Some normative properties of possibility distributions*

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Abstract

In 2001 Carlsson and Fullér introduced the possibilistic mean value, variance and covariance of fuzzy numbers. In 2002 Fullér and Majlender introduced the notations of crisp weighted possibilistic mean value, variance and covariance of fuzzy numbers, which are consistent with the extension principle. In this paper we will show some (normative) properties of possibility distributions.

1 Probability

In probability theory, the dependency between two random variables can be characterized through their joint probability density function. Namely, if X and Y are

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two random variables with probability density functions $f_X(x)$ and $f_Y(y)$, respectively, then the density function, $f_{X,Y}(x,y)$, of their joint random variable (X,Y), should satisfy the following properties

$$\int_{\mathbb{R}} f_{X,Y}(x,t)dt = f_X(x), \quad \int_{\mathbb{R}} f_{X,Y}(t,y)dt = f_Y(y), \tag{1}$$

for all $x, y \in \mathbb{R}$. Furthermore, $f_X(x)$ and $f_Y(y)$ are called the the marginal probability density functions of random variable (X,Y). X and Y are said to be independent if

$$f_{X,Y}(x,y) = f_X(x)f_Y(y),$$

holds for all x, y. The expected value of random variable X is defined as

$$E(X) = \int_{\mathbb{R}} x f_X(x)dx,$$

and if g is a function of X then the expected value of $g(X)$ can be computed as

$$E(g(X)) = \int_{\mathbb{R}} g(x)f_X(x)dx.$$

Furthermore, if h is a function of X and Y then the expected value of $h(X,Y)$ can be computed as

$$E(h(X,Y)) = \int_{\mathbb{R}^2} h(x,y)f_{X,Y}(x,y)dxdy.$$

Especially,

$$\begin{aligned} E(X+Y) = \int_{\mathbb{R}^2} (x+y)f_{X,Y}(x,y)dxdy = \int_{\mathbb{R}} x f_X(x)dx \\ + \int_{\mathbb{R}} y f_Y(y)dy = E(X) + E(Y), \end{aligned}$$

that is, the the expected value of X and Y can be determined according to their individual density functions (that are the marginal probability functions of random variable (X,Y)).

Let $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ with $a \leq b$, then the probability that X takes its value from $[a,b]$ is computed by

$$\mathrm{P}(X \in [a,b]) = \int_a^b f_X(x)dx.$$

The covariance between two random variables X and Y is defined as

$$\begin{aligned} \mathrm{Cov}(X,Y) &= E\big((X-E(X))(Y-E(Y))\big) = E(XY) - E(X)E(Y) \\ &= \int_{\mathbb{R}^2} xy f_{X,Y}(x,y)dxdy - \int_{\mathbb{R}} x f_X(x)dx \int_{\mathbb{R}} y f_Y(y)dy, \end{aligned}$$

and if X and Y are independent then $\mathrm{Cov}(X, Y) = 0$. The variance of random variable X is defined as the covariance between X and itself, that is

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = \int_{\mathbb{R}} x^2 f_X(x)dx - \left(\int_{\mathbb{R}} x f_X(x)dx\right)^2.$$

For any random variables X and Y and real numbers λ and μ the following relationship holds

$$\mathrm{Var}(\lambda X + \mu Y) = \lambda^2 \mathrm{Var}(X) + \mu^2 \mathrm{Var}(Y) + 2\lambda\mu \mathrm{Cov}(X, Y).$$

If X and Y are random variables with finite variances $\mathrm{Var}(X)$ and $\mathrm{Var}(Y)$ then the probabilistic Cauchy-Schwarz inequality can be stated as

$$[\mathrm{Cov}(X, Y)]^2 \leq \mathrm{Var}(X)\mathrm{Var}(Y),$$

where $\mathrm{Cov}(X, Y)$ denotes the covariance between X and Y. Furthermore, the correlation coefficient between X and Y is defined by

$$\rho(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}},$$

and it is clear that $-1 \leq \rho(X, Y) \leq 1$.

