Knowledge Representation and Rough Sets

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by

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Contents

Chapter 1

Introduction

The information systems introduced by Z. Pawlak [41, 42] are used for representing properties of objects by means of attributes and their values. For example, we may express statements concerning the color of objects if the information system includes the attribute “color” and a set of values of this attribute consisting of “green”, “yellow”, etc. The more general nondeterministic information systems in which an object may have several values of an attribute were introduced by E. Orłowska and Z. Pawlak in [37].

Several relations reflecting the indiscernibility, similarity or dissimilarity of objects of a nondeterministic information system have been considered in the literature (see [38, 39, 40], for example). It seems that these relations have many properties in common, and here we introduce general strong and weak preimage relations which are suited for studying such common features.

The information about the objects of an information system yielded by different sets of attributes may depend on each other in various ways. For example, it may turn out that a proper subset of a set of attributes classifies the objects with the same accuracy as the original set. Dependence spaces were introduced by M. Novotný and Z. Pawlak [31] as a general abstract setting for studying such informational dependency. In this thesis we consider cores, dependency relations, independent sets, and reducts especially in terms of dense families of dependence spaces. Dependence spaces induced by strong and weak preimage relations are also studied. In addition to this, we introduce matrices of preimage relations and show how we can by using a matrix represention of preimage relations determine families of sets, which are dense in dependence spaces defined by preimage relations.

In rough set theory it is usually assumed that the knowledge about objects is restricted by some indiscernibility relation (see [43, 45], for example). Indiscernibility relations are equivalences which are interpreted so that two objects are equivalent if we cannot distinguish them by using our information. In an information system an indiscernibility relation arises naturally when one considers a given set of attributes: two objects are equivalent when their values of all attributes in the set are the same.

Some of the natural indiscernibility relations encountered in nondeterministic information systems are not necessarily transitive. Hence, we shall assume that the knowledge about objects is given by a similarity relation R, which is a tolerance on the given universe U of objects. The lower R-approximation of a set $X(\subseteq U)$ of objects consists of elements which necessarily belong to X in view of the knowledge provided by R. The upper R-approximation of X is formed of elements which possibly are in X in light of the knowledge R. Here we study the properties of the above approximations and investigate the structure of the ordered set of rough sets. We also characterize the three types of rough equality relations defined by tolerances (cf. [28, 29]).

Our work is structured as follows. This chapter is concluded by a general overview of the thesis. In Chapter 2 we recall some notions and notation of lattice theory and universal algebra. In the following chapter we consider complete congruences and morphisms of semilattices, which provide the basis of our further study in the subsequent chapters. The fourth chapter is devoted to the study of information systems and preimage relations. In Chapter 5 we examine dependence spaces and in the final chapter we investigate rough sets defined by tolerances.

1.1 Complete Congruences and Morphisms of Semilattices

A closure operator $c\colon P \to P$ on an ordered set $\mathcal{P} = (P, \leq)$ is an extensive, idempotent, and isotone map. Ward [55] has shown that if $\mathcal{P}$ is a complete lattice, then the pointwise defined meet of any set of closure operators on $\mathcal{P}$ is again a closure operator on $\mathcal{P}$. This implies that the set of all closure operators on a complete lattice is again a complete lattice with respect to the pointwise order. Here we generalize this result by showing that if $\mathcal{P}$ is a complete join-semilattice (i.e., $\bigvee S$ exists for all $\emptyset \neq S \subseteq P$), then the set of all closure operators on $\mathcal{P}$ is a complete lattice. Moreover, we describe the joins in this complete lattice in a novel way by applying the Knaster–Tarski Fixpoint Theorem. As a special case we depict the join of continuous closure operators by applying Kleene's Fixpoint Theorem.

Consider a semilattice $\mathcal{P} = (P, \vee)$. We say that a congruence Θ on $\mathcal{P}$ is complete if each Θ-class has a greatest element. The set of all complete congruences on $\mathcal{P}$ may be ordered with the inclusion relation. We prove that this ordered set is isomorphic to the ordered set of all closure operators on $(P, \leq)$. We note that if Θ is a complete congruence on $\mathcal{P}$, then the quotient semilattice $\mathcal{P}/\Theta$, ordered by $a/\Theta \leq b/\Theta$ iff $(a \vee b)/\Theta = b/\Theta$, is isomorphic to the set P_Θ of the greatest elements of Θ-classes ordered by the order inherited from $\mathcal{P}$. We also show that if $(P, \leq)$ is a lattice, a complete join-semilattice, a complete meet-semilattice, or a complete lattice, then so are $\mathcal{P}/\Theta$ and $(P_\Theta, \leq)$. We prove that for a complete join-semilattice $(P, \leq)$, the complete congruences on $(P, \vee)$ are exactly the equivalences which are completely $\vee$-compatible. In addition to this, we describe for any complete join-semilattice $(P, \leq)$ the closure operator $\theta_c\colon \mathrm{Rel}(P) \to \mathrm{Rel}(P)$, which maps each binary relation R on P to the least complete congruence on $(P, \vee)$ containing R.

Let $(P, \leq)$ and $(Q, \leq)$ be ordered sets. A map $f\colon P \to Q$ is a complete join-morphism if for all $S \subseteq P$ such that $\bigvee S$ exists, the join $\bigvee f[S]$ $(= \bigvee\{f(x) \mid x \in S\})$ exists and $\bigvee f[S] = f(\bigvee S)$. We show that if $\mathcal{P} = (P, \leq)$ is a complete join-semilattice, then the kernel Θ_f of any complete join-morphism $f\colon P \to Q$ is a complete congruence on $(P, \vee)$. This means that for a complete join-semilattice $\mathcal{P}$, each complete join-morphism $f\colon P \to Q$ induces a complete congruence Θ_f on $(P, \vee)$ and a closure operator c_f on $(P, \leq)$. We note that $(f[P], \leq)$, $(P/\Theta_f, \leq)$, and $(P_f, \leq)$ are isomorphic complete join-semilattices, where P_f is the set of c_f-closed elements. We also point out that $(f[P], \leq)$, $(P/\Theta_f, \leq)$, and $(P_f, \leq)$ are complete lattices whenever $\mathcal{P}$ is a complete lattice.

Let $\mathcal{P} = (P, \leq)$ be an ordered set and $S \subseteq P$. Novotný [35] has defined an equivalence Θ_S on P by setting

$$\Theta_S = \{(x, y) \in P^2 \mid (\forall z \in S)\ x \leq z \iff y \leq z\}.$$

It is known [35] that if $\mathcal{P}$ is a join-semilattice, then Θ_S is a congruence on $(P, \vee)$. We show that if $\mathcal{P}$ is a complete join-semilattice, then the congruence Θ_S is complete. Consider a congruence Θ on a semilattice $(P, \vee)$. A subset $S (\subseteq P)$ is said to be Θ-dense if $\Theta_S = \Theta$ (cf. [34]). We prove that if Θ is complete, then the Θ-dense subsets of P are exactly the meet-dense (see [5], for example) subsets of $(P_\Theta, \leq)$. We also show that for any complete congruence Θ on a complete join-semilattice there exists at least two Θ-dense sets. This implies that in a finite semilattice $(P, \vee)$ the number of congruence relations and closure operators is at most $2^{|P|-1}$. Furthermore, we point out that this upper bound is the best possible.

In this work a Galois connection $({}^{\blacktriangleright}, {}^{\blacktriangleleft})$ between $(P, \leq)$ and $(P, \geq)$ is called a dual Galois connection on $(P, \leq)$. If $(P, \leq)$ is a complete lattice, then ${}^{\blacktriangleright} \colon P \to P$ is a complete join-morphisms and its kernel $\Theta_{\blacktriangleright}$ is a complete congruence on $(P, \vee)$ such that the greatest element in the $\Theta_{\blacktriangleright}$-class of any $x \in P$ is $x^{\blacktriangleright\blacktriangleleft}$. By duality, the kernel $\Theta_{\blacktriangleleft}$ of ${}^{\blacktriangleleft} \colon P \to P$ is a congruence on $(P, \wedge)$ such that for any $x \in P$, the congruence class $x/\Theta_{\blacktriangleleft}$ has $x^{\blacktriangleleft\blacktriangleright}$ as its least element .

1.2 Information Systems and Preimage Relations

A nondeterministic information system [37] consists of a set U of objects, a set A of attributes, and an indexed set $\{V_a\}_{a \in A}$ of value sets of attributes. Each attribute is a map $a \colon U \to \wp(V_a) - \{\emptyset\}$, which assigns to every object a nonempty set of values of the attribute $a (\in A)$. We exclude the empty set because this assumption guarantees that the similarity relations defined in a nondeterministic information system are reflexive and that the strong and the weak indiscernibility relations defined by an attribute set are included in the corresponding similarity relations.

