, the map

$$\varphi\colon U \to \wp(U), x \mapsto [x]_{\approx} \tag{3.1}$$

may be considered to be of the form $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$, where $\mathcal{B} = (B, \leq)$ equals $(\wp(U), \subseteq)$. The following observations are obvious:

(a) $\approx$ is reflexive $\iff$ φ is extensive;

(b) $\approx$ is symmetric $\iff$ φ is symmetric;

(c) $\approx$ is transitive $\iff$ φ is closed.

Next we introduce the generalizations of lower and upper approximations.

Definition 3.2. Let $\mathcal{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ be any map. For any element $x \in B$, let

$$x^{\blacktriangledown} = \bigvee \{a \in \mathcal{A}(\mathcal{B}) \mid \varphi(a) \leq x\}, \tag{3.2}$$

$$x^{\blacktriangle} = \bigvee \{a \in \mathcal{A}(\mathcal{B}) \mid \varphi(a) \wedge x \neq 0\}. \tag{3.3}$$

The elements $x^{\blacktriangledown}$ and $x^{\blacktriangle}$ are the *lower* and the *upper approximation of* x with respect to φ, respectively. Two elements x and y are *equivalent* if they have the same upper and the same lower approximations. The resulting equivalence classes are called *rough sets*.

It is clear that if φ is a map defined in (3.1), then the functions of Definition 3.2 coincide with the operators (1.1) and (1.2).

The next lemma is useful in our considerations.

Lemma 3.3. Let $\mathcal{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ be any map. Then for all $a \in \mathcal{A}(\mathcal{B})$ and $x \in B$,

(a) $a \le x^{\blacktriangledown} \iff \varphi(a) \le x$;

(b) $a \le x^{\blacktriangle} \iff \varphi(a) \wedge x \neq 0$.

Proof: (a) Suppose that $a \le x^{\blacktriangledown} = \bigvee\{b \in \mathcal{A}(\mathcal{B}) \mid \varphi(b) \le x\}$. If $\varphi(a) \not\le x$, then

$$a \wedge x^{\blacktriangledown} = a \wedge \bigvee\{b \in \mathcal{A}(\mathcal{B}) \mid \varphi(b) \le x\} = \bigvee\{a \wedge b \mid b \in \mathcal{A}(\mathcal{B}) \text{ and } \varphi(b) \le x\} = 0.$$

This implies $a \le (x^{\blacktriangledown})'$ by Lemma 2.5(d), a contradiction! Thus, $\varphi(a) \le x$. On the other hand, if $\varphi(a) \le x$, then $a \le \bigvee\{b \in \mathcal{A}(\mathcal{B}) \mid \varphi(b) \le x\} = x^{\blacktriangledown}$.

Condition (b) can be proved similarly. □

The end of this work is devoted to the study of the operators (3.2) and (3.3) in cases when the map φ is extensive, symmetric, or closed. However, we begin by assuming that φ is arbitrary and present some obvious properties of the maps ${}^{\blacktriangledown}\colon P \to P$ and ${}^{\blacktriangle}\colon P \to P$.

Lemma 3.4. Let $\mathcal{B} = (B, \le)$ be a complete atomic Boolean lattice and let $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ be any map.

(a) $0^{\blacktriangle} = 0$ and $1^{\blacktriangledown} = 1$;

(b) $x \le y$ implies $x^{\blacktriangledown} \le y^{\blacktriangledown}$ and $x^{\blacktriangle} \le y^{\blacktriangle}$.

Note that Lemma 3.4(b) means that the maps ${}^{\blacktriangledown}$ and ${}^{\blacktriangle}$ are order-preserving. For all $S \subseteq B$, we denote $S^{\blacktriangledown} = \{x^{\blacktriangledown} \mid x \in S\}$ and $S^{\blacktriangle} = \{x^{\blacktriangle} \mid x \in S\}$.

Recall that for a semilattice $\mathcal{P} = (P, \circ)$, an equivalence Θ on P is a *congruence* on $\mathcal{P}$ if $x_1 \Theta x_2$ and $y_1 \Theta y_2$ imply $x_1 \circ y_1 \Theta x_2 \circ x_2$, for all $x_1, x_2, y_1, y_2 \in P$.

Proposition 3.5. Let $\mathcal{B} = (B, \le)$ be a complete atomic Boolean lattice and let $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ be any map.

(a) The maps ${}^{\blacktriangle}\colon B \to B$ and ${}^{\blacktriangledown}\colon B \to B$ are mutually dual.

(b) For all $S \subseteq B$, $\bigvee S^{\blacktriangle} = (\bigvee S)^{\blacktriangle}$.

(c) For all $S \subseteq B$, $\bigwedge S^{\blacktriangledown} = (\bigwedge S)^{\blacktriangledown}$.

(d) $(B^{\blacktriangle}, \le)$ is a complete lattice; 0 is the least element and $1^{\blacktriangle}$ is the greatest element of $(B^{\blacktriangle}, \le)$.

(e) $(B^{\blacktriangledown}, \le)$ is a complete lattice; $0^{\blacktriangledown}$ is the least element and 1 is the greatest element of $(B^{\blacktriangledown}, \le)$

(f) The kernel $\Theta_{\blacktriangledown} = \{(x, y) \mid x^{\blacktriangledown} = y^{\blacktriangledown}\}$ of the map ${}^{\blacktriangledown}\colon B \to B$ is a congruence on the semilattice $(B, \wedge)$ such that the $\Theta_{\blacktriangledown}$-class of any x has a least element.

(g) The kernel $\Theta_{\blacktriangle} = \{(x, y) \mid x^{\blacktriangle} = y^{\blacktriangle}\}$ of the map ${}^{\blacktriangle}\colon B \to B$ is a congruence on the semilattice $(B, \vee)$ such that the $\Theta_{\blacktriangle}$-class of any x has a greatest element.

