Fault-tolerant Wavelength Allocations in All-optical Hypercubes

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Abstract

Motivated by wavelength division multiplexing in all-optical networks, we consider the problem of finding an optimal (with respect to the least possible number of wavelengths) set of $f+1$ internally node disjoint dipaths for every pair of distinct nodes in the binary r-dimensional hypercube, where $0 \leq f < r$. This system of dipaths serves as a routing protocol that remains functional in the presence of up to f faults (nodes and/or links) in the hypercube. The importance of such protocols follows from the fact that the major applications for all-optical networks require connections that last for relatively long periods once set up.

In this model, we give nearly optimal All-to-All protocols, in terms of number of wavelengths, for the hypercube network.

Keywords
Optical Networks, Routings, Hypercube, Fault Tolerance.

1 Introduction

Motivation. Optical networking is a very quickly developing new area of research, promising very high speed local or wide area networks in the future. Optics is a key technology in communication networks and it is expected to dominate important applications such as video conferencing, scientific visualization, real-time medical imaging, high speed supercomputing, and distributed computing. A single optical wavelength supports rates of gigabits-per-second. Multiple laser beams that are propagated over the same fiber on distinct optical wavelengths can increase this capacity much further. This is achieved through WDM (Wavelength

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Division Multiplexing) [5], by partitioning the optical bandwidth into a large numbers of channels. It allows multiple data streams to be transferred concurrently (on different wavelengths) along the same optical fiber.

All-optical [8] communication networks are switched networks with reconfigurable wavelength selective optical switches, without wavelength converters, where different signals may travel on the same communication link (but on different wavelengths) into a node, and then exit from it on different links, keeping their original wavelengths. We refer the reader to [2] for an account of the theoretical problems and results obtained for this *all-optical model*. These systems provide all source-destination pairs with end-to-end transparent channels that are identified through a wavelength and a physical path. Maintaining the signal in optical form allows for high data transmission rates in these networks since there is no prohibitive overhead due to conversions to and from the electronic form.

The actual process of setting up switches and routes, and of assigning wavelengths, is done using an electronic backbone control network. A solution consists of settings for the switches in the network (establishing a routing), and an assignment of wavelengths to the requests, so that there is a directed path between the nodes of each request. Moreover, no directed link will carry two different signals on the same wavelength. One may wonder at the use of a relatively slow electronic network to set-up these high-speed connections. In fact, the major applications for such networks require connections that last for relatively long periods once set-up.

Contribution of this work. The cost and feasibility of switching and amplification devices depend on the number of wavelengths they handle. It is evident that the number of wavelengths (optical bandwidth) is a costly source and thus a limiting factor. In addition, in a network where links and/or nodes may fail it is important to establish a routing for a required communication pattern, which guarantees fault-free transmission. The construction has to take into account the available capacity of the optical bandwidth. Hence it is important to construct fault-tolerant routings requiring the least possible number of wavelengths.

In this paper we concentrate on one of the dominating topological structures for parallel/distributed systems - the binary hypercube. The structure has been used in many experimental and commercial machines including NCUBE-2, Intel iPSC, and Connection Machines. We assume that up to f links/nodes may fail in the system, and obtain nearly optimal (with respect to the number of wavelengths) All-to-All f-fault tolerant protocols for the r-dimensional binary hypercube for any f, $0 \leq f < r$. Note that the problem for $f = 0$ was solved in [1, 3, 7]. For related work concerning 0-fault tolerant protocols in all-optical networks we refer the reader to a survey paper [6].

2 Model, Notation, and Results

Our model for optical networks and the wavelength routing problem is adopted from [2].

The Optical Network Model. It has become apparent that the optical amplifiers placed on the fiber will be directed devices. Hence, an all-optical network is modeled as a symmetric directed graph $G = (V_G, A_G)$, where V_G is a set of vertices, and A_G is a set of arcs such that if $\langle u, v\rangle \in A_G$, then $\langle v, u\rangle \in A_G$. Let $P(x,y)$ denote a dipath (directed path) in G from the vertex x to the vertex y, i.e. a sequence of vertices $v_1, \ldots, v_{k+1}$ such that $\langle v_i, v_{i+1}\rangle \in A_G$ is an arc for $1 \le i \le k$, $v_1 = x$ and $v_{k+1} = y$. The number k is the length of the dipath $P(x,y)$. Let $dist(x,y)$ denote the *distance* of x and y in G, that is the length of a shortest dipath $P(x,y)$. If $P(x,y)$ and $P(y,z)$ are two dipaths such that $P(x,y) \cap P(y,z) = \{y\}$, then the *composition* $P(x,y) \cdot P(y,z)$ is the dipath $P(x,y')$ followed by $P(y,z)$, where y' is the last but one vertex of $P(x,y)$.

Wavelength Routing Problem. A *request* in G is an ordered pair of vertices $[x,y]$ in G corresponding to a message to be send from x to y. An *instance* I in G is a set of requests in G. We denote the fact that I is subject to G by (G,I). An f*-wise routing* for an (G,I) is a set of internally vertex-disjoint dipaths $R = \{P_0(x,y), P_1(x,y), \ldots, P_f(x,y) \ : \ [x,y] \in I,\ P_i(x,y) \cap P_j(x,y) = \{x,y\} \text{ for } i \neq j\}$. Let us note that if $f = 0$, then the 0-wise routing for an (G,I) is simply referred to as the routing for (G,I).

- Let G be a symmetric digraph and I an instance in G. The f*-wise routing problem* for (G,I) consists of finding an f-wise routing R for (G,I).
- Let G be a symmetric digraph, I an instance in G, and R an f-wise routing for (G,I). The *wavelength assignment problem* for (G,I,R) consists of finding an assignment of a wavelength to each dipath from R, so that no two dipaths of R sharing an arc are assigned the same wavelength.
- Let G be a symmetric digraph and I an instance in G. The f*-wise wavelength routing problem* for (G,I) consists of finding an f-wise routing R for (G,I) (i.e., solving the f-wise routing problem for (G,I)) and of solving the wavelength assignment problem for (G,I,R).