In a nondeterministic information system $\mathcal{S} = (U, A, \{V_a\}_{a \in A})$ we may define several information relations (see [38, 39, 40], for example). These relations are similar in the following sense. Two objects belong to a certain strong (resp. weak) relation with respect to an attribute set B if and only if their values of all (resp. some) attributes in B are in some given relation. For example, objects x and y are in the strong relation of similarity $sim(B)$ if and only if $a(x) \cap a(y) \neq \emptyset$ for all $a \in B$.

We introduce preimage relations, which allow us to study in a more general setting the common features of strong and weak relations defined in information systems. Let U and Y be nonempty sets, $R \in \mathrm{Rel}(Y)$, and let $f \colon U \to Y$ be a function. The preimage relation of R is defined by

$$f^{-1}(R) = \{(x, y) \in U^2 \mid f(x) R f(y)\}.$$

The notion of preimage relation may be extended in the following way. For any set B of functions $U \to Y$, the strong and the weak preimage relations of B are defined by

$$S_R(B) \;\; = \;\; \{(x, y) \in U^2 \mid (\forall f \in B) f(x) R f(y)\};$$

$$W_R(B) \quad = \quad \{(x,y) \in U^2 \mid (\exists f \in B) f(x) R f(y)\},$$

Because in a nondeterministic information system $\mathcal{S} = (U, A, \{V_a\}_{a \in A})$ each attribute $a \in A$ is a map $a\colon U \to \wp(V_a) - \{\emptyset\}$, strong and weak information relations are preimage relations.

Skowron and Rauszer introduced discernibility matrices in [52]. They presented several results concerning cores, dependency relations, and reducts defined in information systems by applying this notion. Here we introduce matrix representations of preimage relations as a generalization of discernibility matrices.

1.3 Dependence Spaces

We present a generalized definition of dependence spaces. According to Novotný and Pawlak [31], a pair $\mathcal{D} = (A, \Theta)$ is a dependence space, if A is a finite nonempty set and Θ is a congruence on the semilattice $(\wp(A), \cup)$. It can be easily seen that if A is finite, then each congruence on $(\wp(A), \cup)$ is complete, i.e., each congruence class has a greatest element. Therefore our following definition of dependence spaces is justified. We call a pair $\mathcal{D} = (A, \Theta)$ a dependence space if Θ is a complete congruence on $(\wp(A), \cup)$. A family of subsets $\mathcal{H} \subseteq \wp(A)$ is called dense in $\mathcal{D}$ if $\mathcal{H}$ is Θ-dense.

We show that strong and weak preimage relations define dependence spaces. Let A be a set of mappings $U \to Y$ and let R be a binary relation on Y. Then the map $S_R\colon \wp(A) \to \mathrm{Rel}(U)$, $B \mapsto S_R(B)$, is a complete join-morphism $(\wp(A), \subseteq) \to (\mathrm{Rel}(U), \supseteq)$. Similarly, the map $W_R\colon \wp(A) \to \mathrm{Rel}(U)$, $B \mapsto W_R(B)$, is a complete join-morphism $(\wp(A), \subseteq) \to (\mathrm{Rel}(U), \subseteq)$. This implies that the kernels of those maps, denoted by $\Theta_S(A, R)$ and $\Theta_W(A, R)$, respectively, are complete congruences on the semilattice $(\wp(A), \cup)$. We also show how we can determine dense families of the dependence spaces $(A, \Theta_S(A, R))$ and $(A, \Theta_W(A, R))$ by applying preimage matrices.

Consider a set $B \subseteq A$. A subset $C \subseteq B$ is said to be a reduct of B if C is minimal in B/Θ. We characterize the reducts of any subset of a dependence space in terms of dense families.

In a dependence space $\mathcal{D} = (A, \Theta)$, the Θ-class $B/\Theta = \{C \subseteq A \mid B \Theta C\}$ of any $B \subseteq A$ has a greatest element $\mathcal{C}_{\mathcal{D}}(B) = \bigcup B/\Theta$. A subset $C(\subseteq A)$ is said to be dependent on $B(\subseteq A)$, denoted by $B \to C$, if $\mathcal{C}_{\mathcal{D}}(C) \subseteq \mathcal{C}_{\mathcal{D}}(B)$. We present a method based on dense families which for a given dependency $B \to C$ finds all minimal subsets D of B such that $D \to B$.

1.4 Rough Sets

We generalize the lower and upper approximations defined by Pawlak [43]. For any tolerance R on U the lower R-approximation of a set $X(\subseteq U)$ is

$$X_R = \{x \in U \mid x/R \subseteq X\}$$

and its upper R-approximation is

$$X^R = \{x \in U \mid x/R \cap X \neq \emptyset\}.$$

Here x/R is the R-neighborhood $\{y \in U \mid xRy\}$ of x. The set $B_R(X) = X^R - X_R$ is referred to as the R-boundary of X.

The idea is that objects can be observed only by the accuracy given by a tolerance relation R. The set X_R (resp. X^R) consists of elements which surely (resp. possibly) belong to X with respect to the knowledge provided by R. The R-boundary is the actual area of uncertainty. It consists of elements whose membership in X cannot be decided when R-related objects cannot be distinguished from each other.

The kernel $\approx^R$ of the map ${}^R\colon X \mapsto X^R$ is referred to as the upper R-equality, and the kernel $\approx_R$ of the map ${}_R\colon X \mapsto X_R$ is the lower R-equality. We show that the pair $({}^R, {}_R)$ forms a dual Galois connection on $(\wp(U), \subseteq)$. So, $\approx^R$ is a complete congruence on $(\wp(U), \cup)$ and the greatest element in the $\approx^R$-class of any $X(\subseteq U)$ is $(X^R)_R$. Furthermore, the set $\{(x/R)^{\complement} \mid x \in U\}$ is $\approx^R$-dense. Similarly, $\approx_R$ is a congruence on $(\wp(U), \cap)$ such that the $\approx_R$-class of any $X(\subseteq U)$ has $(X_R)^R$ as its smallest element, and the set $\{x/R \mid x \in U\}$ is $\approx_R$-dense.

The relation $\approx^R \cap \approx_R$ is called the R-equality, and it is denoted by $\equiv_R$. Thus, two sets $X(\subseteq U)$ and $Y(\subseteq U)$ are $\equiv_R$-related if and only if exactly the same objects belong surely to X and to Y, and precisely the same objects are possibly in X and in Y.

We say that a relation Θ is a rough bottom equality if there is a tolerance R such that $\Theta = \approx_R$. Rough top equalities and rough equalities are defined in a similar manner.

Here we present a characterization of all three types of rough equalities defined by tolerances. Note that in [28, 29] M. Novotný and Z. Pawlak characterized the three types of rough equalities defined by equivalences on a finite set of objects, and in [54] M. Steinby generalized these characterizations by omitting the assumption of finiteness.

We also generalize Pawlak's notion of rough sets [43] by defining rough sets in terms of tolerances. We call the equivalence classes of $\equiv_R$ R-rough sets. We also study in our generalized setting an approach to rough sets introduced by Iwiński [17], which is based on the fact that any R-rough set $\mathcal{C} \in \wp(U)/\equiv_R$ may be viewed as a pair (X_R, X^R), where $X \in \mathcal{C}$. It is known that for any equivalence $E \in \mathrm{Eq}(U)$ the set $\{(X_E, X^E) \mid X \subseteq U\}$ ordered by the coordinatewise order is a Stone lattice (see e.g. [7, 10, 13, 46]). Here we show that the set $\{(X_R, X^R) \mid X \subseteq U\}$, where $R \in \mathrm{Tol}(U)$, is not necessarily even a semilattice with respect to the coordinatewise order if $|U| \geq 5$.

Chapter 2

Preliminaries

All general lattice theoretical and algebraic notions used in this work can be found in [1, 2, 3, 4, 5, 15, 16], for example.

2.1 General Notation and Conventions

Sets. We assume that the reader is familiar with the following notations: *membership* ($\in$), *set-builder* ($\{- \mid -\}$), *subset* ($\subseteq$), *proper subset* ($\subset$), *union* ($\cup$), *intersection* ($\cap$), *difference* ($-$), *complement* ($^{\complement}$), *ordered n-tuples* ($(x_1, \ldots, x_n)$), and *products of sets* ($A_1 \times \cdots \times A_n$).

The notations A_i, $i \in I$, and $\{A_i\}_{i \in I}$ refer to a *family of sets indexed by a set* I. Given a family $\mathcal{F}$ of sets, the union of $\mathcal{F}$, $\bigcup \mathcal{F}$, is defined by $a \in \bigcup \mathcal{F}$ if and only if $a \in A$ for some $A \in \mathcal{F}$. The *intersection* $\bigcap \mathcal{F}$ of $\mathcal{F}$ is defined by $a \in \bigcap \mathcal{F}$ if and only if $a \in A$ for all $A \in \mathcal{F}$. For a set A, let $\wp(A)$ denote the *power set* of A, that is, the set of all subsets of A.