Proof: (a) Let $a \in \mathcal{A}(\mathcal{B})$. Then

$$a \le (x^{\blacktriangle})' \text{ iff } a \not\le x^{\blacktriangle} \text{ iff } \varphi(a) \wedge x = 0 \text{ iff } \varphi(a) \le x' \text{ iff } a \le (x')^{\blacktriangledown}.$$

This implies that $(x^{\blacktriangle})' = (x')^{\blacktriangledown}$. The other part can be proved similarly.

(b) Let $S \subseteq B$. The map ${}^\blacktriangle \colon B \to B$ is order-preserving, which implies that $\bigvee S^\blacktriangle \leq (\bigvee S)^\blacktriangle$. Let $a \in \mathcal{A}(\mathcal{B})$ and assume that $\varphi(a) \wedge \bigvee S \neq 0$. Then $0 \neq \varphi(a) \wedge \bigvee S = \bigvee\{\varphi(a) \wedge x \mid x \in S\}$, which implies that $\varphi(a) \wedge x \neq 0$ for some $x \in S$. Thus,

$$\{a \in \mathcal{A}(\mathcal{B}) \mid \varphi(a) \wedge \bigvee S \neq 0\} \subseteq \bigcup_{x \in S} \{a \in \mathcal{A}(\mathcal{B}) \mid \varphi(a) \wedge x \neq 0\}$$

and by Lemma 2.3,

$$\begin{aligned}
(\bigvee S)^\blacktriangle &= \bigvee\{a \in \mathcal{A}(\mathcal{B}) \mid \varphi(a) \wedge \bigvee S \neq 0\} \\
&\leq \bigvee \left(\bigcup_{x \in S} \{a \in \mathcal{A}(\mathcal{B}) \mid \varphi(a) \wedge x \neq 0\} \right) \\
&= \bigvee_{x \in S} \left\{ \bigvee \{a \in \mathcal{A}(\mathcal{B}) \mid \varphi(a) \wedge x \neq 0\} \right\} \\
&= \bigvee\{x^\blacktriangle \mid x \in S\} = \bigvee S^\blacktriangle.
\end{aligned}$$

Hence, $\bigvee S^\blacktriangle = (\bigvee S)^\blacktriangle$. Claim (c) can be proved similarly. Assertions (d) and (e) follow easily from (b), (c), and Lemma 3.4(a).

(f) It is clear that $\Theta_\blacktriangledown$ is an equivalence on P. Let $x_1, x_2, y_1, y_2 \in B$ and assume that $(x_1, y_1) \in \Theta_\blacktriangledown$ and $(x_2, y_2) \in \Theta_\blacktriangledown$. Then

$$(x_1 \wedge x_2)^\blacktriangledown = x_1^\blacktriangledown \wedge x_2^\blacktriangledown = y_1^\blacktriangledown \wedge y_2^\blacktriangledown = (y_1 \wedge y_2)^\blacktriangledown.$$

Thus, $\Theta_\blacktriangledown$ is a congruence on $(B, \wedge)$.

Clearly, $\bigwedge [x]_{\Theta_\blacktriangledown}$ is the least element in the congruence class of x since

$$(\bigwedge\{y \mid y^\blacktriangledown = x^\blacktriangledown\})^\blacktriangledown = \bigwedge\{y^\blacktriangledown \mid y^\blacktriangledown = x^\blacktriangledown\} = x^\blacktriangledown$$

by (c). Assertion (g) can be proved similarly. □

Note that Proposition 3.5(b) implies that

$$x^\blacktriangle = (\bigvee\{a \in \mathcal{A}(\mathcal{B}) \mid a \leq x\})^\blacktriangle = \bigvee\{a^\blacktriangle \mid a \in \mathcal{A}(\mathcal{B}) \text{ and } a \leq x\}. \tag{3.4}$$

Let us denote by $c_\blacktriangle(x)$ the greatest element in the $\Theta_\blacktriangle$-class of any $x \in B$. It is easy to see that the map $x \mapsto c_\blacktriangle(x)$ is a closure operator on $\mathcal{B}$. Similarly, if we denote by $c_\blacktriangledown(x)$ the least element of the $\Theta_\blacktriangledown$-class of x, then the map $x \mapsto c_\blacktriangledown(x)$ is an interior operator on $\mathcal{B}$ (see [7] for details).

Remark 3.6. In [12] Lemmon introduced the term modal algebra. A structure $(B, \vee, \wedge, {}', 0, 1, \mu)$ is a *modal algebra* if

(a) $(B, \vee, \wedge, {}', 0, 1)$ is a Boolean algebra and

(b) $\mu \colon B \to B$ is a function such that $\mu(x \vee y) = \mu(x) \vee \mu(y)$ for all $x, y \in B$.

Furthermore, a modal algebra is *normal* if $\mu(0) = 0$. It is clear by Lemma 3.4(a) and Proposition 3.5(b) that $(B, \vee, \wedge, {}', 0, 1, {}^{\blacktriangle})$ is a normal modal algebra in the sense of [12]. Note also that already in [10] Jónsson and Tarski studied Boolean algebras with normal and "completely additive" operators.

Next we show that $(B^{\blacktriangle}, \leq)$ and $(B^{\blacktriangledown}, \leq)$ are dually order-isomorphic.

Lemma 3.7. $(B^{\blacktriangle}, \leq) \cong (B^{\blacktriangledown}, \leq)^{\partial}$.