It is convenient to think of wavelengths as colors. For a given f-wise routing R for an (G,I), the smallest number of colors that are sufficient to solve the wavelength assignment problem for (G,I,R) is denoted by $\vec{\omega}(G,I,R)$. Let $\vec{\omega}(G,I,f)$ denote the smallest $\vec{\omega}(G,I,R)$ over all f-wise routings R for (G,I). If $f = 0$, we abbreviate $\vec{\omega}(G,I,0)$ to $\vec{\omega}(G,I)$.

The Fault Tolerant Model. We consider the model in which arbitrary set of vertices and/or arcs may fail. A routing for (G,I) is said to be f*-fault tolerant* if it can remain functional in the presence of at most f faulty vertices and/or arcs in G. A routing is *functional* as long as it contains a non-faulty communication dipath between each pair of non-faulty vertices from the instance I.

Observation 1 *For any symmetric digraph G, any instance I in G, and any $f \geq 0$, any f-wise routing for (G,I) is f-fault tolerant for (G,I).*

According to the above observation, in order to construct f-fault tolerant routing protocols it is enough to describe f-wise routings.

Special Instances.

- The *All-to-All* instance is $I_A = \{[x,y] \,:\, x,y \in V_G,\ x \neq y\}$.
- The *All-to-All-d* instance is $I_A^d = \{[x,y] \,:\, x,y \in V_G,\ dist(x,y) = d\}$.

Routing Parameters. Let G be a symmetric digraph, I an instance in G, and R an f-wise routing for (G,I). The *arc-congestion* $\vec{\pi}(G,I,R,\alpha)$ of an arc $\alpha \in A_G$ in the routing R is the number of dipaths of R containing α. The *arc-congestion* of G in the routing R for the instance I is the maximum congestion of any arc of G in the routing R and is denoted by $\vec{\pi}(G,I,R)$. Let $\vec{\pi}(G,I,f)$ be the minimum congestion of G in any f-wise routing R for the instance I. If $f = 0$, we abbreviate the above notation to $\vec{\pi}(G,I)$. For the All-to-All instance I_A, $\vec{\pi}(G,I_A)$ is called the *arc-forwarding index* of G and $\vec{\omega}(G,I_A)$ is called the *optical index* of G. Following these lines we call $\vec{\pi}(G,I_A,f)$ the *f-wise arc-forwarding index* of G and $\vec{\omega}(G,I_A,f)$ the *f-wise optical index* of G.

The following observation introduces the relationship between the f-wise wavelength routing problem and the f-wise routing problem.

Observation 2 *For any symmetric digraph G, any instance I in G, and any $f \geq 0$, $\vec{\pi}(G,I,f) \leq \vec{\omega}(G,I,f)$.*

It is natural to ask more precisely about the general relation between $\vec{\pi}(G,I,f)$ and $\vec{\omega}(G,I,f)$, especially the question in which cases equality can be achieved in Observation 2, or in which cases one can at least come close to equality. There has been an ongoing effort in this direction for 0-wise routings on many networks. Especially, One-to-Many and All-to-All instances have been thoroughly investigated in the literature.

The Directed Hypercube. Let H_r denote the r-dimensional binary hypercube in which each edge is replaced by two reversely directed arcs. Let us note that the vertices of H_r are all binary-strings of length r and an ordered pair of vertices, say $u = u_1u_2\ldots u_r$ and $v = v_1v_2\ldots v_r$, is joined by an arc $\langle u,v\rangle$ if, and only if, the corresponding strings differ in precisely one position, say $1 \leq i \leq r$. Subsequently, we say that u and v differ in the *ith dimension*, that the corresponding arc is the *ith dimension arc*, and we will refer to v as $u(i)$. Let symbol $\prec$ denote the lexicographical order on the hypercube vertices. Define $\sigma(x,y): I_A \to \{\prec,\succ\}$ as $\sigma(x,y) = \prec$, iff $x \prec y$, and $\sigma(x,y) = \succ$, otherwise.

2.1 Our Results

First, we study the f-wise arc-forwarding index of the r-dimensional directed hypercube. For any integer f, $0 \le f < r$, we prove that

$$\vec{\pi}(H_r, I_A, f) = (f+1)2^{r-1} + \left\lceil \frac{2}{r} \sum_{i=1}^{f} (f-i+1)\binom{r}{i} \right\rceil .$$

The above result is obtained by describing an f-wise routing $\mathcal{R}_f$ for (H_r, I_A) whose congestion is equal to $\vec{\pi}(H_r, I_A, f)$.

Second, we solve the wavelength assignment problem for $(H_r, I_A, \mathcal{R}_f)$ which gives us an upper-bound for $\vec{\omega}(H_r, I_A, f)$. The bound is nearly optimal. In particular, we prove that for any $0 \le f < \frac{r}{2}$,

$$\vec{\omega}(H_r, I_A, f) \le (f+1)2^{r-1} + (f \ne 0) \cdot \binom{r}{f} + \sum_{i=1}^{f} (f-i+1)\binom{r}{i}$$

$$\le \begin{cases} (f+1)2^{r-1} + \left(\frac{r}{4} + \frac{1}{2}\right) 2^{r-1} \le 2\vec{\pi}(H_r, I_A, f) \text{ for } \frac{r}{4} \le f < \frac{r}{2}, \\ (f+1)2^{r-1} + f.2^{\frac{7}{8}r - \frac{3}{2}} \le \left(1 + 2^{-\frac{1}{8}r}\right) \vec{\pi}(H_r, I_A, f) \le 2\vec{\pi}(H_r, I_A, f) \text{ for } f < \frac{r}{4}, \end{cases}$$

and for any $\frac{r}{2} \le f < r$,

$$\begin{aligned} \vec{\omega}(H_r, I_A, f) &\le \left(2f + 2 - \frac{r}{2}\right) 2^{r-1} - (f+1) + \frac{1}{2} \left\lceil \frac{r}{2} \right\rceil \binom{r}{\lceil \frac{r}{2} \rceil} + \binom{r}{f} \\ &\le \left(2f - \frac{r}{4} + \frac{13}{4}\right) 2^{r-1} \le 2\vec{\pi}(H_r, I_A, f), \end{aligned}$$

where the last inequality comes from the bound $\vec{\pi}(H_r, I_A, f) \ge \left(f + \frac{3}{2}\right) 2^{r-1}$, which holds for $f \ge \frac{r}{2}$.

