Teaching High-School Mathematics with Structured Derivations in Hypertext Format

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Abstract: *In this paper, a way of teaching high-school mathematics using a semiformal representation called structured derivation is presented. Using structured derivation every step of the derived solution to an exercise in presented with facts justifying the step. With this presentation solutions to mathematical problems resemble highly proving of a mathematical problem or a theorem. In teaching standard learning material is used, but it is presented to the students also in a hypertext format browsable on the Internet.*
Key words: *Mathematics, Teaching, Didactics, Proving*

INTRODUCTION

In this paper a method of structured derivations in teaching of high school mathematics is presented and applied. Using structured derivations is seen as a possibility to restore mathematical proving back into the teaching of high school mathematics. In the paper is first analysed the importance and situation of proofs in mathematics teaching. After that the method is presented in more detail technically and in teaching. Before conclusions also the technical details of the system used to present derivations are presented.

PROOFS IN EDUCATION OF MATHEMATICS

The amount of clauses that are proved in high school mathematics has diminished from the 70' in Finnish mathematics. In Finland high-school mathematics proving is studied for the first time in geometry. After that it is used seldom, only proving some clauses in differential analysis.

Proofs are considered an important part of the mathematics teaching [5], even if their amount in teaching has diminished. Including the proofs into teaching is seen as a challenge to teachers in schools [4]. One reason is maybe that their importance to teaching is hard to point out. Students often find proofs cumbersome and additional to other mathematics. If the proofs are totally forgotten, it may "reduce the calculus to the level of the cookbook" [9]. Hanna has pointed out, that one of the main concerns when teaching proving is the easier understanding of mathematics [5]. To this problem is proposed using of the structured derivations as a way of teaching mathematics without forgetting the logical background of mathematics and introducing proofs as naturally as possible belonging to the teaching of all mathematics.

STRUCTURED DERIVATIONS

Structured derivation is based on calculational proof paradigm and it is developed originally for the formal refinement of computer programs and reasoning about their correctness. In the calculational proof paradigm, which has been attributed to Feijen and is described in detail by van Gasteren [3], the mathematical expressions are transformed step by step. In Computer Science community, this proof style, in addition to mathematical proofs, has been used for proving the correctness of programs and refining program specifications. Based on their work with program specifications, Back and von Wright have developed the idea of structured derivations [1][2]. They enlarged the basic calculational proof style with subderivations. The structured proof style is also an alternative presentation of Gentzen's natural deduction. With structured derivations, the benefits of natural deduction and calculational proof are grouped together: the hierarchy of natural deduction and the easy way of reading calculational proofs.

These structured derivations are applied as a more structured way of deriving and presenting solutions to typical problems. Every step of the derived solution to an exercise is presented with facts justifying the step. Furthermore, when solutions are very long, parts of them (subderivations) can be hidden and replaced with a hypertext link giving a more detailed view of the partial solution.

When applying this method in high-school mathematics, writing solutions to typical expression reduction problems means that a new version of an expression is written on a new line and between the lines is written a semi-formal justification for the step. With subderivations, auxiliary details can be added to the solution in an organised way. The idea of subderivations is especially suitable for presentation in hypertext format with a web browser, because the hierarchical nature of structured derivations can be shown or hidden dynamically at will. Thus, problem solving is seen as a rather formal and incremental transformation with well-defined steps from the initial problem description to a final irreducible statement yielding the solution. With this the solving of a problem defines a logical unity, where every part of the solution can be shown truthful and justified. The method has a natural and solid relation to proofs and mathematical problem solving and abstraction. The solving of a task is actually a proof.

STRUCTURED DERIVATIONS IN TEACHING

This method of teaching mathematics at high school level is being examined and tested in Kupittaa High School, in Turku (Finland), with a group of students who take their first courses in high-school mathematics. In addition to the standard mathematical background, basic propositional logic with a few notations has been taught at the beginning of the studies. In addition to the test group, another group of students is simultaneously studying the same course contents in mathematics the traditional way. Courses contain basic algebra, calculus, and some function theory and geometry. Before applying this method a basic knowledge of logical connectives, equivalence, and truth-values had to be taught to the students. Within well-defined time limits, only four or five hours could be used for teaching the foundations of logic.

The teacher uses structured derivations when she is presenting the basic theory face-to-face and writing additional examples on blackboard. The method is also applied to all the pre-made examples of the study material. Whenever the schedule allows, examples are shown using a web browser.

As an example of applying this method in the courses in detail is shown an exercise from the Finnish matriculation examination from the advanced level test in 1995. It is first presented in web browser form, where subderivations corresponding to the underlined explanations are hidden and replaced with a hypertext link giving a more detailed view of the partial solutions. After that it is shown in fully expanded form as written on blackboard. This exercise deals with a barrel shaped like a circular cylinder, with a volume of 200 litres and the length of it's lateral edge of 1.12m. There is 32cm of fuel in the barrel when it is lying on it's side. According to the exercise, a customer paid 600 finnmarks (FIM) for the barrel and the fuel together. The question then is, was that a good buy if the fuel costs 4.67 marks per litres and the price of the barrel is 100 marks. The problem solution also refers to a figure (Fig. 1).