Let us write $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. The *cardinality* of a set A is denoted by $|A|$. A set A is said to be *finite* if $|A| = n$ for some $n \in \mathbb{N}_0$; otherwise A is *infinite*. In particular, the empty set $\emptyset$ is finite and its cardinality is 0.

Relations. An *n-ary relation* R on a set A is a subset of A^n. If $n = 2$, then R is called a *binary relation*. We denote by $\mathrm{Rel}(A)$ the set of all binary relations on the set A. We sometimes write aRb for $(a, b) \in R$. The *inverse* of a relation $R \in \mathrm{Rel}(A)$ is $R^{-1} = \{(b, a) \mid aRb\}$. A relation $R \in \mathrm{Rel}(A)$ is

- *antisymmetric*, if for all $a, b \in A$, aRb and bRa imply $a = b$;

- *irreflexive*, if for all $a \in A$, $(a, a) \notin R$;
- *reflexive*, if for all $a \in A$, aRa;
- *symmetric*, if for all $a, b \in A$, aRb implies bRa;
- *transitive*, if for all $a, b, c \in A$, aRb and bRc imply aRc.

If R is reflexive and symmetric, it is called a *tolerance relation*. The set of all tolerance relations on A is denoted by $\mathrm{Tol}(A)$. For $R \in \mathrm{Tol}(\mathrm{A})$, the set $a/R = \{b \in A \mid aRb\}$ is called the R*-neighborhood* of a. A tolerance relation is an *equivalence relation* if it is transitive. We denote by $\mathrm{Eq}(A)$ the set of all equivalence relations on A. The *diagonal relation* of A,

$$\Delta_A = \{(a, a) \mid a \in A\},$$

and the *all relation* of A,

$$\nabla_A = A \times A,$$

are equivalences on A. Let $E \in \mathrm{Eq}(A)$. For all $a \in A$, the E-neighborhood a/E is called the *equivalence class of* a *modulo* E. The *quotient set of* A *modulo* E is the set $A/E = \{a/E \mid a \in A\}$.

Functions. A *function* f from a set A to a set B, denoted by $f\colon A \to B$, is a subset of $A \times B$ such that for each $a \in A$, there exists exactly one $b \in B$ with $(a, b) \in f$, in which case we write $f(a) = b$ or $f\colon a \mapsto b$. The terms *map* and *mapping* are often used instead of function. The set of all functions from A to B is denoted by B^A. For a function $f\colon A \to B$, we write for all $S \subseteq A$,

$$f[S] = \{f(x) \mid x \in S\}.$$

The set $f[A]$ is called the *range* of f. The *preimage set* of $Y (\subseteq B)$ is

$$f^{-1}(Y) = \{x \in A \mid f(x) \in Y\}.$$

The map $f\colon A \to B$ is *injective* (or *one-to-one*) if $f(a_1) = f(a_2)$ implies $a_1 = a_2$, and f is *surjective* (or *onto*) if for every $b \in B$, there exists an element $a \in A$ with $f(a) = b$; that is, $f[A] = B$. Furthermore, f is *bijective* if it is both injective and surjective. A map $f\colon A \to A$ is *idempotent* if $f(f(a)) = f(a)$ for all $a \in A$.

For two maps $f: A \to B$ and $g: B \to C$, let $g \circ f: A \to C$ be the map defined by $(g \circ f)(a) = g(f(a))$. The map $g \circ f$ is called the *composition* (or *product*) of f and g. The map $1_A: A \to A, a \mapsto a$, is called the *identity map* of A. A map $g: B \to A$ is the *inverse map* of $f: A \to B$ if $g \circ f = 1_A$ and $f \circ g = 1_B$. It is well-known that $f: A \to B$ has an inverse map if and only if f is a bijection. The inverse of a bijection f is denoted by f^{-1}.

For an equivalence $E \in \mathrm{Eq}(A)$, the *canonical map* of E is the map $v_E: A \to A/E, a \mapsto a/E$. On the other hand, for any map $f: A \to B$, the *kernel* of the map f, which is defined by

$$\Theta_f = \{(x, y) \in A \times A \mid f(x) = f(y)\},$$

is an equivalence on A.

2.2 Orders and Lattices

Order Relations. Let P be a set. An *order* (or a *partial order*) on P is a binary relation $\leq$ such that, for all $a, b, c \in P$,

(1) $a \leq a$,
(2) $a \leq b$ and $b \leq a$ imply $a = b$, and
(3) $a \leq b$ and $b \leq c$ imply $a \leq c$,

that is to say, the relation $\leq$ is reflexive, antisymmetric, and transitive. A set P equipped with an order relation $\leq$ is said to be an *ordered set* (or a *partially ordered set*). Many authors use the shorthand *poset*. An order-relation $\leq$ gives rise to a relation $<$ of *strict order*: $a < b$ iff $a \leq b$ and $a \neq b$. Let $\mathcal{P} = (P, \leq)$ be an ordered set and let $S \subseteq P$. Then S inherits the order relation $\leq$ from $\mathcal{P}$: for any $a, b \in S$, $a \leq b$ in S if and only if $a \leq b$ in P.

Let $x, y \in P$. We say that x is *covered* by y (or y *covers* x), and write $x \prec y$, if $x \leq y$ and $x \leq z < y$ implies $x = z$. The *Hasse diagram* of an ordered set $(P, \leq)$ represents the elements with small circles $\circ$, and the circles representing two elements x, y are connected by a straight line if one covers the other. Moreover, if x is covered by y, then the circle representing x is lower than the circle representing y.

Example 2.2.1. Let $P = \{a, b, c, d\}$ and define the order on P so that the covering relation consists of the pairs (a, b), (a, c), (b, d), (c, d). The Hasse diagram of $(P, \leq)$ is presented in Figure 1.

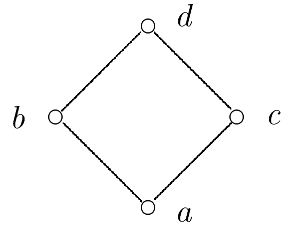


Figure 1.

Let $(P, \leq)$ be an ordered set and let $S \subseteq P$. Then $a \in S$ is a *maximal element* of S, if $a \leq x \in S$ implies $a = x$. The set of all maximal elements in S is denoted by $\max S$. Furthermore, $a \in S$ is the *greatest element* of S, if $x \leq a$ for all $x \in S$. *Minimal elements*, the set $\min S$, and the *least element* of S are defined dually, that is to say, by reversing the order.

The greatest element of P, if it exists, is called the *top element* of P and written $\top$. Similarly, the least element of P, if such an element exists, is called the *bottom element* and it is denoted by $\bot$. If $(P, \leq)$ has top and bottom elements, it is *bounded*.

Lattices and Semilattices as Ordered Sets. If $S \subseteq P$, then an element $x \in P$ is an *upper bound* of S if $a \leq x$ for all $a \in S$. A *lower bound* is defined dually. The set of all upper bounds of S is denoted by S^u, and the set of all lower bounds by S^l.

If S^u has a least element, this is called the *least upper bound* of S. Dually, if S^l has a greatest element, this is called the *greatest lower bound* of S. The least upper bound of S is also called the *supremum* of S and is denoted by $\sup S$. Similarly, the greatest lower bound of S is also called the *infimum* of S and is denoted by $\inf S$.

We write $a \vee b$ (read as "a *join* b") in place of $\sup\{a, b\}$ and $a \wedge b$ (read as "a *meet* b") in place of $\inf\{a, b\}$. Similarly, we write $\bigvee S$ (the "*join* of S") and $\bigwedge S$ (the "*meet* of S") instead of $\sup S$ and $\inf S$, respectively. It is sometimes necessary to indicate that the join or meet is being found in a particular ordered set $\mathcal{P} = (P, \leq)$, in which case we write $\bigvee_P S$ or $\bigwedge_P S$. Obviously, $\emptyset^u = P$ and $\bigvee \emptyset$ exists if and only if P has a bottom element $\bot$; in this case $\bigvee \emptyset = \bot$. Dually, $\bigwedge \emptyset = \top$ whenever P has a top element. If P has a top element, then $P^u = \{\top\}$ and $\bigvee P = \top$. By duality, $\bigwedge P = \bot$ whenever P has a bottom element.

The next lemma is an immediate consequence of the definitions of least upper bounds and greatest lower bounds.

Lemma 2.2.2. *Let $(P, \leq)$ be an ordered set and suppose S and T are subsets of P such that $\bigvee S$, $\bigvee T$, $\bigwedge S$, and $\bigwedge T$ exist in P. If $S \subseteq T$, then $\bigvee S \leq \bigvee T$ and $\bigwedge T \leq \bigwedge S$.* □

Let $(P, \leq)$ be a nonempty ordered set. Then $(P, \leq)$ is called a *join-semilattice*, if for all $a, b \in S$, the join $a \vee b$ exists. Similarly, $(P, \leq)$ is a *meet-semilattice*, if for all $a, b \in P$, the meet $a \wedge b$ exists. Furthermore, $(P, \leq)$ is a *lattice* if it is both a join- and a meet-semilattice.