3 The f-Wise Routing for (H_r, I_A)

Since the f-wise property of a routing for the instance I_A is implicitly related to the $(f+1)$-vertex/arc connectivity, and since H_r is r-vertex/arc connected, we may assume w.l.o.g., that $0 \le f < r$. We begin by giving a lower bound to the f-wise arc-forwarding index of H_r.

Binomial Identity. We use the following basic identity which can be found, for example, in [4].

$$\binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1}, k \neq 0 \tag{1}$$

Lemma 1 *For any f, $0 \leq f < r$, it holds that*

$$(f+1)2^{r-1} + \left\lceil \frac{2}{r}\sum_{i=1}^{f}(f-i+1)\binom{r}{i} \right\rceil \leq \vec{\pi}(H_r, I_A, f).$$

PROOF. We first lower-bound the total congestion count in any f-wise layout for (H_r, I_A).

In any f-wise layout there are $f+1$ internally vertex disjoint dipaths joining any ordered pair of vertices. Consider any two vertices in H_r of distance d, $1 \leq d \leq r$. The total contribution of the corresponding $f+1$ dipaths to the congestion count is at least $d^2 + (f-d+1)(d+2)$ if $d \leq f$, since there are at most d internally vertex disjoint dipaths joining any pair of vertices of distance d and all other internally vertex disjoint dipaths joining these vertices have length at least $d+2$. Similarly, if $d > f$, then the total contribution to the congestion count is at least $(f+1)d$. Since there are $2^r\binom{r}{d}$ distinct ordered pairs of vertices of distance d in H_r, the total contribution through all these pairs is at least $2^r\binom{r}{d}((d+2)f-d+2)$ if $d \leq f$, and is at least $2^r\binom{r}{d}(f+1)d$ if $d > f$.

Hence the total congestion count is at least

$$\begin{aligned} &\sum_{i=1}^{f} 2^r\binom{r}{i}((i+2)f-i+2) + \sum_{i=f+1}^{r} 2^r\binom{r}{i}(f+1)i = \\ \overset{\text{by (1)}}{=}\ &(f+1)r2^{2r-1} + 2^{r+1}\sum_{i=1}^{f}(f-i+1)\binom{r}{i} = L \end{aligned}$$

Since there are $r2^r$ arcs in H_r, the average congestion of any f-wise layout for (H_r, I_A) is at least $L/(r2^r)$. This means that

$$(f+1)2^{r-1} + \left\lceil \frac{2}{r}\sum_{i=1}^{f}(f-i+1)\binom{r}{i} \right\rceil \leq \vec{\pi}(H_r, I_A, f).$$

¶

In what follows we construct an f-wise routing for (H_r, I_A) that achieves the above lower bound.

Characteristic. For any pair of distinct vertices u and v of H_r, define the *characteristics* $C = (C_1, C_2, \ldots, C_d)$ with $C_1 < C_2 < \cdots < C_d$ as the ordered list of all dimensions in which u and v differ. Let $\mathbf{C}$ denote the set of all characteristics in H_r, and $\mathbf{C}_d$ the set of all characteristics in H_r with d elements. Obviously, $|\mathbf{C}| = 2^r - 1$, and $|\mathbf{C}_d| = \binom{r}{d}$. For convenience, let $\mathbf{C}_{a,b} = \cup_{d=a}^{b} \mathbf{C}_d$.

Regular r, f-System of Functions. Consider integers r and f such that $f < r$. The *r, f-system of functions* $t_0, t_1, \ldots, t_{f-1}$ is a system of functions, where $t_i : \mathbf{C}_{1,f-i} \times \{\prec, \succ\} \to \{1, \ldots, r\}$, and for each characteristic $C \in \mathbf{C}_d$ and each *orientation* $\phi \in \{\prec, \succ\}$, the elements of C and elements $t_0(C, \phi), \ldots, t_{f-d}(C, \phi)$ are all distinct.

Let $\{s_i\}_{i=0}^{p}$ be an arbitrary system of functions with $s_i : D_i \to \{1, \ldots, r\}$. The *distribution* $e = (e_1, \ldots, e_r)$ of $\{s_i\}_{i=0}^{p}$ is a vector of integers, where

$$e_j = |\{(x,i) \ : \ x \in D_i, \ i = 0, \ldots, p, \ s_i(x) = j\}|$$

for all $j = 1, \ldots, r$. Numbers e_j represent how many times functions $s_0, \ldots, s_p$ return value j. We say that a system of functions is *regular*, if $\max_{j=1}^{r} e_j - \min_{j=1}^{r} e_j \leq 1$ and that a system of functions has an *(a,b)-cyclic distribution*, iff $e_a = e_{a+1} = \cdots = e_{b-1} = \alpha + 1$ and $e_b = e_{b+1} = \cdots = e_{a-1} = \alpha$ for some $\alpha \geq 0$. Note, that all operations with indices are done modulo r. In what follows we prove that for any $f < r$ there is a regular r, f-system of functions.