2 Possibility

A fuzzy set A in $\mathbb{R}$ is said to be a fuzzy number if it is normal, fuzzy convex and has an upper semi-continuous membership function of bounded support. The family of all fuzzy numbers will be denoted by $\mathcal{F}$. A γ-level set of a fuzzy set A in $\mathbb{R}^m$ is defined by $[A]^\gamma = \{x \in \mathbb{R}^m : A(x) \geq \gamma\}$ if $\gamma > 0$ and $[A]^\gamma = \mathrm{cl}\{x \in \mathbb{R}^m : A(x) > \gamma\}$ (the closure of the support of A) if $\gamma = 0$. If $A \in \mathcal{F}$ is a fuzzy number then $[A]^\gamma$ is a convex and compact subset of $\mathbb{R}$ for all $\gamma \in [0, 1]$.

Fuzzy numbers can be considered as possibility distributions. Let $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ with $a \leq b$, then the possibility that $A \in \mathcal{F}$ takes its value from $[a, b]$ is defined by [7]

$$\mathrm{Pos}(A \in [a, b]) = \max_{x \in [a,b]} A(x).$$

A fuzzy set B in $\mathbb{R}^m$ is said to be a joint possibility distribution of fuzzy numbers $A_i \in \mathcal{F}$, $i = 1, \dots, m$, if it satisfies the relationship

$$\max_{x_j \in \mathbb{R},\ j \neq i} B(x_1, \dots, x_m) = A_i(x_i),\ \forall x_i \in \mathbb{R}, i = 1, \dots, m.$$

Furthermore, A_i is called the i-th marginal possibility distribution of B, and the projection of B on the i-th axis is A_i for $i = 1, \dots, m$. We emphasise here that the joint possibility distribution always uniquely defines its marginal distributions (the shadow of B on the i-th axis is exactly A_i), but not vice versa.

Let B denote a joint possibility distribution of $A_1, A_2 \in \mathcal{F}$. Then B should satisfy the relationships

$$\max_y B(x_1, y) = A_1(x_1), \quad \max_y B(y, x_2) = A_2(x_2), \ \forall x_1, x_2 \in \mathbb{R}.$$

If $A_i \in \mathcal{F}$, $i = 1, \dots, m$, and B is their joint possibility distribution then the relationships $B(x_1, \dots, x_m) \le \min\{A_1(x_1), \dots, A_m(x_m)\}$ and $[B]^\gamma \subseteq [A_1]^\gamma \times \cdots \times [A_m]^\gamma$, hold for all $x_1, \dots, x_m \in \mathbb{R}$ and $\gamma \in [0, 1]$.

For $m = 2$ then any γ-level set of $[B]^\gamma$ should be contained by the rectangle determined by the Cartesian product of the γ-level sets of marginal distributions $[A_1]^\gamma \times [A_2]^\gamma$, and it should reach each side of that rectangle.

In the following the biggest (in the sense of subsethood of fuzzy sets) joint possibility distribution will play a special role among joint possibility distributions: it defines the concept of independence of fuzzy numbers.

Definition 2.1. *Fuzzy numbers $A_i \in \mathcal{F}$, $i = 1, \dots, m$, are said to be independent if their joint possibility distribution, B, is given by*

$$B(x_1, \dots, x_m) = \min\{A_1(x_1), \dots, A_m(x_m)\},$$

or, equivalently, $[B]^\gamma = [A_1]^\gamma \times \cdots \times [A_m]^\gamma$, for all $x_1, \dots, x_m \in \mathbb{R}$ and $\gamma \in [0, 1]$.

Marginal probability distributions are determined from the joint one by the principle of ’falling integrals’ and marginal possibility distributions are determined from the joint possibility distribution by the principle of ’falling shadows’.