If $\bigvee S$ exists for all $\emptyset \neq S \subseteq P$, then $(P, \leq)$ is called a *complete join-semilattice* and if $\bigwedge S$ exists for all $\emptyset \neq S \subseteq P$, then $(P, \leq)$ is called a *complete meet-semilattice*. Moreover, $(P, \leq)$ is a *complete lattice* if $\bigvee S$ and $\bigwedge S$ exist for all $S \subseteq P$. It can be easily seen that $\mathcal{P} = (P, \leq)$ is a complete lattice if and only if $\mathcal{P}$ is both a complete join-semilattice and a complete meet-semilattice. Now the following lemma holds (see e.g. [5]).

Lemma 2.2.3. *If $(P, \leq)$ is a complete join-semilattice, then $\bigwedge S$ exists in P for every subset S of P which has a lower bound in P; indeed, $\bigwedge S = \bigvee S^l$.* □

To show that an ordered set is a complete lattice requires only half as much work as the definitions would have us to believe. The following lemma (see [3, 4, 5], for example), which follows easily from Lemma 2.2.3, is usually stated in its dual form.

Lemma 2.2.4. *If $(P, \leq)$ is an ordered set such that $\bigvee S$ exists in P for every subset S of P, then $(P, \leq)$ is a complete lattice.* □

Note that in the above lemma the existence of $\bigvee \emptyset$ guarantees a bottom element $\bot$, and since $\bot \in P$, $\bigwedge S$ exists for all $S \subseteq P$ by Lemma 2.2.3. Hence, adjoining a bottom element to a complete join-semilattice creates a complete lattice.

Example 2.2.5. The ordered set $(\mathbb{N}, \leq)$ is a complete meet-semilattice, in which $\bigwedge S = \min S$ for all $\emptyset \neq S \subseteq \mathbb{N}$. Now $(\mathbb{N}, \leq)$ is not a complete lattice, since $\bigvee S$ does not exist for any infinite $S \subseteq \mathbb{N}$.

Let us consider the set $\mathbb{N} \cup \{\infty\}$, in which the order $\leq$ is defined by

$$n \leq m \iff n \leq m \text{ holds in } \mathbb{N} \text{ or } m = \infty.$$

Obviously, $(\mathbb{N} \cup \{\infty\}, \leq)$ is a complete lattice.

Let $(P, \leq)$ be a lattice and $\emptyset \neq S \subseteq P$. We say that $(S, \leq)$ is a *sublattice* of S if $a, b \in S$ implies $a \vee b \in S$ and $a \wedge b \in S$. Similarly, if $(P, \leq)$ is a complete lattice and $\emptyset \neq S \subseteq P$, then $(S, \leq)$ is a *complete sublattice* of $(P, \leq)$ if $\bigvee H \in S$ and $\bigwedge H \in S$ for all $H \subseteq S$. The *subsemilattices* and the *complete subsemilattices* may be defined similarly.

Let $\mathcal{P} = (P, \leq)$ and $\mathcal{Q} = (Q, \leq)$ be ordered sets. A map $f \colon P \to Q$ is said to be *order-preserving* (or *isotone*), if $a \leq b$ in $\mathcal{P}$ implies $f(a) \leq f(b)$ in $\mathcal{Q}$. The map f is an *order-embedding*, if $a \leq b$ in $\mathcal{P}$ if and only if $f(a) \leq f(b)$ in $\mathcal{Q}$. Note that an order-embedding is always an injection. An order-embedding f onto Q is called an *order-isomorphism* between $\mathcal{P}$ and $\mathcal{Q}$. When there exists an order-isomorphism between $\mathcal{P}$ and $\mathcal{Q}$, we say that $\mathcal{P}$ and $\mathcal{Q}$ are *order-isomorphic* and write $\mathcal{P} \cong \mathcal{Q}$.

A map $f \colon P \to Q$ is *order-reversing* (or *antitone*) if $a \leq b$ implies $f(a) \geq f(b)$. The map f is a *dual order-embedding*, if $a \leq b$ in $\mathcal{P}$ if and only if $f(a) \geq f(b)$ in $\mathcal{Q}$. A dual order-embedding onto Q is called a *dual order-isomorphism*; in such a case $\mathcal{P}$ and $\mathcal{Q}$ are said to be *dually order-isomorphic*. A map $g \colon P \to P$ is *extensive*, if $x \leq g(x)$ for all $x \in P$.

Each order-isomorphism preserves all existing joins and meets, which fact is stated in the following lemma.

Lemma 2.2.6. *Let $(P, \leq)$ and $(Q, \leq)$ be ordered sets. If $f \colon P \to Q$ is an order-isomorphism and $S \subseteq P$ is such that $\bigvee S$ exists in P, then $\bigvee f[S]$ exists in Q and $\bigvee f[S] = f(\bigvee S)$, and dual statements hold for $\bigwedge S$.* □

Axiom of Choice, Zorn's Lemma, and Chain Conditions. The Axiom of Choice asserts that there always exists a map which picks one element from each member of a given family of nonempty sets. This can be formally stated as follows.

Axiom of Choice. Given a nonempty family $\mathcal{H} = \{A_i\}_{i \in I}$ of nonempty sets, there exists a *choice function* for $\mathcal{H}$, that is to say, a map

$$f \colon I \to \bigcup_{i \in I} A_i$$

such that $f(i) \in A_i$ for every $i \in I$.

Let $\mathcal{P} = (P, \leq)$ be an ordered set. Then $S \subseteq P$ is a *chain* in $\mathcal{P}$ if, for all $x, y \in S$, either $x \leq y$ or $y \leq x$. For any $n \geq 1$, we denote by $\mathbf{n}$ the n-element chain obtained by ordering the set $\{0, 1, \ldots, n-1\}$ so that $0 < 1 < \cdots < n-1$.

Zorn's Lemma. Let $\mathcal{H}$ be a nonempty family of sets such that $\bigcup_{i \in I} A_i \in \mathcal{H}$ whenever $\{A_i\}_{i \in I}$ is a nonempty chain in $(\mathcal{H}, \subseteq)$. Then $\mathcal{H}$ has a maximal element.

The following fact is well-known.

Proposition 2.2.7. *The Axiom of Choice and Zorn's Lemma are equivalent.* □

Let $\mathcal{P} = (P, \leq)$ be an ordered set. We say that $\mathcal{P}$ has *no infinite chains* if every chain in $\mathcal{P}$ is finite, and that $\mathcal{P}$ satisfies the *ascending chain condition* (ACC), if given any sequence $a_1 \leq a_2 \leq \cdots \leq a_n \cdots$ of elements of P, there exists a $k \in \mathbb{N}$ such that $a_k = a_{k+1} = \ldots$ The dual of the ascending chain condition is the *descending chain condition* (DCC). It is obvious that every finite ordered set satisfies both the ACC and the DCC. A proof of the following lemma can be found in [5, pp. 38–39], for example.

Lemma 2.2.8. *Let $\mathcal{P} = (P, \leq)$ be an ordered set.*

(a) *$\mathcal{P}$ satisfies the* ACC *if and only if every nonempty subset S of P has a maximal element.*

(b) *$\mathcal{P}$ has no infinite chains if and only if $\mathcal{P}$ satisfies the ACC and the DCC.* □

The next lemma is presented in [5] for lattices, but we show that it holds also for semilattices. This lemma says that if a join-semilattice $\mathcal{P}$ satisfies the ACC, then $\mathcal{P}$ is a complete join-semilattice. This implies also that a lattice with no infinite chains is complete.

Lemma 2.2.9. *If $\mathcal{P} = (P, \leq)$ is a join-semilattice which satisfies the ACC, then every nonempty subset S of P has a finite subset F such that $\bigvee F = \bigvee S$.*

Proof. Let S be a nonempty subset of P. By Lemma 2.2.8(a), the nonempty subset

$$B = \{\bigvee F \mid F \text{ is a finite nonempty subset of } S\}$$

of P has a maximal element $\bigvee F$ for some finite $\emptyset \neq F \subseteq S$. If $a \in S$, then $\bigvee(F \cup \{a\}) \in B$. By Lemma 2.2.2, $\bigvee(F \cup \{a\}) \geq \bigvee F$. Since $\bigvee F$ is maximal, this implies $a \leq \bigvee(F \cup \{a\}) = \bigvee F$ and hence $\bigvee S \leq \bigvee F$. As $F \subseteq S$, we obtain $\bigvee F \leq \bigvee S$ by Lemma 2.2.2. □

Closure Operators and Closure Systems. A family $\mathcal{L}$ of subsets of a set A is said to be a *closure system* if $\mathcal{L}$ is closed under intersections, which means that for all $\mathcal{H} \subseteq \mathcal{L}$, we have $\bigcap \mathcal{H} \in \mathcal{L}$. If $\mathcal{L}$ is a closure system on A, then the ordered set $(\mathcal{L}, \subseteq)$ is a complete $\cap$-subsemilattice of $(\wp(A), \subseteq)$. According to the dual of Lemma 2.2.4 it is also a complete lattice but the join need to be $\cup$. Therefore, $(\mathcal{L}, \subseteq)$ is not usually a $\cup$-subsemilattice of $(\wp(A), \subseteq)$.