Configuration. Let ρ be the *cyclic shift* on the set $\{1, \ldots, r\}$, i.e. $\rho(i) = i + 1$ for $i < r$ and $\rho(r) = 1$. Clearly, the map ρ can be extended to the set of all characteristics $\mathbf{C}$. Let us define the equivalence relation on $\mathbf{C}$: $C_1 \sim C_2$ iff there exists k such that $C_2 = \rho^k(C_1)$, where $C_1, C_2 \in \mathbf{C}$. This relation factorizes the set $\mathbf{C}$ (resp. $\mathbf{C}_d$) to the set of equivalence classes $\hat{\mathbf{C}}$ (resp. $\hat{\mathbf{C}}_d$). The equivalence classes are called *configurations*. One can imagine each configuration $\hat{C} \in \hat{\mathbf{C}}$ as a ring with r beads colored by two colors, say black and white, see Figure 1.

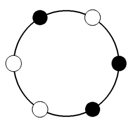

Fig. 1

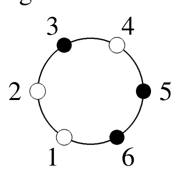

Fig. 2

If we number beads clockwise starting from an arbitrary bead, then the black beads represent a characteristic. For example, in Figure 2 the characteristic $(3,5,6)$ is depicted, which belongs to the equivalence class

$$\hat{C} = \{(1,3,4), (2,4,5), (3,5,6), (1,4,6), (1,2,5), (2,3,6)\}.$$

If we get the same configuration by rotating the ring representing configuration $\hat{C}$ by m beads, we say that $\hat{C}$ has *symmetry* m. Further, a configuration $\hat{C}$ is *m-symmetric*, if $m > 0$ is the smallest symmetry it has. It is easy to observe that any m-symmetric configuration contains exactly m characteristics and that $m \mid r$.

Lemma 2 *For any $0 \leq f < r$ there exists a regular r,f-system of functions, which can be constructed in the linear time.*

PROOF. We construct an r,f-system of functions that has $(0,b)$-cyclic distribution for some b. Such a system is obviously regular.

Take an arbitrary $j_0 \in \{1,\ldots,r\}$, an m-symmetric configuration $\hat{C} \in \hat{\mathbf{C}}_d$, $1 \leq d < r$, and $\phi \in \{\prec,\succ\}$. We will show how to define functions $\{t_i\}_{i=0}^{f-d}$ on $\hat{C} \times \{\phi\}$, such that they have (j_0, j_0')-cyclic distribution for some j_0'. Obviously, there exists a characteristic $C_0 \in \hat{C}$ such that $j_0 \notin C_0$. Let $C_k = \rho^k(C_0)$ for $k = 1,\ldots,m-1$; so $\hat{C} = \{C_0,\ldots,C_{m-1}\}$.

Consider the numbered ring representing C_0. The ring has $r-d$ white beads. Since the ring is m-symmetric, each part of the ring with m beads has the same number of white beads. We divide the ring into $\frac{r}{m}$ such disjoint parts, and so each part consists of exactly $w := \frac{m}{r}(r-d)$ white beads. Consider the part which contains j_0 and let $j_1,\ldots,j_{w-1} \in \{1,\ldots,r\}$ be the corresponding numbers of other white beads in this part. Hence $j_0, j_1,\ldots,j_{w-1}$ do not belong to C_0, and are distinct modulo m. Let $j_w = j_0$.
Let $f-d+1 = p \cdot \frac{r}{m} + q$, where $0 \leq q < \frac{r}{m}$, so

$$p = \left\lfloor \frac{m}{r}(f-d+1) \right\rfloor \leq \frac{m}{r}(f-d+1) \leq w,$$

where $p = w$ implies $q = 0$. We define values of functions $t_0,\ldots,t_{f-d}$ on the set $\hat{C} \times \{\phi\} \subseteq \mathbf{C}_d \times \{\phi\}$ as follows:

$$t_{i+(l-1)\cdot\frac{r}{m}}(C_k,\phi) \equiv j_l + k + im \bmod r, \tag{2}$$

for $i = 0,\ldots,\frac{r}{m}-1$, $l = 1,\ldots,p$, and if $q > 0$, let

$$t_{i+p\cdot\frac{r}{m}}(C_k,\phi) \equiv j_0 + k + im \bmod r, \tag{3}$$

for $i = 0,\ldots,q-1$. Note that since $p \leq w$, all values j_l in (2) are defined, and if $q > 0$, then j_0 differs from all these values. Moreover, since the returned values of the above functions must be from $\{1,\ldots,r\}$, the functions are well defined using the congruence relation in (2) and (3). We first show that values in C_k and values $t_0(C_k,\phi),\ldots,t_{f-d}(C_k,\phi)$ are all distinct for any $k = 0,\ldots,m-1$. Indeed, since for any $0 \leq l < w$, $j_l \notin C_0$, we have also $(j_l + k) \bmod r \notin C_k$, and since $\hat{C}$ is m-symmetric, for all i it holds that $(j_l + k + im) \bmod r \notin C_k$, so $t_0(C_k,\phi),\ldots,t_{f-d}(C_k,\phi) \notin C_k$. Assume that $t_{i+(l-1)\cdot\frac{r}{m}}(C_k,\phi) = t_{i'+(l'-1)\cdot\frac{r}{m}}(C_k,\phi)$ for some $0 \leq i,i' \leq \frac{r}{m}-1$, and $1 \leq l,l' \leq p$. This implies that $j_l + im \equiv j_{l'} + i'm \bmod r$, and also $j_l \equiv j_{l'} \mod m$. But j_l's are distinct modulo m, so $l = l'$ and the first equality simplifies to $(i-i')m \equiv 0 \mod r$, which gives $i = i'$. Note that if $m = r$, then by assumption, we must have $i = i' = 0$.