Proof: We show that $x^{\blacktriangle} \mapsto (x')^{\blacktriangledown}$ is the required dual order-isomorphism. Obviously, $x^{\blacktriangle} \mapsto (x')^{\blacktriangledown}$ is onto $B^{\blacktriangledown}$. Assume that $x^{\blacktriangle} \leq y^{\blacktriangle}$. Then for all $a \in \mathcal{A}(\mathcal{B})$, $\varphi(a) \wedge x \neq 0$ implies $\varphi(a) \wedge y \neq 0$. Suppose that $(y')^{\blacktriangledown} \not\leq (x')^{\blacktriangledown}$. This means that there exists an $a \in \mathcal{A}(\mathcal{B})$ such that $\varphi(a) \leq y'$ but $\varphi(a) \not\leq x'$. Since $\varphi(a) \not\leq x'$ is equivalent to $\varphi(a) \wedge x \neq 0$, we have that $\varphi(a) \wedge y \neq 0$. But this means that $\varphi(a) \not\leq y'$, a contradiction! Hence, $(y')^{\blacktriangledown} \leq (x')^{\blacktriangledown}$.

On the other hand, assume that $(y')^{\blacktriangledown} \leq (x')^{\blacktriangledown}$. Let us suppose that $x^{\blacktriangle} \not\leq y^{\blacktriangle}$. This means that there exists an $a \in \mathcal{A}(\mathcal{B})$ such that $\varphi(a) \wedge x \neq 0$ and $\varphi(a) \wedge y = 0$. But this implies that $\varphi(a) \leq y'$ and $a \leq (y')^{\blacktriangledown}$. Hence, $a \leq (x')^{\blacktriangledown}$ and $\varphi(a) \leq x'$. This is equivalent to $\varphi(a) \wedge x = 0$, a contradiction! Thus, $x^{\blacktriangledown} \leq y^{\blacktriangledown}$. □

Next we study the properties of approximations more closely in cases when the corresponding map $\varphi \colon \mathcal{A}(\mathcal{B}) \to B$ is extensive, symmetric, or closed.

Extensiveness

Here we study the functions ${}^{\blacktriangle} \colon B \to B$ and ${}^{\blacktriangledown} \colon B \to B$ defined by an extensive mapping φ. We show that $(B^{\blacktriangledown}, \leq)$ and $(B^{\blacktriangle}, \leq)$ are bounded by 0 and 1. Furthermore, each element of B is proved to be between its approximations.

Proposition 3.8. Let $\mathcal{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi \colon \mathcal{A}(\mathcal{B}) \to B$ be an extensive map. Then

(a) $0^{\blacktriangledown} = 0^{\blacktriangle} = 0$ and $1^{\blacktriangledown} = 1^{\blacktriangle} = 1$;

(b) $x^{\blacktriangledown} \leq x \leq x^{\blacktriangle}$ for all $x \in \mathcal{B}$.

Proof: (a) Since $\varphi(a) \geq a > 0$ for all $a \in \mathcal{A}(\mathcal{B})$, $\{a \in \mathcal{A}(\mathcal{B}) \mid \varphi(a) \leq 0\} = \emptyset$. Hence, $0^{\blacktriangledown} = 0$. Similarly, $\{a \in \mathcal{A}(\mathcal{B}) \mid \varphi(a) \wedge 1 \neq 0\} = \mathcal{A}(\mathcal{B})$ and thus $1^{\blacktriangle} = 1$. By Lemma 3.4(a), $0^{\blacktriangle} = 0$ and $1^{\blacktriangledown} = 1$.

(b) Let $a \in \mathcal{A}(\mathcal{B})$. Assume that $a \leq x$. If $\varphi(a) \leq x'$, then $a \leq \varphi(a) \leq x'$. Hence, $a \leq x \wedge x' = 0$, a contradiction! This means that $\varphi(a) \not\leq x'$ and $\varphi(a) \wedge x \neq 0$. Thus,

$$\{a \in \mathcal{A}(\mathcal{B}) \mid a \leq x\} \subseteq \{a \in \mathcal{A}(\mathcal{B}) \mid \varphi(a) \wedge x \neq 0\}$$

and

$$x = \bigvee \{a \in \mathcal{A}(\mathcal{B}) \mid a \leq x\} \leq \bigvee \{a \in \mathcal{A}(\mathcal{B}) \mid \varphi(a) \wedge x \neq 0\} = x^{\blacktriangle}.$$

On the hand, if $\varphi(a) \leq x$, then $a \leq \varphi(a) \leq x$, and hence

$$\{a \in \mathcal{A}(\mathcal{B}) \mid \varphi(a) \leq x\} \subseteq \{a \in \mathcal{A}(\mathcal{B}) \mid a \leq x\}.$$

This implies that $x^{\blacktriangledown} \leq x$. □

Symmetry

In this subsection we assume that φ is symmetric. First we show that the pair $({}^\blacktriangle, {}^\blacktriangledown)$ is a dual Galois connection.

Proposition 3.9. Let $\mathcal{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ be a symmetric map. Then the pair $({}^\blacktriangle, {}^\blacktriangledown)$ is a dual Galois connection on $\mathcal{B}$.

Proof: As we have noted, the maps ${}^\blacktriangledown\colon B \to B$ and ${}^\blacktriangle\colon B \to B$ are order-preserving. Let $a \in \mathcal{A}(\mathcal{B})$. If $a \leq x^{\blacktriangledown\blacktriangle}$, then $\varphi(x) \wedge x^\blacktriangledown \neq 0$. This implies that there exists an atom $b \in \mathcal{A}(\mathcal{B})$ such that $b \leq \varphi(a) \wedge x^\blacktriangledown$. Thus $b \leq \varphi(a)$ and $b \leq x^\blacktriangledown$. Since φ is symmetric, also $a \leq \varphi(b)$ holds. Because $b \leq x^\blacktriangledown$ is equivalent to $\varphi(b) \leq x$, we get $a \leq x$ and hence

$$\{a \in \mathcal{A}(\mathcal{B}) \mid a \leq x^{\blacktriangledown\blacktriangle}\} \subseteq \{a \in \mathcal{A}(\mathcal{B}) \mid a \leq x\},$$

which implies $x^{\blacktriangledown\blacktriangle} \leq x$.