A *closure operator* on a set A is an extensive, idempotent, and order-preserving map $\mathcal{C}\colon \wp(A) \to \wp(A)$, that is to say,

(a) $B \subseteq \mathcal{C}(B)$,
(b) $\mathcal{C}(\mathcal{C}(B)) = \mathcal{C}(B)$, and
(c) $B \subseteq C$ implies $\mathcal{C}(B) \subseteq \mathcal{C}(C)$

for all $B, C \subseteq A$. A subset B of A is *closed* (with respect to $\mathcal{C}$) if $\mathcal{C}(B) = B$. A closure system $\mathcal{L}$ on A defines a closure operator $\mathcal{C}_\mathcal{L}$ on A by the rule

$$\mathcal{C}_\mathcal{L}(B) = \bigcap \{L \in \mathcal{L} \mid B \subseteq L\}.$$

Conversely, if $\mathcal{C}$ is a closure operator on A, then the family

$$\mathcal{L}_\mathcal{C} = \{B \subseteq A \mid \mathcal{C}(B) = B\}$$

of closed subsets of A is a closure system. The relationship between closure systems and closure operators is bijective. The closure operator induced by the closure system $\mathcal{L}_\mathcal{C}$ is $\mathcal{C}$ itself, and similarly the closure system induced by the closure operator $\mathcal{C}_\mathcal{L}$ is $\mathcal{L}$. In symbols,

$$\mathcal{C}_{(\mathcal{L}_\mathcal{C})} = \mathcal{C} \quad \text{and} \quad \mathcal{L}_{(\mathcal{C}_\mathcal{L})} = \mathcal{L}.$$

Note that if $\mathcal{L}$ is a closure system on A, then in the complete lattice $(\mathcal{L}, \subseteq)$, $\bigvee \mathcal{H} = \mathcal{C}_\mathcal{L}(\bigcup H)$ for all $\mathcal{H} \subseteq \mathcal{L}$.

An *interior operator* $\mathcal{I}\colon \wp(A) \to \wp(A)$ satisfies the following three conditions: (a) $\mathcal{I}(B) \subseteq B$, (b) $B \subseteq C$ implies $\mathcal{I}(B) \subseteq \mathcal{I}(C)$, and (c) $\mathcal{I}(\mathcal{I}(B)) = \mathcal{I}(B)$. It is known (see e.g. [24]) that each closure operator $\mathcal{C}\colon \wp(A) \to \wp(A)$ defines an interior operator $\mathcal{I}_\mathcal{C}\colon \wp(A) \to \wp(A)$ by the rule $\mathcal{I}_\mathcal{C}(B) = \mathcal{C}(B^\complement)^\complement$, and similarly every interior operator yields a closure operator. A system $\mathcal{N}$ of subsets of A is said to be an *interior system* if $\mathcal{N}$ is closed under unions. The relationship between interior systems and interior operators is also bijective. If $\mathcal{N}$ is an interior system, then the ordered set $(\mathcal{N}, \subseteq)$ is a complete lattice such that $\bigvee \mathcal{H} = \bigcup \mathcal{H}$ and $\bigwedge \mathcal{H} = \mathcal{I}_\mathcal{N}(\bigcap \mathcal{H})$ for all $\mathcal{H} \subseteq \mathcal{N}$, where $\mathcal{I}_\mathcal{N}$ is the interior operator corresponding to $\mathcal{N}$.

A closure operator $\mathcal{C}$ on a set A is called *algebraic* if for all $B \subseteq A$,

$$\mathcal{C}(B) = \bigcup \{\mathcal{C}(F) \mid F \text{ is a finite subset of } B\}.$$

Example 2.2.10. Let A be a set. The set of all equivalences on A is a closure system because $\bigcap \mathcal{H} \in \mathrm{Eq}(A)$ for all $\mathcal{H} \subseteq \mathrm{Eq}(A)$. The corresponding closure operator is

$$^E\colon \mathrm{Rel}(A) \to \mathrm{Rel}(A), R \mapsto \bigcap \{E \in \mathrm{Eq}(A) \mid R \subseteq E\}.$$

Hence, $(\mathrm{Eq}(A), \subseteq)$ is a complete lattice in which

$$\bigwedge \mathcal{H} = \bigcap \mathcal{H} \quad \text{and} \quad \bigvee \mathcal{H} = (\bigcup \mathcal{H})^E.$$

Moreover, the closure operator $^E\colon \mathrm{Rel}(A) \to \mathrm{Rel}(A)$ is algebraic.

We can also give the following description of R^E.

Proposition 2.2.11. *If A is a set, $R \in \mathrm{Rel}(A)$, and $x, y \in A$, then*

$$\begin{aligned} (x,y) \in R^E \quad & \textit{iff} \quad (\exists n \in \mathbb{N}_0)(\exists c_0, \ldots, c_n \in A) c_0 = x, c_n = y, \\ & \textit{and} \quad c_i R c_{i+1} \textit{ or } c_{i+1} R c_i \textit{ for all } 0 \leq i \leq n-1. \end{aligned}$$

□

2.3 Algebras

General Concepts. For a nonempty set A and a nonnegative integer $n \in \mathbb{N}_0$, we define $A^0 = \{\emptyset\}$ and for $n > 0$, A^n is the set of n-tuples of elements from A. An *n-ary operation* on A is any map f from A^n to A; n is the *arity* (or rank) of f. A *finitary operation* is an n-ary operation for some n. The image of $(a_1, \ldots, a_n)$ under an n-ary operation f is denoted by $f(a_1, \ldots, a_n)$. A map f on A is called a *constant* if its arity is zero. It is completely determined by the image $f(\emptyset)$ in A. Hence, it is convenient to identify it with this element of A. An operation f on A is *unary, binary* or *ternary* if its arity is 1, 2, or 3, respectively.

A *language* (or *type*) of algebras is a set Σ of *function symbols* such that a nonnegative integer n is assigned to each member f of Σ. This integer is called the *arity* (or *rank*) of f, and f is said to be an *n-ary function symbol*. The subset of n-ary function symbols in Σ is denoted by Σ_n.

Let Σ be a set of function symbols. A *Σ-algebra* is an ordered pair $\mathcal{A} = (A, F)$, where A is a set and F is a family of finitary operations of A indexed by the language Σ such that corresponding to each n-ary function symbol $f \in \Sigma$ there is an n-ary operation $f^{\mathcal{A}}$ on A. The set A is called the *universe* of $\mathcal{A}$ and the $f^{\mathcal{A}}$'s are called the *fundamental operations of* $\mathcal{A}$. Usually we write (A, Σ) instead of (A, F) and we often drop the upper index from $f^{\mathcal{A}}$.

Let $\mathcal{A} = (A, \Sigma)$ and $\mathcal{B} = (B, \Sigma)$ be two Σ-algebras. A *homomorphism* (or *morphism*) from $\mathcal{A}$ to $\mathcal{B}$ is a mapping $\varphi: A \to B$ such that

$$\varphi(f^{\mathcal{A}}(a_1, \ldots, a_n)) = f^{\mathcal{B}}(\varphi(a_1), \ldots, \varphi(a_n))$$

holds for all $n \geq 0$, $f \in \Sigma_n$ and $a_1, \ldots, a_n \in A$.

If φ is injective, then it is called an *embedding*. The map φ is an *isomorphism* if φ is injective and onto. We say that $\mathcal{A}$ is *isomorphic* to $\mathcal{B}$, denoted by $\mathcal{A} \cong \mathcal{B}$, if there is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$.

Lemma 2.3.1. *The composition of homomorphisms is again a homomorphism, and similar statements apply for embeddings and isomorphisms. Furthermore, the inverse of an isomorphism is an isomorphism.* □

Let $\mathcal{A} = (A, \Sigma)$ be a Σ-algebra and let $\Theta \in \mathrm{Eq}(A)$. Then Θ is a *congruence* on $\mathcal{A}$ if Θ satisfies for each n-ary function symbol $f \in \Sigma$ and any elements $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$ the following *compatibility property*:

$$\text{if } a_i \Theta b_i \text{ for all } 1 \leq i \leq n, \text{ then } f^{\mathcal{A}}(a_1, \ldots, a_n) \Theta f^{\mathcal{A}}(b_1, \ldots, b_n).$$

If Θ is a congruence on an algebra $\mathcal{A}$, then the *quotient algebra of $\mathcal{A}$ modulo Θ*, denoted by $\mathcal{A}/\Theta$, is the algebra whose universe is A/Θ and whose fundamental operations satisfy

$$f^{\mathcal{A}/\Theta}(a_1/\Theta, \ldots, a_n/\Theta) = f^{\mathcal{A}}(a_1, \ldots, a_n)/\Theta$$

where $a_1, \ldots, a_n \in A$ and f is an n-ary function symbol in Σ. Note that the quotient algebras of $\mathcal{A}$ are of the same type as $\mathcal{A}$. The set of all congruences on an algebra $\mathcal{A}$ is denoted by $\mathrm{Con}(\mathcal{A})$.