We next show that for each $l = 1,\ldots,p$, functions $\{t_{i+(l-1)\cdot\frac{r}{m}}\}_{i=0}^{\frac{r}{m}-1}$ defined on $\hat{C} \times \{\phi\}$ contributes to the distribution by the r-vector $(1,\ldots,1)$. Indeed, the sequence

$$
\begin{aligned}
&t_{0+(l-1)\cdot\frac{r}{m}}(C_0,\phi), t_{0+(l-1)\cdot\frac{r}{m}}(C_1,\phi), \ldots, t_{0+(l-1)\cdot\frac{r}{m}}(C_{m-1},\phi),\\
&t_{1+(l-1)\cdot\frac{r}{m}}(C_0,\phi), t_{1+(l-1)\cdot\frac{r}{m}}(C_1,\phi), \ldots, t_{1+(l-1)\cdot\frac{r}{m}}(C_{m-1},\phi),\\
&\ldots\\
&t_{\frac{r}{m}-1+(l-1)\cdot\frac{r}{m}}(C_0,\phi), t_{\frac{r}{m}-1+(l-1)\cdot\frac{r}{m}}(C_1,\phi), \ldots, t_{\frac{r}{m}-1+(l-1)\cdot\frac{r}{m}}(C_{m-1},\phi)
\end{aligned}
$$

is an arithmetic sequence modulo r with step 1.

In the same sequel, one can show that functions $\{t_{i+p\cdot\frac{r}{m}}\}_{i=0}^{q-1}$ on $\hat{C}\times\{\phi\}$ have (j_0, j_0')-cyclic distribution, where $j_0' \equiv j_0 + qm \bmod r$. Hence functions $\{t_i\}_{i=0}^{f-d}$ on $\hat{C}\times\{\phi\}$ have (j_0, j_0')-cyclic distribution.

Now, let $\hat{C}_1, \hat{C}_2, \ldots, \hat{C}_K$ be a sequence of all configurations, such that $\hat{C}_i \subseteq \mathbf{C}_{1,f}$. Defining the r,f-system of functions is equivalent to defining the system on the sequence $\hat{C}_1\times\{\prec\}, \hat{C}_2\times\{\prec\}, \ldots, \hat{C}_K\times\{\prec\}, \hat{C}_1\times\{\succ\}, \hat{C}_2\times\{\succ\}, \ldots, \hat{C}_K\times\{\succ\}$. By the previous construction, we can define functions on $\hat{C}_1\times\{\prec\}$, such that they have $(0,a_1)$-cyclic distribution for some a_1. Then we define functions on $\hat{C}_2\times\{\prec\}$ such that they have (a_1,a_2)-cyclic distribution. Clearly, the functions on $\hat{C}_1\times\{\prec\}\cup\hat{C}_2\times\{\prec\}$ have $(0,a_2)$-cyclic distribution. We continue inductively and construct the r,f-system of functions with $(0,b)$-cyclic distribution for some value b. ¶

Consider a regular r,f-system of functions $\{t_i\}_{i=0}^{f-1}$. If we permute values $t_0(C,\phi),\ldots,t_{f-d}(C,\phi)$ for any $C\in\mathbf{C}_d$, and $\phi\in\{\prec,\succ\}$, we get a regular system of functions, again. Obviously, for each $d=1,\ldots,f-1$, and $C\in\mathbf{C}_d$, there exists a permutation of $t_0(C,\succ),\ldots,t_{f-d}(C,\succ)$ such that $t_i(C,\prec)\neq t_i(C,\succ)$ for $i=0,\ldots,f-d$. Hence, we have the following corollary.

Corollary 1 *For any $0\le f<r$, there exists a regular r,f-system of functions satisfying*

$$t_i(C,\prec)\neq t_i(C,\succ),\ \textit{for all}\ 1\le d<f,\ C\in\mathbf{C}_d,\ 0\le i\le f-d. \tag{4}$$

Description of the Layout. Let u and v be any two vertices of H_r of characteristic C with d elements. For each k, $0\le k\le d-1$, define $P(u,v;k) := (u_0,u_1,\ldots,u_d)$, where $u_0=u, u_1=u_0(C_{k+1}), u_2=u_1(C_{k+2}),\ldots,u_{d-k}=u_{d-k-1}(C_d), u_{d-k+1}=u_{d-k}(C_1), u_{d-k+2}=u_{d-k+1}(C_2),\ldots,u_d=u_{d-1}(C_k))$. Let us note that the k-th dipath is constructed so that in u we first change the C_{k+1}-th dimension, then the C_{k+2}-th dimension, and so on.

Obviously, the above defined dipaths are shortest $u-v$ dipaths and have the length d. Now, for any two vertices u and v define the system of r dipaths in H_r as follows. For $i=0,1,\ldots,d-1$, set $\mathcal{P}_i(u,v) := P(u,v;i)$. For $i=d,d+1,\ldots,r-1$, set $\mathcal{P}_i(u,v) := \langle u,u'\rangle\cdot P(u',v';0)\cdot\langle v',v\rangle$, where $u'=u(t_{i-d}(C,\sigma(u,v)))$, $v'=$

$v(t_{i-d}(C,\sigma(u,v)))$, and $\{t_i\}_{i=0}^{r-d-1}$ is a system of partial functions from $\mathbf{C}\times\{\prec,\succ\}$ to $\{1,\ldots,r\}$.

For every f, $0\leq f<r$, define the layout $\mathcal{R}_f$ for (H_r,I_A) as follows.

$$\mathcal{R}_f=\{\mathcal{P}_i(x,y),i=0,\ldots,f\ :\ [x,y]\in I_A\},$$

where $\{t_i\}_{i=0}^{f-1}$ is a regular r,f-system of functions. According to Lemma 2 such a system always exists, so $\mathcal{R}_f$ is well defined. For the notational convenience we define the following sublayouts of $\mathcal{R}_f$. For any $1\leq d\leq r$, $0\leq k\leq f$, and any characteristic C, let

$$\begin{aligned}\mathcal{R}_{f,k}^d &= \{\mathcal{P}_k(x,y)\in\mathcal{R}_f\ :\ [x,y]\in I_A^d\},\\ \mathcal{R}_{f,k}(C) &= \{\mathcal{P}_k(x,y)\ :\ \text{the characteristic of } [x,y] \text{ is } C\}.\end{aligned}$$

Obviously, $\mathcal{R}_f=\cup_{1\leq d\leq r}\cup_{0\leq k\leq f}\mathcal{R}_{f,k}^d$.