Let us denote $y = x'$. Then $y^{\blacktriangledown\blacktriangle} \leq y$ implies

$$x = y' \leq (y^{\blacktriangledown\blacktriangle})' = ((y^\blacktriangledown)')^\blacktriangledown = (y')^{\blacktriangle\blacktriangledown} = x^{\blacktriangle\blacktriangledown}.$$

$\square$

By the previous proposition, the pair $({}^\blacktriangle, {}^\blacktriangledown)$ is a dual Galois connection whenever the map φ is symmetric. This means that these maps have the properties of Proposition 2.2. In particular, the map $x \mapsto x^{\blacktriangle\blacktriangledown}$ is a closure operator and the map $x \mapsto x^{\blacktriangledown\blacktriangle}$ is an interior operator. Let us denote

$$B^{\blacktriangle\blacktriangledown} = \{x^{\blacktriangle\blacktriangledown} \mid x \in B\} \quad \text{and} \quad B^{\blacktriangledown\blacktriangle} = \{x^{\blacktriangledown\blacktriangle} \mid x \in B\}.$$

Proposition 3.10. Let $\mathcal{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ be a symmetric map. Then

(a) $a^\blacktriangle = \varphi(a)$ for all $a \in \mathcal{A}(\mathcal{B})$;

(b) $x^\blacktriangle = \bigvee\{\varphi(a) \mid a \in \mathcal{A}(\mathcal{B}) \text{ and } a \leq x\}$ for every $x \in B$;

(c) $B^{\blacktriangle\blacktriangledown} = B^\blacktriangledown$ and $B^{\blacktriangledown\blacktriangle} = B^\blacktriangle$;

(d) $(B^\blacktriangle, \leq) \cong (B^\blacktriangledown, \leq)$.

Proof: (a) Let $a \in \mathcal{A}(\mathcal{B})$. It obvious that $\{b \in \mathcal{A}(\mathcal{B}) \mid b \leq \varphi(a)\} = \{b \in \mathcal{A}(\mathcal{B}) \mid a \leq \varphi(b)\}$, since φ is symmetric. Thus,

$$\begin{aligned} \varphi(a) &= \bigvee\{b \in \mathcal{A}(\mathcal{B}) \mid b \leq \varphi(a)\} \\ &= \bigvee\{b \in \mathcal{A}(\mathcal{B}) \mid a \leq \varphi(b)\} \\ &= \bigvee\{b \in \mathcal{A}(\mathcal{B}) \mid a \wedge \varphi(b) \neq 0\} \\ &= a^\blacktriangle. \end{aligned}$$

Claim (b) follows easily from (a) and (3.4).

(c) It is clear that $B^{\blacktriangle\blacktriangledown} \subseteq B^\blacktriangledown$. On the other hand, if $x \in B^\blacktriangledown$, then $x = y^\blacktriangledown$ for some $y \in B$ and thus $x^{\blacktriangle\blacktriangledown} = y^{\blacktriangledown\blacktriangle\blacktriangledown} = y^\blacktriangledown = x$, that is, $x \in B^{\blacktriangle\blacktriangledown}$. The other part can be proved similarly.

Claim (d) is obvious, since $(B^{\blacktriangle\blacktriangledown}, \leq) \cong (B^{\blacktriangledown\blacktriangle}, \leq)$ by Proposition 2.2(c). $\square$

We denoted by $c_\blacktriangle(x)$ the greatest element in the $\Theta_\blacktriangle$-class of x. Similarly, $c_\blacktriangledown(x)$ denotes the least element in the x's $\Theta_\blacktriangledown$-class. We can now write the following corollary.

Corollary 3.11. Let $\mathcal{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ be a symmetric map. Then

(a) $c_\blacktriangle(x) = x^{\blacktriangle\blacktriangledown}$ for all $x \in B$,

(b) $c_\blacktriangledown(x) = x^{\blacktriangledown\blacktriangle}$ for all $x \in B$,

(c) $(B^\blacktriangledown, \leq) \cong (B^\blacktriangledown, \leq)^\partial$ and $(B^\blacktriangle, \leq) \cong (B^\blacktriangle, \leq)^\partial$.

Proof: (a) Since $x^\blacktriangle = x^{\blacktriangle\blacktriangledown\blacktriangle}$, we have $(x, x^{\blacktriangle\blacktriangledown}) \in \Theta_\blacktriangle$. Furthermore, $x \leq x^{\blacktriangle\blacktriangledown}$ holds by the definition of dual Galois connections. Equality (b) can be proved similarly.

By Lemma 3.7 and Proposition 3.10(d),

$$(B^\blacktriangledown, \leq) \cong (B^\blacktriangle, \geq) \cong (B^\blacktriangledown, \geq) = (B^\blacktriangle, \leq),$$

which proves (c). □

Note that we have now showed that

$$\{x^{\blacktriangle\blacktriangledown} \mid x \in B\} = \{c_\blacktriangle(x) \mid x \in B\} = \{x^\blacktriangledown \mid x \in B\}$$

and

$$\{x^{\blacktriangledown\blacktriangle} \mid x \in B\} = \{c_\blacktriangledown(x) \mid x \in B\} = \{x^\blacktriangle \mid x \in B\}.$$

In the following example the map φ is extensive and closed, but not symmetric.