Proposition 2.3.2. $(\mathrm{Con}(\mathcal{A}), \subseteq)$ *is a complete sublattice of* $(\mathrm{Eq}(A), \subseteq)$. □

Thus, the congruence lattice of $\mathcal{A}$ is the lattice whose universe is $\mathrm{Con}(\mathcal{A})$, and joins and meets are the same as when working with equivalence relations.

Lemma 2.3.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be Σ-algebras.*

(a) *The kernel Θ_f of any morphism $f\colon \mathcal{A} \to \mathcal{B}$ is a congruence on $\mathcal{A}$.*

(b) *If Θ is a congruence on $\mathcal{A}$, then the canonical map v_Θ is a homomorphism from $\mathcal{A}$ onto the quotient algebra $\mathcal{A}/\Theta$.* □

In the literature the following Homomorphism Theorem is also referred to as "The First Isomorphism Theorem".

Theorem 2.3.4. (Homomorphism Theorem) *Let $\mathcal{A} = (A, \Sigma)$ and $\mathcal{B} = (B, \Sigma)$ be two Σ-algebras, and let $f\colon \mathcal{A} \to \mathcal{B}$ be a homomorphism onto B. Then the map $\varphi\colon A/\Theta_f \to B$, $a/\Theta_f \mapsto f(a)$ is an isomorphism between $\mathcal{A}/\Theta_f$ and $\mathcal{B}$. Furthermore, if $v_{(\Theta_f)}$ denotes the canonical map from A to A/Θ_f, then the kernel of $v_{(\Theta_f)}$ is Θ_f and the diagram in Figure 2 commutes, that is, $f = \varphi \circ v_{(\Theta_f)}$.* □

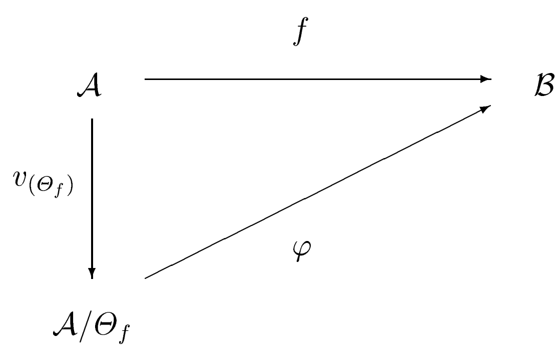


Figure 2.

Lattices and Semilattices as Algebras. In Section 2.2 we saw that for a lattice L we may define the binary operations join and meet on L by

$$a \vee b = \sup\{a, b\} \quad \text{and} \quad a \wedge b = \inf\{a, b\}$$

for all $a, b \in L$. Next we present the algebraic properties of the operations $\vee$ and $\wedge$. First we note the connections between $\vee$, $\wedge$, and $\leq$.

Lemma 2.3.5. *If $(L, \leq)$ is a lattice and $a, b \in L$, then the following are equivalent:*

(a) *$a \leq b$;*

(b) *$a \vee b = b$;*

(c) *$a \wedge b = a$.* □

The next proposition presents the characteristic properties of the operations $\vee$ and $\wedge$.

Proposition 2.3.6. *If $(L, \leq)$ is a lattice, then $\vee$ and $\wedge$ satisfy for all $a, b, c \in L$,*

(L1)	$(a \vee b) \vee c = a \vee (b \vee c)$	
(L1)$^\partial$	$(a \wedge b) \wedge c = a \wedge (b \wedge c)$	(associative laws)
(L2)	$a \vee b = b \vee a$	
(L2)$^\partial$	$a \wedge b = b \wedge a$	(commutative laws)
(L3)	$a \vee a = a$	
(L3)$^\partial$	$a \wedge a = a$	(idempotency laws)
(L4)	$a \vee (a \wedge b) = a$	
(L4)$^\partial$	$a \wedge (a \vee b) = b$	(absorption laws). $\square$

We say that an algebra $(L, \vee, \wedge)$ is a *lattice*, if L is nonempty set and $\vee$ and $\wedge$ are binary operations on L which satisfy (L1)–(L4) and (L1)$^\partial$–(L4)$^\partial$.

If an ordered set $(L, \leq)$ is a lattice, then by Proposition 2.3.6 the algebra $(L, \vee, \wedge)$ is a lattice. Similarly, if an algebra $(L, \vee, \wedge)$ is a lattice and we set $a \leq b$ if and only if $a \vee b = b$ ($a, b \in L$), then the ordered set $(L, \leq)$ is a lattice in which the original operations agree with the induced operations, that is, $a \vee b = \sup\{a, b\}$ and $a \wedge b = \inf\{a, b\}$.

Let $(L, \vee, \wedge)$ be a lattice. We say that L has a *unit* (or *identity*) element if there exists an element $1 \in L$ such that $a \wedge 1 = a$ for all $a \in L$. Dually, L is said to have a *zero* if there exists a $0 \in L$ such that $a = a \vee 0$ for all $a \in L$. The lattice $(L, \vee, \wedge)$ has a unit if and only if $(L, \leq)$ has a top element $\top$ and in that case $1 = \top$. A dual statement holds for 0 and $\bot$. A lattice $(L, \vee, \wedge)$ possessing a 0 and a 1 is obviously bounded.

Also semilattices may be defined both as algebras and as ordered sets. A *semilattice* is an algebra $(P, \circ)$, where $\circ$ is an associative, commutative and idempotent binary operation.

The different notions of semilattices are related as follows. Let $(P, \circ)$ be a semilattice. The condition

$$a \leq b \text{ if and only if } a \circ b = b$$

defines a partial order $\leq$ on P such that $(P, \leq)$ is a join-semilattice and $a \vee b = a \circ b$. Similarly, the condition

$$a \leq b \text{ if and only if } a \circ b = a$$

defines a partial order $\leq$ on P such that $(P, \leq)$ is a meet-semilattice and $a \wedge b = a \circ b$. Conversely, if $(P, \leq)$ is a join-semilattice, then $(P, \vee)$ is a semilattice as an algebra, and an analogical statement holds for meet-semilattices. Moreover, an ordered set $(P, \leq)$ is a join-semilattice if and only if $(P, \geq)$ is a meet-semilattice.

We adopt the convention that in a semilattice denoted by $(P, \vee)$ the order relation is defined by $a \leq b$ iff $a \vee b = b$, but in a semilattice denoted by $(P, \wedge)$ the order relation is defined by $a \leq b$ iff $a \wedge b = a$.

Example 2.3.7. If A is a set, then the algebra $(\wp(A), \cup)$ is a semilattice. Because $B \subseteq C$ if and only if $B \cup C = C$, the corresponding join- and meet-semilattices are $(\wp(A), \subseteq)$ and $(\wp(A), \supseteq)$, respectively. Similarly, the algebra $(\wp(A), \cap)$ is a semilattice and the corresponding join- and meet-semilattices are $(\wp(A), \supseteq)$ and $(\wp(A), \subseteq)$.

If Θ is a congruence on a semilattice $\mathcal{P} = (P, \vee)$, then the quotient semilattice $\mathcal{P}/\Theta$ is a semilattice such that the join of a/Θ and b/Θ is $(a \vee b)/\Theta$. In the corresponding join-semilattice $(P/\Theta, \leq)$ the order relation $\leq$ is defined by

$$a/\Theta \leq b/\Theta \iff (a \vee b)/\Theta = b/\Theta.$$

A subset S of an ordered set $(P, \leq)$ is called *convex*, if $x \leq z \leq y$ implies $z \in S$ whenever $x, y \in S$. It is well-known and obvious that every congruence class of a congruence on a semilattice is a convex subset.

Example 2.3.8. If Θ is a congruence on $(\wp(A), \cup)$, the operator $\vee$ of the quotient semilattice $(\wp(A)/\Theta, \vee)$ is defined by $B/\Theta \vee C/\Theta = (B \cup C)/\Theta$. The order relation $\leq$ on $\wp(A)/\Theta$ is defined by

$$B/\Theta \leq C/\Theta \text{ if and only if } (B \cup C)/\Theta = C/\Theta.$$

The following proposition is usually presented for lattices (see [5], for example), but it holds also for semilattices.