Routing Properties of the Layout $\mathcal{R}_f$.

Lemma 3 *The layout $\mathcal{R}_f$ is f-wise in H_r for any f, where $0\leq f<r$.*

PROOF. It is enough to prove that the dipaths $\mathcal{P}_i(u,v)$, $i=0,1,\ldots,r-1$, are pairwise internally vertex disjoint for any pair of vertices u and v from H_r. Consider any pair of vertices $u=u_1u_2\ldots u_r$ and $v=v_1v_2\ldots v_r$ of characteristic C with d elements.

The dipaths $\mathcal{P}_i(u,v)$ and $\mathcal{P}_j(u,v)$ are internally vertex disjoint for any $0\leq i<j<d$. Indeed, every internal vertex, say $x=x_1x_2\ldots x_r$, of $\mathcal{P}_j(u,v)$ has $x_{C_{j+1}}=v_{C_{j+1}}$ and $x_{C_j}=u_{C_j}$, while every internal vertex, say $y=y_1y_2\ldots y_r$, of $\mathcal{P}_i(u,v)$ has either $y_{C_{j+1}}=v_{C_{j+1}}$ and $y_{C_j}=v_{C_j}$ or $y_{C_{j+1}}=u_{C_{j+1}}$. The claim follows from the fact that $u_{C_j}\neq v_{C_j}$ and $u_{C_{j+1}}\neq v_{C_{j+1}}$.

The dipaths $\mathcal{P}_i(u,v)$ and $\mathcal{P}_j(u,v)$ are internally vertex disjoint for any $0\leq i<d$ and $d\leq j<r$. Indeed, denote by $\overline{b}$ the complement of bit $b\in\{0,1\}$, i.e. $\overline{b}=1-b$. Every internal vertex, say $x=x_1x_2\ldots x_r$, of $\mathcal{P}_j(u,v)$ has $x_{t_{j-d}(C,\sigma(u,v))}=\overline{u_{t_{j-d}(C,\sigma(u,v))}}$, while every internal vertex, say $y=y_1y_2\ldots y_r$, of $\mathcal{P}_i(u,v)$ has $y_{t_{j-d}(C,\sigma(u,v))}=u_{t_{j-d}(C,\sigma(u,v))}$, since $t_{j-d}(C,\sigma(u,v))\notin C$.

Finally, the dipaths $\mathcal{P}_i(u,v)$ and $\mathcal{P}_j(u,v)$ are internally vertex disjoint for any $d\leq i<j<r$. Indeed, the values $t_{i-d}(C,\sigma(u,v))$, $t_{j-d}(C,\sigma(u,v))$ are distinct and do not belong to C. Hence every internal vertex, say $x=x_1x_2\ldots x_r$, of $\mathcal{P}_j(u,v)$ has $x_{t_{j-d}(C,\sigma(u,v))}=\overline{u_{t_{j-d}(C,\sigma(u,v))}}$, while every internal vertex, say $y=y_1y_2\ldots y_r$, of $\mathcal{P}_i(u,v)$ has $y_{t_{j-d}(C,\sigma(u,v))}=u_{t_{j-d}(C,\sigma(u,v))}$. ¶

Lemma 4 *For any f, $0\leq f<r$, it holds that*

$$\vec{\pi}(H_r,I_A,\mathcal{R}_f)\leq(f+1)2^{r-1}+\left\lceil\frac{2}{r}\sum_{i=1}^{f}(f-i+1)\binom{r}{i}\right\rceil.$$

PROOF. We split the layout $\mathcal{R}_f$ into $(f+1)r$ sublayouts $\mathcal{R}^d_{f,k}$, $0 \le k \le f$, $1 \le d \le r$, and calculate the congestion for every sublayout separately.

Let us fix d, $1 \le d \le r$. First, consider sublayouts $\mathcal{R}^d_{f,k}$ for any $k = 0,1,\ldots,$ $\min\{d-1, f\}$. For any such fixed k, all dipaths in $\mathcal{R}^d_{f,k}$ are the shortest ones.

Fix a characteristic $C \in \mathbf{C}_d$, and consider $\mathcal{R}_{f,k}(C)$. Obviously, in the layout $\mathcal{R}_{f,k}(C)$ every arc of H_r is used at most once. More precisely, the congestion of an arc is one if, and only if, the arc is an ith dimension arc, where $i \in C$. Indeed, for any fixed C and any fixed k, fixing an ith dimension arc ($i \in C$) determines whole dipath uniquely.

This means that the congestion of an ith dimension arc of H_r in the sublayout $\mathcal{R}^d_{f,k}$ is equal to the total number of characteristic from $\mathbf{C}_d$ containing i. Obviously, this number is equal to $\binom{r-1}{d-1}$. Since this number does not depend on i, this means that the congestion of any arc of H_r in $\mathcal{R}^d_{f,k}$ is $\binom{r-1}{d-1}$.