Example 3.12. Let $B = \{0, a, b, c, d, e, f, 1\}$ and let the order $\leq$ be defined as in Figure 2. The set of atoms of the complete atomic Boolean lattice $\mathcal{B} = (B, \leq)$ is $\{a, b, c\}$. Let the map $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ be defined by

$$\varphi(a) = d, \qquad \varphi(b) = b, \qquad \varphi(c) = f.$$

Then obviously φ is extensive and closed. The map φ is not symmetric, since $b \leq \varphi(a)$, but $a \not\leq \varphi(b)$, for example.

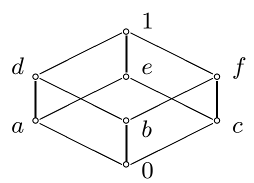

Figure 2. The ordered set $(B, \leq)$

In Table 2 is presented the lower and upper approximations of the elements of B. The dually order-isomorphic ordered sets $(B^\blacktriangle, \leq)$ and $(B^\blacktriangledown, \leq)$ are shown in Figure 3. Note that $(B^\blacktriangle, \leq) \not\cong (B^\blacktriangledown, \leq)$, $(B^\blacktriangledown, \leq) \not\cong (B^\blacktriangledown, \leq)^\partial$, and $(B^\blacktriangle, \leq) \not\cong (B^\blacktriangle, \leq)^\partial$.

x	$x^{\blacktriangledown}$	$x^{\blacktriangle}$
0	0	0
a	0	a
b	b	$a \vee b \vee c = 1$
c	0	c
d	$a \vee b = d$	$a \vee b \vee c = 1$
e	0	$a \vee c = e$
f	$b \vee c = f$	$a \vee b \vee c = 1$
1	$a \vee b \vee c = 1$	$a \vee b \vee c = 1$

Table 2. Lower and upper approximations

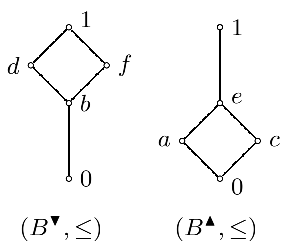

Figure 3. The ordered sets $(B^{\blacktriangledown}, \leq)$ and $(B^{\blacktriangle}, \leq)$

Closedness

We end this work by studying the case in which φ is closed. First we present the following observation.

Lemma 3.13. Let $\mathcal{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi \colon \mathcal{A}(\mathcal{B}) \to B$ be a closed map. Then for all $x \in B$,

(a) $x^{\blacktriangle\blacktriangle} \leq x^{\blacktriangle}$;

(b) $x^{\blacktriangledown} \leq x^{\blacktriangledown\blacktriangledown}$.

Proof: (a) Let $a \in \mathcal{A}(\mathcal{B})$. If $a \leq x^{\blacktriangle\blacktriangle}$, then $\varphi(a) \wedge x^{\blacktriangle} \neq 0$. This means that there exists an atom $b \in \mathcal{A}(\mathcal{B})$ such that $b \leq \varphi(a)$ and $b \leq x^{\blacktriangle}$. Now $b \leq x^{\blacktriangle}$ is equivalent to $\varphi(b) \wedge x \neq 0$ and since φ is closed, $b \leq \varphi(a)$ implies $\varphi(b) \leq \varphi(a)$. Thus, also $\varphi(a) \wedge x \neq 0$ holds and so $a \leq x^{\blacktriangle}$. Hence,

$$\{a \in \mathcal{A}(\mathcal{B}) \mid a \leq x^{\blacktriangle\blacktriangle}\} \subseteq \{a \in \mathcal{A}(\mathcal{B}) \mid a \leq x^{\blacktriangle}\}$$

and $x^{\blacktriangle\blacktriangle} \leq x^{\blacktriangle}$.

(b) Let us denote $y = x'$. Then $y^{\blacktriangle\blacktriangle} \leq y^{\blacktriangle}$ and

$$x^{\blacktriangledown} = (y')^{\blacktriangledown} = (y^{\blacktriangle})' \leq (y^{\blacktriangle\blacktriangle})' = ((y^{\blacktriangle})')^{\blacktriangledown} = (y')^{\blacktriangledown\blacktriangledown} = x^{\blacktriangledown\blacktriangledown}.$$

$\square$

Now we can write the following proposition.

Proposition 3.14. Let $\mathcal{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ be an extensive and closed map.

(a) The map ${}^{\blacktriangle}\colon B \to B$ is a closure operator.

(b) The map ${}^{\blacktriangledown}\colon B \to B$ is an interior operator.

(c) $(B^{\blacktriangledown}, \leq)$ and $(B^{\blacktriangle}, \leq)$ are complete sublattices of $(B, \leq)$.

Proof: (a) The map ${}^{\blacktriangle}\colon B \to B$ is extensive because φ is extensive, and it is order-preserving by Lemma 3.4(b). By Lemma 3.13(a) $x^{\blacktriangle\blacktriangle} \leq x^{\blacktriangle}$, and $x^{\blacktriangle} \leq x^{\blacktriangle\blacktriangle}$ holds since φ is extensive. Thus, $x^{\blacktriangle} = x^{\blacktriangle\blacktriangle}$. Claim (b) follows from Lemma 2.7 and Proposition 3.5(a).

(c) Let $S^{\blacktriangledown} = \{x^{\blacktriangledown} \mid x \in S\}$ be a subset of $B^{\blacktriangledown}$. Then by Proposition 3.5(c),

$$\bigwedge S^{\blacktriangledown} = (\bigwedge S)^{\blacktriangledown},$$

which implies that $\bigwedge S^{\blacktriangledown} \in B^{\blacktriangledown}$. Next we will show that also $\bigvee S^{\blacktriangledown} \in B^{\blacktriangledown}$.