Proposition 2.3.9. *If $(P, \leq)$ and $(Q, \leq)$ are join-semilattices and $\varphi \colon P \to Q$ is a map, then the following are equivalent.*

(a) *φ is an order-isomorphism between the ordered sets $(P, \leq)$ and $(Q, \leq)$.*

(b) *φ is an isomorphism between the algebras $(P, \vee)$ and $(Q, \vee)$.*

Proof. Obviously, φ is a bijection in both cases. By Lemma 2.2.6 it is clear that (a) implies (b).

Conversely, suppose (b) holds. If $a \leq b$, then $\varphi(a) \vee \varphi(b) = \varphi(a \vee b) = \varphi(b)$. Thus $\varphi(a) \leq \varphi(b)$ by Lemma 2.3.5. On the other hand, if $\varphi(a) \leq \varphi(b)$, then $\varphi(b) = \varphi(a) \vee \varphi(b) = \varphi(a \vee b)$. Because φ is a bijection, this implies $b = a \vee b$ and $a \leq b$. □

Chapter 3

Complete Congruences and Morphisms of Semilattices

3.1 Closure Operators on Ordered Sets

In this section we consider closure operators on ordered sets, and particularly on complete join-semilattices. Ward [55] has shown that if $\mathcal{P} = (P, \leq)$ is a complete lattice, then the pointwise defined meet of any set of closure operators on $\mathcal{P}$ is again a closure operator. This means that the set of all closure operators on a complete lattice is a complete lattice with respect to the pointwise order. We generalize this result by showing that if $\mathcal{P}$ is a complete join-semilattice, then the set of all closure operators on $\mathcal{P}$ is a complete lattice. Moreover, we describe the joins in this complete lattice in a new way by using the Knaster–Tarski Fixpoint Theorem. We conclude this section by describing as a special case the join of continuous closure operators by applying Kleene's Fixpoint Theorem.

Definition. Let $\mathcal{P} = (P, \leq)$ be an ordered set. Then a function $c\colon P \to P$ is called a *closure operator* on $\mathcal{P}$, if for all $a, b \in P$,

(a) $a \leq c(a)$, (extensive)
(b) $c(c(a)) = c(a)$, and (idempotent)
(c) $a \leq b$ implies $c(a) \leq c(b)$. (order-preserving)

An element $a \in P$ is called *closed* if $c(a) = a$. The set of all closed elements of P is denoted by P_c.

In the next lemma we present some basic properties of closure operators. Equality (a) can be found in [5] and conditions (d) and (e) are presented in [12, 36].

Lemma 3.1.1. *If $c\colon P \to P$ and $k\colon P \to P$ are closure operators on an ordered set $(P, \leq)$ and $S \subseteq P_c$, then*

(a) $P_c = \{c(a) \mid a \in P\}$*;*

(b) *if $P_c = P_k$, then $c = k$;*

(c) $c(x) = \bigwedge_P \{a \in P_c \mid x \leq a\}$ *for any $x \in P$;*

(d) *if $\bigvee S$ exists in P, then $\bigvee S$ exists in P_c and $\bigvee_{P_c} S = c(\bigvee_P S)$;*

(e) *if $\bigwedge S$ exists in P, then $\bigwedge S$ exists in P_c and $\bigwedge_{P_c} S = \bigwedge_P S$.*

Proof. (b) If $P_c = P_k$, then $c(x) \in P_k$ for all $x \in P$, which implies $k(x) \leq k(c(x)) = c(x)$. Similarly, $k(x) \in P_c$ implies $c(x) \leq k(x)$. So, $c(x) = k(x)$ for all $x \in P$.

(c) If $x \leq a \in P_c$, then $c(x) \leq c(a) = a$, which shows that $c(x)$ is a lower bound of $\{a \in P_c \mid x \leq a\}$. Since $c(x)$ itself is in $\{a \in P_c \mid x \leq a\}$, this implies that $c(x) = \bigwedge_P \{a \in P_c \mid x \leq a\}$. □

This lemma has the following immediate consequences.

Corollary 3.1.2. *Let $\mathcal{P} = (P, \leq)$ be an ordered set and let $c\colon P \to P$ be a closure operator.*

(a) *If $\mathcal{P}$ is a join-semilattice, then $(P_c, \leq)$ is a join-semilattice such that*

$$a \vee b = c(a \vee_P b)$$

for all $a, b \in P_c$.

(b) *If $\mathcal{P}$ is a meet-semilattice, then $(P_c, \leq)$ is a meet-semilattice such that*

$$a \wedge b = a \wedge_P b$$

for all $a, b \in P_c$.

(c) *If $\mathcal{P}$ is a complete join-semilattice, then $(P_c, \leq)$ is a complete join-semilattice such that*

$$\bigvee S = c(\bigvee_P S)$$

for all $\emptyset \neq S \subseteq P_c$.

(d) *If $\mathcal{P}$ is a complete meet-semilattice, then $(P_c, \leq)$ is a complete meet-semilattice such that*

$$\bigwedge S = \bigwedge_P S$$

for all $\emptyset \neq S \subseteq P_c$.

In particular, if $(P, \leq)$ is a lattice, then $(P_c, \leq)$ is lattice in which

$$a \vee b = c(a \vee_P b) \quad \text{and} \quad a \wedge b = a \wedge_P b$$

for all $a, b \in P_c$. Similarly, if $(P, \leq)$ is a complete lattice, then $(P_c, \leq)$ is a complete lattice in which

$$\bigvee S = c(\bigvee_P S) \quad \text{and} \quad \bigwedge S = \bigwedge_P S$$

for all $S \subseteq P_c$ (cf. [2, 55], for example).

If X is any set and $\mathcal{P} = (P, \leq)$ is an ordered set, we may order the set P^X of all maps from X to P by the *pointwise order:*

$$f \leq g \text{ in } P^X \text{ if and only if for all } x \in X,\ f(x) \leq g(x) \text{ in } P.$$

We denote the ordered set $(P^X, \leq)$ by $\mathcal{P}^X$. It inherits some properties of $\mathcal{P}$ listed in the next obvious lemma.

Lemma 3.1.3. *Let X be a set and $\mathcal{P} = (P, \leq)$ an ordered set.*

(a) *If $\mathcal{P}$ is a join-semilattice, then $\mathcal{P}^X$ is a join-semilattice such that for all $\varphi_1, \varphi_2 \in P^X$ and $x \in X$,*

$$(\varphi_1 \vee \varphi_2)(x) = \varphi_1(x) \vee \varphi_2(x).$$

(b) *If $\mathcal{P}$ is a meet-semilattice, then $\mathcal{P}^X$ is a meet-semilattice such that for all $\varphi_1, \varphi_2 \in P^X$ and $x \in X$,*

$$(\varphi_1 \wedge \varphi_2)(x) = \varphi_1(x) \wedge \varphi_2(x).$$

(c) *If $\mathcal{P}$ is a complete join-semilattice, then $\mathcal{P}^X$ is a complete join-semilattice such that for $\emptyset \neq \{\varphi_i\}_{i \in I} \subseteq P^X$ the supremum $\varphi = \bigvee_{i \in I} \varphi_i$ is defined so that for any $x \in X$,*

$$\varphi(x) = \bigvee_{i \in I} \varphi_i(x).$$

(d) *If $\mathcal{P}$ is a complete meet-semilattice, then $\mathcal{P}^X$ is a complete meet-semilattice such that for $\emptyset \neq \{\varphi_i\}_{i \in I} \subseteq P^X$ the infimum $\varphi = \bigwedge_{i \in I} \varphi_i$ is defined so that for any $x \in X$*

$$\varphi(x) = \bigwedge_{i \in I} \varphi_i(x).$$

□

It is now clear that if $\mathcal{P}$ is a lattice, then $\mathcal{P}^X$ is a lattice in which

$$(\varphi_1 \vee \varphi_2)(x) = \varphi_1(x) \vee \varphi_2(x) \quad \text{and} \quad (\varphi_1 \wedge \varphi_2)(x) = \varphi_1(x) \wedge \varphi_2(x).$$

Similarly, if $\mathcal{P}$ is a complete lattice, then $\mathcal{P}^X$ is a complete lattice in which

$$(\bigvee_{i \in I} \varphi_i)(x) = \bigvee_{i \in I} \varphi_i(x) \quad \text{and} \quad (\bigwedge_{i \in I} \varphi_i)(x) = \bigwedge_{i \in I} \varphi_i(x).$$

Let $(P, \leq)$ be an ordered set. We denote by $\langle P \to P \rangle$ the set of all extensive and order-preserving maps $f \colon P \to P$. It is clear that the identity map $1_P \colon x \mapsto x$ of P is the least element of $\langle P \to P \rangle$ with respect to the order of $\mathcal{P}^P$. On the other hand, if $\mathcal{P}$ has a top element $\top$, then the map $f_\top \colon x \mapsto \top$ is the greatest element of $\langle P \to P \rangle$.