Second, consider sublayouts $\mathcal{R}^d_{f,k}$ for $k = d, d+1, \ldots, f$ (remember, d is fixed). For any such fixed k, all dipaths in $\mathcal{R}^d_{f,k}$ are of the form $\langle x, x'\rangle \cdot P(x', y'; 0) \cdot \langle y', y\rangle$, where $x' = x(t_{k-d}(C, \sigma(x,y)))$, $y' = y(t_{k-d}(C, \sigma(x,y)))$, and $[x,y] \in I^d_A$ of a characteristic C. We first calculate the congestion forced by dipaths $P(x', y'; 0)$. One can see that $\cup_{[x,y]\in I^d_A}[x', y'] = I^d_A$. Indeed, x' and y' are of the same characteristic as x and y are, and moreover, $\sigma(x', y') = \sigma(x,y)$. Hence, $x = x'(t_{k-d}(C, \sigma(x', y')))$, $y = y'(t_{k-d}(C, \sigma(x', y')))$. As a consequence we conclude that $\cup_{[x,y]\in I^d_A} P(x', y'; 0) = \cup_{[x,y]\in I^d_A} P(x, y; 0) = \mathcal{R}^d_{f,0}$, hence the congestion of any arc of H_r in $\mathcal{R}^d_{f,k}$ forced by dipaths $P(x', y'; 0)$ is $\binom{r-1}{d-1}$.

It follows that the congestion of any arc of H_r in $\mathcal{R}_f$ forced by the dipaths considered thus far is equal to $(f+1)2^{r-1}$, since d is from the interval $[1, r]$.

Finally, we have to consider the reminding arcs (arcs of the form $\langle x, x'\rangle$ and $\langle y', y\rangle$, $[x,y] \in I^d_A$) of the dipaths from $\mathcal{R}^d_{f,k}$, where $1 \le d \le k \le f$. The above arcs depend only on the characteristic of $[x,y]$ and the returned values of functions $\{t_i\}_{i=0}^{f-1}$.

For any fixed characteristic $C \in \mathbf{C}_{1,f}$, any $\phi \in \{\prec, \succ\}$, and any fixed i, $0 \le i \le f-d$, where d is the number of elements in C, let

$$\mathcal{L}(C, i, \phi) = \{\langle x, x'\rangle, \langle y', y\rangle \ : \ \text{characteristic of } [x,y] \text{ is } C \text{ and } \sigma(x,y) = \phi\},$$

where $x' = x(t_i(C,\phi))$, $y' = y(t_i(C,\phi))$. Since $[x', y']$ has the same characteristic as $[x,y]$, and since $\sigma(x', y') = \phi$, it follows that all the arcs in the layout $\mathcal{L}(C, i, \phi)$ are the $t_i(C,\phi)$th dimension arcs. In particular, every $t_i(C,\phi)$th dimension arc has congestion one in $\mathcal{L}(C, i, \phi)$. Hence, in order to calculate the congestion of an jth dimension arc forced by reminding edges in $\mathcal{R}_f$ it is enough to know how many times $t_i(C,\phi) = j$ for any $\phi \in \{\prec, \succ\}$, any characteristic $C \in \mathbf{C}_{1,f}$, and any i, $0 \le i \le f-d$, where d is the number of elements in C. Since $\{t_i\}_{i=0}^{f-1}$ is a regular r,f-system of functions, we know that the value j is returned at most

$\lceil \frac{2}{r} \sum_{i=1}^{f} (f-i+1)\binom{r}{i} \rceil$ times. Indeed, for any fixed i, $1 \leq i \leq f$, we have $\binom{r}{i}$ distinct characteristics, and we apply $t_0, t_1, \ldots, t_{f-i}$ to every of these characteristics two times (once with $\prec$, and once with $\succ$) in the layout $\mathcal{R}_f$.

Now we can sum-up all partial congestions and see that the congestion of any arc of H_r in layout $\mathcal{R}_f$ is at most $(f+1)2^{r-1} + \lceil \frac{2}{r} \sum_{i=1}^{f} (f-i+1)\binom{r}{i} \rceil$, as claimed. ¶

Corollary 2 *For any f, $0 \leq f < r$, it holds that*

$$\vec{\pi}(H_r, I_A, f) = (f+1)2^{r-1} + \left\lceil \frac{2}{r} \sum_{i=1}^{f} (f-i+1)\binom{r}{i} \right\rceil .$$

In what precedes, we proved that the routing $\mathcal{R}_f$, $0 \leq f < r$ is f-wise and thus by Observation 1, it is f-fault tolerant in H_r. In what follows, we will show that it also requires nearly optimal number of wavelengths to solve the wavelength assignment problem for it.

4 The Wavelength Assignment for $\mathcal{R}_f$

Description of the Coloring. Consider a characteristic $C \in \mathbf{C}_d$. It is easy to observe that if $k < d$, any two dipaths in $\mathcal{R}_{f,k}(C)$ are arc disjoint, so we color all dipaths in $\mathcal{R}_{f,k}(C)$ by the same color. Denote by $\bar{C}$ the complement of characteristic C, i.e. $\bar{C} = \{1, \ldots, r\} - C$. If $k < d$ and $k < r - d$, then any dipath in $\mathcal{R}_{f,k}(C)$ is arc disjoint with any dipath in $\mathcal{R}_{f,k}(\bar{C})$, since they use arcs of distinct dimensions. Hence, in this case we color all dipaths in $\mathcal{R}_{f,k}(C) \cup \mathcal{R}_{f,k}(\bar{C})$ by the same color.