Let $A \in \mathcal{F}$ be fuzzy number with $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$, $\gamma \in [0, 1]$. A function $f\colon [0, 1] \to \mathbb{R}$ is said to be a weighting function [4] if f is non-negative, monotone increasing and satisfies the following normalization condition

$$\int_0^1 f(\gamma) d\gamma = 1. \qquad (2)$$

In [4] the f-weighted possibilistic mean (or expected) value of fuzzy number A was defined as

$$E_f(A) = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} f(\gamma) d\gamma. \qquad (3)$$

It should be noted that if $f(\gamma) = 2\gamma$, $\gamma \in [0, 1]$ then

$$E_f(A) = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} 2\gamma d\gamma = \int_0^1 [a_1(\gamma) + a_2(\gamma)] \gamma d\gamma.$$

That is the f-weighted possibilistic mean value defined by (3) can be considered as a generalization of possibilistic mean value introduced in [1]. From the definition of a weighting function it can be seen that $f(\gamma)$ might be zero for certain (unimportant) γ-level sets of A. So by introducing different weighting functions we can give different (case-dependent) importances to γ-levels sets of fuzzy numbers.

Example 1. *Let $A = (a, b, \alpha, \beta)$ be a fuzzy number of trapezoidal form with peak $[a, b]$, left-width $\alpha > 0$ and right-width $\beta > 0$, and let $f(\gamma) = (n+1)\gamma^n$, $n \geq 0$. The γ-level of A is computed by $[A]^\gamma = [a - (1-\gamma)\alpha, b + (1-\gamma)\beta]$, $\forall \gamma \in [0, 1]$. Then the weighted possibilistic mean values of A are computed by*

$$E_f(A) = \frac{1}{2}\left(a - \frac{\alpha}{n+2} + b + \frac{\beta}{n+2}\right) = \frac{a+b}{2} + \frac{\beta - \alpha}{2(n+2)}.$$

So,

$$\lim_{n\to\infty} E_f(A) = \lim_{n\to\infty}\left(\frac{a+b}{2} + \frac{\beta-\alpha}{2(n+2)}\right) = \frac{a+b}{2}.$$

Let A and B be fuzzy numbers and let f be a weighting function. In [4] the f-weighted possibilistic variance of A was defined by

$$\mathrm{Var}_f(A) = \int_0^1 \left(\frac{a_2(\gamma) - a_1(\gamma)}{2}\right)^2 f(\gamma) d\gamma, \tag{4}$$

and the f-weighted covariance of A and B is defined as

$$\mathrm{Cov}_f(A, B) = \int_0^1 \frac{a_2(\gamma) - a_1(\gamma)}{2} \cdot \frac{b_2(\gamma) - b_1(\gamma)}{2} f(\gamma) d\gamma. \tag{5}$$

It should be noted that if $f(\gamma) = 2\gamma$, $\gamma \in [0, 1]$ then

$$\begin{aligned}\mathrm{Var}_f(A) &= \int_0^1 \left(\frac{a_2(\gamma) - a_1(\gamma)}{2}\right)^2 2\gamma d\gamma \\ &= \frac{1}{2}\int_0^1 [a_2(\gamma) - a_1(\gamma)]^2 \gamma d\gamma = \mathrm{Var}(A),\end{aligned}$$

and

$$\begin{aligned}\mathrm{Cov}_f(A, B) &= \int_0^1 \frac{a_2(\gamma) - a_1(\gamma)}{2} \cdot \frac{b_2(\gamma) - b_1(\gamma)}{2} f(\gamma) d\gamma \\ &= \frac{1}{2}\int_0^1 (a_2(\gamma) - a_1(\gamma)) \cdot (b_2(\gamma) - b_1(\gamma))\, 2\gamma d\gamma = \mathrm{Cov}(A, B).\end{aligned}$$

Where $\mathrm{Var}(A)$ and $\mathrm{Cov}(A, B)$ denote the possibilistic variance and covariance introduced by Carlsson and Fullér in [1]. That is the f-weighted possibilistic variance and covariance defined by (4) and (5) can be considered as a generalization of It can easily be verified that the weighted covariance is a symmetrical bilinear operator.