It is clear that for all $x \in S$,

$$x^{\blacktriangledown} \leq \bigvee S^{\blacktriangledown} \text{ and } x^{\blacktriangledown} = x^{\blacktriangledown\blacktriangledown} \leq (\bigvee S^{\blacktriangledown})^{\blacktriangledown}$$

Thus, $(\bigvee S^{\blacktriangledown})^{\blacktriangledown}$ is an upper bound of $S^{\blacktriangledown}$ and $\bigvee S^{\blacktriangledown} \leq (\bigvee S^{\blacktriangledown})^{\blacktriangledown}$. Because φ is extensive, $(\bigvee S^{\blacktriangledown})^{\blacktriangledown} \leq \bigvee S^{\blacktriangledown}$. Hence, $\bigvee S^{\blacktriangledown} = (\bigvee S^{\blacktriangledown})^{\blacktriangledown}$ and $\bigvee S^{\blacktriangledown} \in B^{\blacktriangledown}$. The other part can be proved similarly. □

It is known that every sublattice of a distributive lattice is distribute (see [3], for example). Therefore, we can write the following corollary.

Corollary 3.15. Let $\mathcal{B} = (B, \leq)$ be a complete atomic Boolean lattice. If $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ is extensive and closed map, then $(B^{\blacktriangledown}, \leq)$ and $(B^{\blacktriangle}, \leq)$ are distributive.

Example 3.16. It is known that if $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ is extensive and symmetric, then $(B^{\blacktriangledown}, \leq)$ and $(B^{\blacktriangle}, \leq)$ are not necessarily distributive (see [7] for an example). By Corollary 3.15, these ordered sets are distributive if φ is extensive and closed. Next we show that $(B^{\blacktriangledown}, \leq)$ and $(B^{\blacktriangle}, \leq)$ are not always distributive when φ is merely closed.

Let us consider a set $U = \{a, b, c, d, e, f\}$ and let the relation $\approx$ on U be a transitive binary relation on U such that,

$$\begin{aligned}
[a]_{\approx} &= [b]_{\approx} = [c]_{\approx} = \emptyset, \\
[d]_{\approx} &= \{a\}, \\
[e]_{\approx} &= \{b, c\}, \\
[f]_{\approx} &= \{a, b\}.
\end{aligned}$$

Now obviously the map $\varphi\colon U \to \wp(U), x \mapsto [x]_{\approx}$ is closed, but not extensive nor symmetric. The set $\{X^{\blacktriangle} \mid X \subseteq U\}$ consisting of the upper approximations of the elements of $\wp(U)$ is

$$\{\emptyset, \{e\}, \{d, f\}, \{e, f\}, \{d, e, f\}\}$$

The ordered set $(\{X^{\blacktriangle} \mid X \subseteq U\}, \subseteq)$ is presented in Figure 4. Obviously, this ordered set is isomorphic with $\mathbf{N_5}$, which means that it is not distributive (see [3], for example).

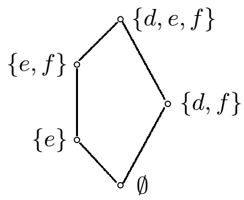

Figure 4. The ordered set $(\{X^{\blacktriangle} \mid X \subseteq U\}, \subseteq)$

Lemma 3.17. Let $\mathcal{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ be extensive, symmetric, and closed. Then for all $x \in B$,

$$x^{\blacktriangledown\blacktriangle} = x^{\blacktriangledown} \text{ and } x^{\blacktriangle\blacktriangledown} = x^{\blacktriangle}.$$

Proof: Since φ is extensive, $x^{\blacktriangledown} \leq x^{\blacktriangledown\blacktriangle}$ for all $x \in P$. Let $a \in \mathcal{A}(\mathcal{B})$ and suppose that $a \leq x^{\blacktriangledown\blacktriangle}$, which means that $\varphi(a) \wedge x^{\blacktriangledown} \neq 0$. This implies that there exists an atom $b \in \mathcal{A}(\mathcal{B})$ such that $b \leq \varphi(a)$ and $b \leq x^{\blacktriangledown}$. Since φ is symmetric, $b \leq \varphi(a)$ implies $a \leq \varphi(b)$. The fact that φ is closed yields $\varphi(b) \leq \varphi(a)$ and $\varphi(a) \leq \varphi(b)$, that is, $\varphi(a) = \varphi(b)$. Because $b \leq x^{\blacktriangledown}$ we obtain $\varphi(a) = \varphi(b) \leq x$. Hence, $a \leq x^{\blacktriangledown}$ and so also $x^{\blacktriangledown\blacktriangle} \leq x^{\blacktriangledown}$ holds.

The other part is now obvious:

$$x^{\blacktriangle} = (x^{\blacktriangle})'' = ((x')^{\blacktriangledown})' = ((x')^{\blacktriangledown\blacktriangle})' = x^{\blacktriangle\blacktriangledown}.$$

□

An element $x \in B$ is said to be *exact* if $x^{\blacktriangle} = x^{\blacktriangledown}$. Now the following proposition holds.

Proposition 3.18. Let $\mathcal{B} = (B, \leq)$ be a complete atomic Boolean lattice. If $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ is extensive and symmetric, then the set of all exact elements is a complete Boolean sublattice of $\mathcal{B}$.

Proof: By Proposition 3.8(a), 0 and 1 are exact. If x is exact, then $(x')^{\blacktriangledown} = (x^{\blacktriangle})' = (x^{\blacktriangledown})' = (x')^{\blacktriangle}$.