Finally, we analyze case $k \geq d$. Take any two distinct dipaths $\mathcal{P}_k(u_1, v_1)$, and $\mathcal{P}_k(u_2, v_2)$ from $\mathcal{R}_{f,k}(C)$, $C \in \mathbf{C}_d$, such that $\phi = \sigma(u_1, v_1) = \sigma(u_2, v_2)$. So we have $\mathcal{P}_k(u_1, v_1) = \langle u_1, u_1' \rangle \cdot \mathcal{P}_0(u_1', v_1') \cdot \langle v_1', v_1 \rangle$ and $\mathcal{P}_k(u_2, v_2) = \langle u_2, u_2' \rangle \cdot \mathcal{P}_0(u_2', v_2') \cdot \langle v_2', v_2 \rangle$, where $u_1' = u_1(e)$, $v_1' = v_1(e)$, $u_2' = u_2(e)$, $v_2' = v_2(e)$, and $e = t_{k-d}(C, \phi)$. Clearly, $\mathcal{P}_0(u_1', v_1')$ and $\mathcal{P}_0(u_2', v_2')$ are arc-disjoint, since they are the shortest dipaths of the same characteristic C. Also arcs $\langle u_1, u_1' \rangle$, $\langle v_1', v_1 \rangle$ (resp. $\langle u_2, u_2' \rangle$, $\langle v_2', v_2 \rangle$) cannot be contained in the dipath $\mathcal{P}_0(u_2', v_2')$ (resp. $\mathcal{P}_0(u_1', v_1')$), since they are arcs of dimension $e \notin C$. Hence, the only possibility, where the above dipaths could intersect are arcs $\langle u_1, u_1' \rangle$, $\langle v_2', v_2 \rangle$ or arcs $\langle u_2, u_2' \rangle$, $\langle v_1', v_1 \rangle$. Assume, for instance, $\langle u_1, u_1' \rangle = \langle v_2', v_2 \rangle$. This implies $v_2' = u_1$, $u_1' = v_2$, and also $u_2' = v_1$, $v_1' = u_2$. We must have $\sigma(v_2, v_1) = \sigma(v_2', v_1') = \sigma(u_1, u_2)$, which is a contradiction. One can come to a similar contradiction with the second pair of arcs. So, the dipaths are arc-disjoint. Note, that the same holds without the assumption that the pairs $[u_1, v_1]$, $[u_2, v_2]$ have the same orientation, if we assume that $t_k(C, \prec) \neq t_k(C, \succ)$. According to Corollary 1, we can assume that the system of functions $\{t_i\}_{i=0}^{f-1}$ has property (4), and we can color all dipaths in $\mathcal{R}_{f,k}(C)$ by the same color, if $d < f$. If $d = f$, then $k = f$ and we need 2 colors for dipaths $\mathcal{P}_f(u, v) \in \mathcal{R}_{f,f}(C)$: one for corresponding dipaths with $u \prec v$ and another one for dipaths with $u \succ v$. This gives us the following upper bound on the f-wise optical index.

Theorem 1 *For any* $0 \le f < \frac{r}{2}$*, it holds that*

$$\vec{\omega}(H_r, I_A, f) \le (f+1)2^{r-1} + \sum_{i=1}^{f}(f-i+1)\binom{r}{i} + (f \ne 0)\cdot\binom{r}{f},$$

and for any $\frac{r}{2} \le f < r$*, it holds that*

$$\begin{aligned}\vec{\omega}(H_r, I_A, f) &\le \left(2f+2-\frac{r}{2}\right)2^{r-1} - (f+1) + \frac{1}{2}\left\lceil\frac{r}{2}\right\rceil\binom{r}{\lceil\frac{r}{2}\rceil} + \binom{r}{f}\\ &\le \left(2f - \frac{r}{4} + \frac{13}{4}\right)2^{r-1}.\end{aligned}$$

PROOF. Let $\mathcal{R}_{f,k} = \bigcup_{d=1}^{r}\mathcal{R}_{f,k}^{d}$. Consider, for instance, r odd. According to the description of coloring, to color paths in $\mathcal{R}_{f,k}$, where $k \le \frac{r-1}{2}$, we can use $2^{r-1} + \sum_{i=1}^{k}\binom{r}{i}$ colors: to color paths in $\cup_{d=\frac{r+1}{2}}^{r}\mathcal{R}_{f,k}^{d}$ we use 2^{r-1} colors; for paths in $\mathcal{R}_{f,k}^{d}$, where $k < d \le \frac{r-1}{2}$, we can use the same set of colors as for paths in $\mathcal{R}_{f,k}^{r-d}$ and for other paths we use new colors. If $k = d = f \ne 0$, then according to the description we have to use another set of $\binom{r}{f}$ colors. So for $f \le \frac{r-1}{2}$, we can color our layout with $\sum_{k=0}^{f}\left[2^{r-1} + \sum_{i=1}^{k}\binom{r}{i}\right] + (f \ne 0)\cdot\binom{r}{f}$ colors, which gives us the first claim of the theorem.

Notice, that to color paths in $\mathcal{R}_{f,k}$, where $k \ge \frac{r+1}{2}$, we can use the same number of colors as we used for coloring paths in $\mathcal{R}_{f,\frac{r-1}{2}}$. This is $2^{r-1} + \sum_{i=1}^{\frac{r-1}{2}}\binom{r}{i} = 2.2^{r-1} - 1$ colors. All together we can color the layout $\mathcal{R}_f$, where $f \ge \frac{r+1}{2}$ with

$$\begin{aligned}&\sum_{k=0}^{\frac{r-1}{2}}\left[2^{r-1} + \sum_{i=1}^{k}\binom{r}{i}\right] + \left(f+1-\frac{r+1}{2}\right)(2.2^{r-1}-1) + \binom{r}{f} =\\ &\left(2f+2-\frac{r}{2}\right)2^{r-1} - (f+1) + \frac{r+1}{4}\binom{r}{\frac{r+1}{2}} + \binom{r}{f}\end{aligned}$$

colors, which gives the claim. The evaluation for even r is very similar. The last bound for the optical index comes from the inequality $\binom{n}{m} \le 2^{n-1}$. ¶

5 Conclusion

We studied the problem of finding a set of $f+1$, $0 \le f < r$ internally vertex-disjoint dipaths for every pair of distinct vertices in the binary r-dimensional hypercube, which can be colored by the fewest number of colors so that dipaths of the same color do not share an arc. These systems of colored dipaths represent WDM based routing protocols that remain functional in the presence of up to f faults (nodes and/or links) in the hypercube network.

We computed the f-wise arc-forwarding index of the hypercube, and constructed nearly optimal All-to-All protocols, in terms of number of wavelengths, for these networks. Future work may concern other topologies.

Last Note. A referee suggested a shorter proof of the main result using flows. However, our proof is more constructive.

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