Example 2. *Let $A = (a, b, \alpha, \beta)$ be a trapezoidal fuzzy number and let $f(\gamma) = (n+1)\gamma^n$ be a weighting function. Then,*

$$\begin{aligned}\mathrm{Var}_f(A) &= (n+1)\int_0^1 \left[\frac{a_2(\gamma) - a_1(\gamma)}{2}\right]^2 \gamma^n d\gamma \\ &= \left[\frac{b-a}{2} + \frac{\alpha+\beta}{2(n+2)}\right]^2 + \frac{(n+1)(\alpha+\beta)^2}{4(n+2)^2(n+3)}.\end{aligned}$$

So,

$$\lim_{n\to\infty} \mathrm{Var}_f(A) = \frac{b-a}{2}.$$

The following theorem shows that the variance of linear combinations of fuzzy numbers can easily be computed (in a similar manner as in probability theory).

Theorem 2.1. *[4] Let f be a weighting function, let A and B be fuzzy numbers and let λ and μ be real numbers. Then the following properties hold,*

$$\mathrm{Var}_f(\lambda A + \mu B) = \lambda^2 \mathrm{Var}_f(A) + \mu^2 \mathrm{Var}_f(B) + 2|\lambda||\mu|\mathrm{Cov}_f(A, B).$$

Example 3. *Let $A = (a, b, \alpha, \beta)$ and $B = (a', b', \alpha', \beta')$ be fuzzy numbers of trapezoidal form. Let $f(\gamma) = (n+1)\gamma^n$, $n \geq 0$, be a weighting function then the power-weighted covariance between A and B is computed by*

$$\begin{aligned}\mathrm{Cov}_f(A, B) = &\left[\frac{b-a}{2} + \frac{\alpha+\beta}{2(n+2)}\right]\left[\frac{b'-a'}{2} + \frac{\alpha'+\beta'}{2(n+2)}\right] \\ &+ \frac{(n+1)(\alpha+\beta)(\alpha'+\beta')}{4(n+2)^2(n+3)}.\end{aligned}$$

So,

$$\lim_{n\to\infty} \mathrm{Cov}_f(A, B) = \frac{b-a}{2} \cdot \frac{b'-a'}{2}.$$

If $a = b$ and $a' = b'$, i.e. we have two triangular fuzzy numbers, then their covariance becomes

$$\mathrm{Cov}_f(A, B) = \frac{(\alpha+\beta)(\alpha'+\beta')}{2(n+2)(n+3)}.$$

3 On possibilistic dependencies

The main drawback of definition (5) is that $\mathrm{Cov}_f(A, B) \geq 0$ for any pair of fuzzy numbers. However, in probability theory the covariance can definitely be negative. To overcome this difficulty we introduced the definition of central value of a fuzzy number and a dependency relation between γ-level sets of fuzzy numbers via their joint possibility distributions in [5] as follows

Definition 3.1. *[5] Let $A \in \mathcal{F}$ be a fuzzy number with $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$, $\gamma \in [0, 1]$. The central value of $[A]^\gamma$ is defined by*

$$\mathcal{C}([A]^\gamma) = \frac{1}{\int_{[A]^\gamma} dx} \int_{[A]^\gamma} x dx.$$

It is easy to see that the central value of $[A]^\gamma$ is computed as

$$\mathcal{C}([A]^\gamma) = \frac{1}{a_2(\gamma) - a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} x dx = \frac{a_1(\gamma) + a_2(\gamma)}{2}.$$

Definition 3.2. *Let $A_1, \ldots, A_n \in \mathcal{F}$ be fuzzy numbers, and let $g : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Then, $g(A_1, \ldots, A_n)$ is defined by the sup–min extension principle [6] as follows*

$$g(A_1, \ldots, A_n)(y) = \sup_{g(x_1,\ldots,x_n)=y} \min\{A_1(x_1), \ldots, A_n(x_n)\}.$$

Definition 3.3. *[5] Let $A_1, \ldots, A_n \in \mathcal{F}$ be fuzzy numbers, let B be their joint possibility distribution and let $\gamma \in [0, 1]$. The central value of the γ-level set of $g(A_1, \ldots, A_n)$ with respect to their joint possibility distribution B is defined by*

$$\mathcal{C}_B\left([g(A_1, \ldots, A_n)]^\gamma\right) = \frac{1}{\int_{[B]^\gamma} dx} \int_{[B]^\gamma} g(x) dx,$$

where $g(x) = g(x_1, \ldots, x_n)$.