Let S be a subset of the set of exact elements. Since φ is extensive, $x^{\blacktriangledown} = x = x^{\blacktriangle}$ for all exact elements x. Then

$$(\bigvee S)^{\blacktriangle} = \bigvee S^{\blacktriangle} = \bigvee S$$

and

$$(\bigvee S)^{\blacktriangledown} = (\bigvee S^{\blacktriangle})^{\blacktriangledown} = (\bigvee S)^{\blacktriangle\blacktriangledown} \geq \bigvee S \geq (\bigvee S)^{\blacktriangledown}.$$

Thus, $(\bigvee S)^{\blacktriangle} = \bigvee S = (\bigvee S)^{\blacktriangledown}$. Similarly, we can show that $\bigwedge S$ is exact. □

By Lemma 3.17 it is clear that if $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ is an extensive, symmetric, and closed map, then $B^{\blacktriangledown} = \{x^{\blacktriangledown} \mid x \in B\}$ equals $B^{\blacktriangle} = \{x^{\blacktriangle} \mid x \in B\}$. For simplicity, let us denote $\mathcal{E} = B^{\blacktriangledown} = B^{\blacktriangle}$. Note that $\mathcal{E}$ is now the set of exact elements.

Proposition 3.19. Let $\mathcal{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ be an extensive, symmetric, and closed map. The ordered set $(\mathcal{E}, \leq)$ is a complete atomic Boolean sublattice of $\mathcal{B}$. The set of atoms is $\{a^{\blacktriangle} \mid a \in \mathcal{A}(\mathcal{B})\}$.

Proof: Because $\mathcal{E}$ is the set of exact elements, $(\mathcal{E}, \leq)$ is a complete Boolean lattice by Proposition 3.18.

We have still to show that $(\mathcal{E}, \leq)$ is atomic. First we show that $\{a^{\blacktriangle} \mid a \in \mathcal{A}(\mathcal{B})\}$ is the set of atoms. Let $a \in \mathcal{A}(\mathcal{B})$ and suppose that $0 < x < a^{\blacktriangle}$ for some $x \in \mathcal{E}$. This means that there exists an atom $b \in \mathcal{A}(\mathcal{B})$ such that $b \leq x$, which implies that $b \leq \varphi(b) = b^{\blacktriangle} \leq x^{\blacktriangle} = x < a^{\blacktriangle} = \varphi(a)$. Since φ is symmetric and closed, we have $a \leq \varphi(b)$ and $\varphi(a) \leq \varphi(b)$. Hence, $a^{\blacktriangle} = \varphi(a) = \varphi(b) = b^{\blacktriangle}$. But this implies $a^{\blacktriangle} = b^{\blacktriangle} \leq x < a^{\blacktriangle}$, a contradiction! Thus, $a^{\blacktriangle}$ is an atom of $(\mathcal{E}, \leq)$. It is now also obvious that $\{a^{\blacktriangle} \mid a \in \mathcal{A}(\mathcal{B})\}$ is the set of all atoms.

Because $a \leq x$ iff $a^{\blacktriangle} \leq x$ for all $a \in \mathcal{A}(\mathcal{B})$ and $x \in \mathcal{E}$, $x = x^{\blacktriangle} = \bigvee\{a^{\blacktriangle} \mid a \in \mathcal{A}(\mathcal{B}) \text{ and } a \leq x\}$, which means that every element of $\mathcal{E}$ is the supremum of the atoms below it. Thus, $(\mathcal{E}, \leq)$ is atomic. $\square$

The following corollary follows from Proposition 3.19.

Corollary 3.20. Let $\mathcal{B} = (B, \leq)$ be a complete atomic Boolean lattice. If $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ is symmetric and closed, then $(B^{\blacktriangle}, \leq)$ and $(B^{\blacktriangledown}, \leq)$ are complete atomic Boolean lattices.

Proof: Let $A^* = \{a \in \mathcal{A}(\mathcal{B}) \mid \varphi(a) \neq 0\}$ and let $B^* = \{\bigvee X \mid X \subseteq A^*\}$. It is easy to see that $(B^*, \leq)$ is a complete atomic Boolean lattice and that A^* is the set of atoms of $(B^*, \leq)$.

Let us denote by φ^* the map φ restricted to A^*. First we show that $\varphi^*(a) \in B^*$ for all $a \in A^*$. Namely, if we assume that $0 \neq \varphi^*(a) \in B - B^*$ for some $a \in A^*$, then there exists a $b \in \mathcal{A}(\mathcal{B}) - A^*$ such that $b \leq \varphi^*(a) = \varphi(a)$. But since φ is symmetric, we obtain $0 \neq a \leq \varphi(b)$ and $b \in A^*$, a contradiction!

Obviously $\varphi^* : A^* \to B^*$ symmetric and closed. We show that φ^* is also extensive, that is, $a \leq \varphi^*(a)$ for $a \in A^*$. Let $a \in A^*$. Then $0 \neq \varphi^*(a) \in B^*$, which implies that there is an atom $b \in A^*$ such that $b \leq \varphi^*(a)$. But since φ^* is symmetric, also $a \leq \varphi^*(b)$ holds. Because φ^* is closed, $b \leq \varphi^*(a)$ implies $\varphi^*(b) \leq \varphi^*(a)$. Hence, $a \leq \varphi^*(b) \leq \varphi^*(a)$ and so φ^* is also extensive.