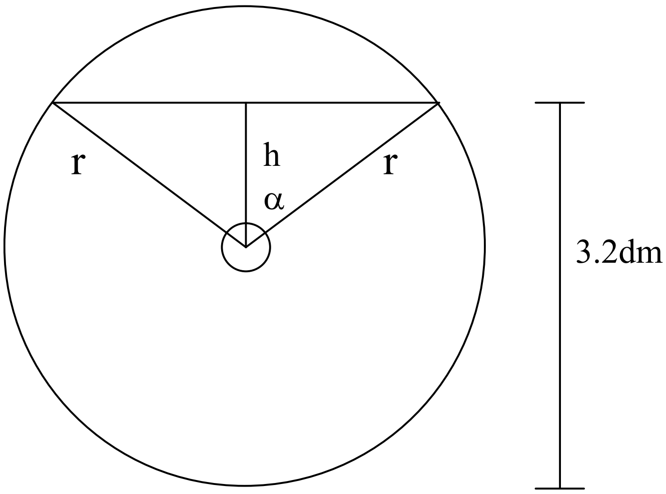

Figure 1. A picture related to the fuel exercise.

The buyer makes a good buy
≡ { set up condition }
”price of fuel and barrel each sold separately “ > FIM 600
≡ { price of barrel is FIM 100, one litre of fuel costs FIM 4.67 }
4.67 FIM/dm^3 · ”amount of fuel” + FIM 100 > FIM 600
≡ { determine amount of fuel }
4.67 FIM/ dm^3 · 142.7 dm^3+ FIM 100 > FIM 600
≡ { compute }
FIM 766.43 > FIM 600
≡ { simplify }
T

The buyer makes a good buy
≡ { set up condition }
”price of fuel and barrel each sold separately “ > FIM 600
≡ { price of barrel is FIM 100, one litre of fuel costs FIM 4.67 }
4.67 FIM/dm^3 · ”amount of fuel” + FIM 100 > FIM 600
≡ { determine amount of fuel }

- ”amount of fuel”
  = { based on the shape of the cylinder }
  ” area of larger segment” · 11.2 dm
  = { determine larger segment }
  - ” larger segment”
    = { definition of segment }
    $(360° - 2\alpha) / 360° \cdot \pi r^2$ + ”central triangle”
    = { determine r, α and central triangle }
    - r
      = { $V = \pi r^2 \cdot h$, isolate r }
      $\sqrt{V/(\pi \cdot h)}$
      = { V = 200 dm^3, h=11.2 dm }
      $\sqrt{200dm^3/(\pi \cdot 11.2dm)}$
      = { compute }
      2.3841 dm
    - α

= { Fig. 1 }
arccos ((3.2 dm – r) / r)
= { r = 2.3841 dm }
69.987°
• "central triangle"
= { $A = ½(r^2 \sin(2\alpha))$ }
$½ \cdot (2.38412\ dm^2 \cdot \sin(2 \cdot 69.987°))$
= { compute }
$1.828\ dm^2$
$(360° - 2 \cdot 69.987°) / 360° \cdot \pi \cdot 2.38412\ dm^2 + 1.828\ dm^2$
= { compute }
$12.742\ dm^2$

$12.742 dm^2 \cdot 11.2\ dm$
= { compute }
$142.7\ dm^3$

4.67 FIM / $dm^3 \cdot 142.7\ dm^3$+ FIM 100 > FIM 600
≡ { compute }
FIM 766.43 > FIM 600
≡ { simplify }
T

TECHNICAL CONSIDERATIONS

The web pages are basically constructed using standard html augmented with images for representing complex mathematical expressions and Java scripts introducing the collapsible lists feature. Every line of the derivation is written in a TeX file, which is then converted into html [7][8]. As mathematical expressions also images can be inserted with appropriate way to html-document. Structured derivations are presented as collapsible lists based on the Java scripts [6], slightly modified for enhanced appearance. Every derivation starts out as an empty list, and items are then added to the list. Every other line in the derivation is an image converted from TeX files and the other line is a simple plain text comment. When subderivations are used, a sublist is created and added to the main list. The items in the lists are html code, either comments or references to images, all written in the html file representing the solution. Having generated necessary images and written the corresponding html file, the solution can be rendered and browsed using any browser with support for Java scripts.

EVALUATION AND DISCUSSION

At first, the students were disappointed with the new method, because they felt that they were forced to write auxiliary text when solving the exercises. The use of logic, on the other hand, has not been any kind of burden, especially in solving inequalities, logic proved out to be a very effective tool.

Until now it has been attributed that when teaching with structured derivations the outlook of answers is cleared and they can be analysed in more detail and more easily. The need for explaining their decisions in their answers forces students to think the problem in its entirety. The use of logic has cleared the students' solutions and unified their structure. Also they have to think what are the methods they are using in their answers and why they use them. Also, using this method makes it easier for the students to follow the teaching process and teacher's workload is diminished with clearer answers in exams. The test has also brought up the idea that the most talented and diligent students mostly exploit the use of this method. The most important difficulties experienced have been related to the hasty course schedules; e.g., too few examples can be shown on the web. There have been also some technical problems regarding the complex process of preparing the web pages.

Applying this method in teaching indicates that to enhance the students' mathematical skills and problem solving it would be better to introduce logic into mathematics teaching much earlier and much more widely. The critical thinking of argumentation and deduction requires rehearsing and it can't be introduced to students' working rapidly. In this research it has been detected at least so far, that proving and proofs can be brought back to high school mathematics with a few notions and closely related to other mathematical topics.

To summarise, it seems rather certain at this point, that this method does not at least impair the performance of the students. Furthermore in the future also the differences in the students' other skills, e.g., in programming, could be analysed, as students from the test group will have the same programming courses as other students. Then the structure and logic of the programs written by the students from the test group could be examined in more detail compared to programming of other students.

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