In [5] we proved that the central value operator is linear.

Theorem 3.1. *[5] Let $A, B \in \mathcal{F}$ be fuzzy numbers, let C be their joint possibility distribution and let $\gamma \in [0, 1]$. Then*

$$\mathcal{C}_C([A + B]^\gamma) = \mathcal{C}_C([A]^\gamma) + \mathcal{C}_C([B]^\gamma).$$

Definition 3.4. *[5] Let $A, B \in \mathcal{F}$ be fuzzy numbers, let C be their joint possibility distribution, and let $\gamma \in [0, 1]$. The dependency relation between the γ-level sets of A and B is defined by*

$$\mathrm{Rel}_C([A]^\gamma, [B]^\gamma) = \mathcal{C}_C\big(\big[(A - \mathcal{C}_C([A]^\gamma))(B - \mathcal{C}_C([B]^\gamma))\big]^\gamma\big),$$

which can be written in the form,

$$\mathrm{Rel}_C([A]^\gamma,[B]^\gamma) =$$
$$\frac{1}{\int_{[C]^\gamma} dxdy}\int_{[C]^\gamma} xydxdy - \frac{1}{\int_{[C]^\gamma} dx}\int_{[C]^\gamma} xdx \times \frac{1}{\int_{[C]^\gamma} dy}\int_{[C]^\gamma} ydy.$$

The covariance of A and B with respect to a weighting function f is defined as [5]

$$\begin{aligned}\mathrm{Cov}_f(A,B) &= \int_0^1 \mathrm{Rel}_C([A]^\gamma,[B]^\gamma)f(\gamma)d\gamma\\ &= \int_0^1 \big[\mathcal{C}_C([AB]^\gamma) - \mathcal{C}_C([A]^\gamma)\cdot\mathcal{C}_C([B]^\gamma)\big]f(\gamma)d\gamma.\end{aligned}$$

In [5] we proved that if $A,B\in\mathcal{F}$ are independent then $\mathrm{Cov}_f(A,B)=0$. The variance of a fuzzy number A is defined as [5]

$$\mathrm{Var}_f(A) = \mathrm{Cov}_f(A,A) = \int_0^1 \frac{(a_2(\gamma)-a_1(\gamma))^2}{12}f(\gamma)d\gamma.$$

In [5] we proved that the that the 'principle of central values' leads us to the same relationships in possibilistic environment as in probabilitic one. It is why we can claim that the principle of 'central values' should play an important role in defining possibilistic dependencies.

Theorem 3.2. *[5] Let A, B and C be fuzzy numbers, and let $\lambda,\mu\in\mathbb{R}$. Then*

$$\mathrm{Cov}_f(\lambda A+\mu B, C) = \lambda\mathrm{Cov}_f(A,C) + \mu\mathrm{Cov}_f(B,C),$$

where all terms in this equation are defined through joint possibility distributions.

Theorem 3.3. *[5] Let A and B be fuzzy numbers, and let $\lambda,\mu\in\mathbb{R}$. Then*

$$\mathrm{Var}_f(\lambda A+\mu B) = \lambda^2\mathrm{Var}_f(A) + \mu^2\mathrm{Var}_f(B) + 2\lambda\mu\mathrm{Cov}_f(A,B).$$

and if A and B are independent then $\mathrm{Var}(A+B)=\mathrm{Var}(A)+\mathrm{Var}(B)$.

Furthermore, in [2] we have shown the following theorem.