We can now apply Proposition 3.19, which says that the set of upper approximations $(B^{\vartriangle}, \leq)$ defined by the map $\varphi^* : A^* \to B^*$ is a complete atomic Boolean lattice. For all $x \in B$,

$$\begin{aligned} x^{\blacktriangle} &= \bigvee\{a \in \mathcal{A}(\mathcal{B}) \mid \varphi(a) \wedge x \neq 0\} \\ &= \bigvee\{a \in \mathcal{A}(\mathcal{B}) \mid \varphi^*(a) \wedge x \neq 0\} \\ &= \bigvee\{a \in A^* \mid \varphi^*(a) \wedge x \neq 0\} = x^{\vartriangle}. \end{aligned}$$

This means that $B^{\blacktriangle} = B^{\vartriangle}$. Because $(B^{\vartriangle}, \leq)$ is a complete atomic Boolean lattice, $(B^{\blacktriangle}, \leq)$ is a complete atomic Boolean lattice. Since $(B^{\blacktriangle}, \leq) \cong (B^{\blacktriangledown}, \leq)$, also $(B^{\blacktriangledown}, \leq)$ is a complete atomic Boolean lattice. $\square$

We end this work by the following proposition.

Proposition 3.21. Let $\mathcal{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ be extensive, symmetric, and closed. Then for all $x \in B$,

$$x^{\blacktriangledown} = \bigvee_B \{b \in \mathcal{E} \mid b \leq x\} \quad \text{and} \quad x^{\blacktriangle} = \bigwedge_B \{b \in \mathcal{E} \mid x \leq b\}.$$

Proof: Because $b \leq x$ implies $b \leq x^{\blacktriangledown}$ for all $b \in \mathcal{E}$, $x^{\blacktriangledown}$ is an upper bound of $\{b \in \mathcal{E} \mid b \leq x\}$. Thus,

$$\bigvee_B \{b \in \mathcal{E} \mid b \leq x\} \leq x^{\blacktriangledown}$$

For all $a \in \mathcal{A}(\mathcal{B})$ such that $\varphi(a) \leq x$ we have

$$a \leq \varphi(a) \leq \bigvee_B \{\varphi(a) \mid a \in \mathcal{A}(\mathcal{B}) \text{ and } \varphi(a) \leq x\},$$

which implies

$$x^{\blacktriangledown} = \bigvee_B \{a \in \mathcal{A}(\mathcal{B}) \mid \varphi(a) \leq x\} \leq \bigvee_B \{\varphi(a) \mid a \in \mathcal{A}(\mathcal{B}) \text{ and } \varphi(a) \leq x\}.$$

By Proposition 3.10(a), $\varphi(a) = a^{\blacktriangle}$, which means that $\varphi(a) \in \mathcal{E}$. Thus,

$$\{\varphi(a) \mid a \in \mathcal{A}(\mathcal{B}) \text{ and } \varphi(a) \leq x\} \subseteq \{b \in \mathcal{E} \mid b \leq x\},$$

which yields

$$x^{\blacktriangledown} \leq \bigvee_B \{\varphi(a) \mid a \in \mathcal{A}(\mathcal{B}) \text{ and } \varphi(a) \leq x\} \leq \bigvee_B \{b \in \mathcal{E} \mid b \leq x\}$$

This means that $x^{\blacktriangledown} = \bigvee_B \{b \in \mathcal{E} \mid b \leq x\}$.

On the other hand,

$$\begin{aligned}
x^{\blacktriangle} &= ((x')^{\blacktriangledown})' \\
&= (\textstyle\bigvee_B \{b \in \mathcal{E} \mid b \leq x'\})' \\
&= (\textstyle\bigvee_B \{b \mid b \in \mathcal{E} \text{ and } b' \geq x\})' \\
&= \textstyle\bigwedge_B \{b' \mid b \in \mathcal{E} \text{ and } b' \geq x\} \\
&= \textstyle\bigwedge_B \{b' \mid b' \in \mathcal{E} \text{ and } b' \geq x\} \\
&= \textstyle\bigwedge_B \{b \mid b \in \mathcal{E} \text{ and } b \geq x\}.
\end{aligned}$$

□

Now we may present a result which follows directly from Theorem 2 of Gehrke and Walker [5]. We state that if φ is extensive, symmetric, and closed, then the pointwise ordered set of all rough sets $\{(x^{\blacktriangledown}, x^{\blacktriangle}) \mid x \in B\}$ is order-isomorphic to $\mathbf{2}^I \times \mathbf{3}^J$, where $I = \{a^{\blacktriangle} \mid a \in \mathcal{A}(\mathcal{B}) \text{ and } \varphi(a) = a\}$ and $J = \{a^{\blacktriangle} \mid a \in \mathcal{A}(\mathcal{B}) \text{ and } \varphi(a) \neq a\}$. Note that I is the set of atoms of $\mathcal{E}$ that are atoms of $\mathcal{B}$ and J consists of the atoms of $\mathcal{E}$ which are not atoms of $\mathcal{B}$

Note also that it is known that the pointwise ordered set of all rough sets $\{(x^{\blacktriangledown}, x^{\blacktriangle}) \mid x \in B\}$ is not necessarily even a semilattice if $\varphi\colon \mathcal{A}(\mathcal{B}) \to B$ is extensive and symmetric, but not closed (see [7]).

Conclusions

We have studied the structure of the ordered set of rough approximations. We have proved that the ordered sets $(B^{\blacktriangle}, \leq)$ and $(B^{\blacktriangledown}, \leq)$ are always complete lattices. Furthermore, we have shown that if φ is extensive and symmetric, these ordered sets are not necessarily distributive and same holds if φ is merely closed. When φ is extensive and closed, $(B^{\blacktriangle}, \leq)$ and $(B^{\blacktriangledown}, \leq)$ are complete distributive sublattices of $\mathcal{B}$. These ordered sets are complete atomic Boolean lattices if φ is symmetric and closed, and if φ is extensive, symmetric, and closed, then $(B^{\blacktriangle}, \leq)$ and $(B^{\blacktriangledown}, \leq)$ are mutually equal complete atomic Boolean sublattices of $\mathcal{B}$.

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