Theorem 3.4. *Let $A,B\in\mathcal{F}$ be fuzzy numbers (with $\mathrm{Var}_f(A)\neq 0$ and $\mathrm{Var}_f(B)\neq 0$) with joint possibility distribution C. Then, the correlation coefficient between A and B, defined by*

$$\rho_f(A,B) = \frac{\mathrm{Cov}_f(A,B)}{\sqrt{\mathrm{Var}_f(A)\mathrm{Var}_f(B)}}.$$

satisfies the property

$$-1\leq\rho_f(A,B)\leq 1.$$

for any weighting function f.

Let us consider three interesting cases. In [5] we proved that if A and B are independent, that is, their joint possibility distribution is $A \times B$ then $\rho_f(A, B) = 0$.

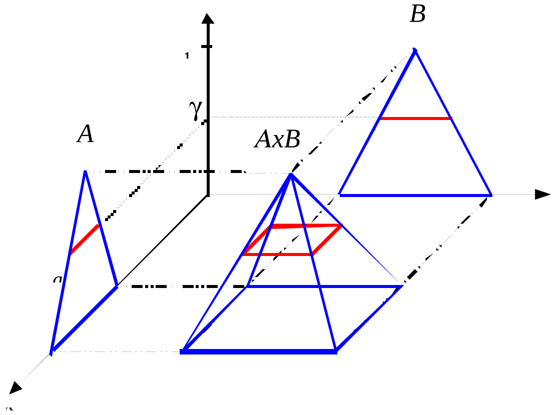

Figure 1: Independent possibility distributions.

Consider now the case depicted in Fig. 2. It can be shown [2] that in this case $\rho_f(A, B) = 1$.

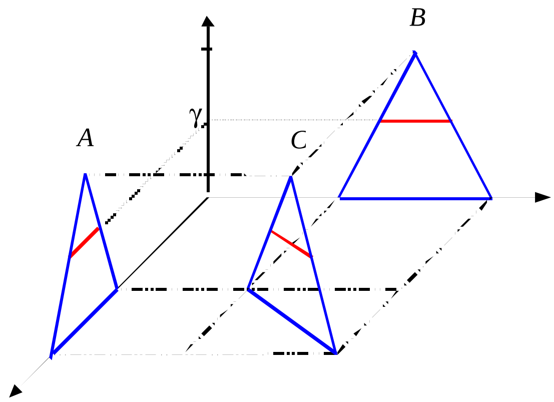

Figure 2: The case of $\rho_f(A, B) = 1$.

Consider now the case depicted in Fig. 3. It can be shown [2] that in this case

$$\rho_f(A,B) = \frac{\mathrm{Cov}_f(A,B)}{\sqrt{\mathrm{Var}_f(A)\mathrm{Var}_f(B)}}$$

$$= -\frac{\int_0^1 \frac{(a_2(\gamma)-a_1(\gamma))(b_2(\gamma)-b_1(\gamma))}{12} f(\gamma)d\gamma}{\sqrt{\int_0^1 \frac{(a_2(\gamma)-a_1(\gamma))^2}{12} f(\gamma)d\gamma}\sqrt{\int_0^1 \frac{(b_2(\gamma)-b_1(\gamma))^2}{12} f(\gamma)d\gamma}}$$

$$= -1.$$

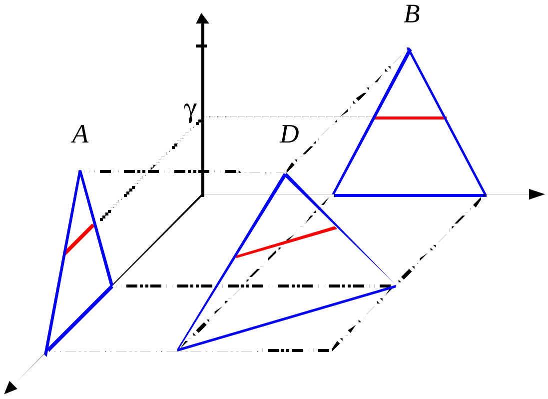

Figure 3: The case of $\rho_f(A,B) = -1$.

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