Lemma 3.1.4. *Let $\mathcal{P} = (P, \leq)$ be an ordered set.*

(a) *If $\mathcal{P}$ is a join-semilattice, then $\langle P \to P \rangle$ is a join-subsemilattice of $\mathcal{P}^P$.*

(b) *If $\mathcal{P}$ is a meet-semilattice, then $\langle P \to P \rangle$ is a meet-subsemilattice of $\mathcal{P}^P$.*

(c) *If $\mathcal{P}$ is a complete join-semilattice, then $\langle P \to P \rangle$ is a complete join-subsemilattice of $\mathcal{P}^P$. Moreover, $(\langle P \to P \rangle, \leq)$ is a complete lattice.*

(d) *If $\mathcal{P}$ is a complete meet-semilattice, then $\langle P \to P \rangle$ is a complete meet-subsemilattice of $\mathcal{P}^P$.*

Proof. We prove (c). Statements (a), (b), and (d) can be proved similarly.

Assume that the ordered set $\mathcal{P} = (P, \leq)$ is a complete join-semilattice. Then by Lemma 3.1.3, $\mathcal{P}^P$ is a complete join-semilattice. Let $\emptyset \neq \{\varphi_i\}_{i \in I} \subseteq \langle P \to P \rangle$. We denote $\varphi = \bigvee_{\mathcal{P}^P} \{\varphi_i \mid i \in I\}$. Because each φ_i is extensive, $x \leq \varphi_i(x)$ for all $i \in I$ which implies $x \leq \bigvee_{\mathcal{P}^P} \{\varphi_i(x) \mid i \in I\} = \varphi(x)$. Moreover, it is known that the join of order-preserving maps is again order-preserving (see [5], for example). Hence, $\varphi \in \langle P \to P \rangle$. Because $\langle P \to P \rangle$ has a bottom element $1_P \colon x \mapsto x$, also $\bigvee \emptyset$ exists in $\langle P \to P \rangle$ and equals 1_P. This implies by Lemma 2.2.4 that $(\langle P \to P \rangle, \leq)$ is a complete lattice. $\square$

It is clear that if $\mathcal{P} = (P, \leq)$ is a lattice, then $\langle P \to P \rangle$ is a sublattice of $\mathcal{P}^P$, and if $\mathcal{P}$ is a complete lattice, then $\langle P \to P \rangle$ is a complete sublattice of $\mathcal{P}^P$.

Let us denote by $\mathrm{Clo}(\mathcal{P})$ the set of all closure operators on $\mathcal{P} = (P, \leq)$. Because $\mathrm{Clo}(\mathcal{P}) \subseteq P^P$, $\mathrm{Clo}(\mathcal{P})$ has an order inherited from $\mathcal{P}^P$. Obviously, $1_P \colon x \mapsto x$ is the least element in $\mathrm{Clo}(\mathcal{P})$, and if $\mathcal{P}$ has a top element $\top$, then $f_\top \colon x \mapsto \top$ is the greatest element in $\mathrm{Clo}(\mathcal{P})$.

As we already mentioned, Ward [55] has shown that for a complete lattice $\mathcal{P}$, the ordered set $(\mathrm{Clo}(\mathcal{P}), \leq)$ is a complete lattice. Our next lemma shows that analogous statements hold for meet-semilattices and complete meet-semilattices.

Lemma 3.1.5. *Let $\mathcal{P} = (P, \leq)$ be an ordered set.*

(a) *If $\mathcal{P}$ is a meet-semilattice, then $\mathrm{Clo}(\mathcal{P})$ is a meet-semilattice with respect to the pointwise order.*

(b) *If $\mathcal{P}$ is a complete meet-semilattice, then $\mathrm{Clo}(\mathcal{P})$ is a complete meet-semilattice with respect to the pointwise order.*

Proof. (b) Suppose $\mathcal{P}$ is a complete meet-semilattice and $\emptyset \neq H \subseteq \mathrm{Clo}(\mathcal{P})$. We will show that $c = \bigwedge_{P^P} H$ belongs to $\mathrm{Clo}(\mathcal{P})$. By the previous lemma c is extensive and order-preserving. It is also clear that $c(x) \leq c(c(x))$ for all $x \in P$. Let $g \in H$ and $x \in P$. Then $c(x) = \bigwedge_{f \in H} f(x) \leq g(x)$. Because $c(c(x)) = \bigwedge_{f \in H} f(c(x))$, we get $c(c(x)) \leq g(c(x)) \leq g(g(x)) = g(x)$. Hence, $c(c(x)) \leq \bigwedge_{f \in H} f(x) = c(x)$. Thus, $c(c(x)) = c(x)$ and hence c is a closure operator. Assertion (a) may be proved analogously. □

The ordered set $(\mathrm{Clo}(\mathcal{P}), \leq)$ is not necessarily a join-semilattice even if $\mathcal{P}$ is a lattice and a complete meet-semilattice, as we see in the following example.

Example 3.1.6. Let us consider the ordered set $\mathcal{P} = (\mathbb{N}, \leq)$. It is well-known that $\mathcal{P}$ is a lattice in which $m \vee n = \max\{m, n\}$ and $m \wedge n = \min\{m, n\}$ for all $m, n \in \mathbb{N}$. Furthermore, $\mathcal{P}$ is a complete meet-semilattice such that $\bigwedge S = \min S$ for any $\emptyset \neq S \subseteq \mathbb{N}$.

Let us define two closure operators c_1 and c_2 on $\mathcal{P}$ by

$$c_1(n) = \begin{cases} n+1 & \text{if } n \text{ is odd,} \\ n & \text{otherwise;} \end{cases}$$

and

$$c_2(n) = \begin{cases} n+1 & \text{if } n \text{ is even,} \\ n & \text{otherwise.} \end{cases}$$

It is easy to see that $c_1 \wedge c_2 = 1_{\mathbb{N}}$ in $\mathrm{Clo}(\mathcal{P})$, but $(\mathrm{Clo}(\mathcal{P}), \leq)$ is not a join-semilattice because there is no $c \in \mathrm{Clo}(\mathcal{P})$ such that $c_1, c_2 \leq c$.

Next we intend to present a condition under which $(\mathrm{Clo}(\mathcal{P}), \leq)$ is a complete lattice. It is done with the help of fixpoints. An element $a (\in P)$ is a *fixpoint* of a mapping $f \colon P \to P$ if $f(a) = a$. If $(P, \leq)$ is an ordered set and $f \colon P \to P$ has a *least fixpoint*, i.e., a fixpoint a such that $a \leq x$ for all fixpoints x of f, we denote this by $\mu(f)$. The following well-known result can be found in [5], for example.

Theorem 3.1.7. (Knaster–Tarski Fixpoint Theorem) *If $(P, \leq)$ is a complete lattice, then every order-preserving map $f\colon P \to P$ has fixpoints. In particular, f has a least fixpoint and*

$$\mu(f) = \bigwedge\{x \in P \mid f(x) \leq x\}.$$

□

Let $(P, \leq)$ be an ordered set. For each $x \in P$, we denote $[x) = \{y \in P \mid x \leq y\}$. If $(P, \leq)$ is a lattice, then $[x)$ is called a *principal filter*. Obviously, $[x)$ inherits the order from $(P, \leq)$ and if $(P, \leq)$ is a complete join-semilattice, then $([x), \leq)$ is a complete lattice which has the bottom element x. For any extensive map $f\colon P \to P$, let $f_x\colon [x) \to [x)$ be the restriction of f to $[x)$.

Our next proposition shows how we can find for an extensive and order-preserving map f the smallest closure operator above f.

Proposition 3.1.8. *Let $\mathcal{P} = (P, \leq)$ be a complete join-semilattice and suppose $f \in \langle P \to P\rangle$. The function $\overline{f}\colon P \to P$ defined so that $\overline{f}(x) = \mu(f_x)$, for all $x \in P$, is the smallest closure operator above f in $\mathcal{P}^P$.*

Proof. Let $x \in P$. Because $\mathcal{P}$ is a complete join-semilattice, $([x), \leq)$ is a complete lattice. This implies by the Knaster-Tarski Fixpoint Theorem that the function $f_x\colon [x) \to [x)$ has a least fixpoint $\mu(f_x)$. Next we show that $\overline{f}\colon P \to P, x \mapsto \mu(f_x)$, is a closure operator.

It is obvious that $\mu(f_x) \in [x)$, which implies $x \leq \overline{f}(x)$. If $x \leq y$, then $f_x(a) = f_y(a)$ for all $a \in [y)$. Since $\alpha = \overline{f}(y)$ is a fixpoint of f_y and $\alpha \in [y)$, we obtain $f_x(\alpha) = f_y(\alpha) = \alpha$. Thus, α is also a fixpoint of f_x, which implies $\overline{f}(x) \leq \alpha = \overline{f}(y)$. It is clear that $f_{\overline{f}(x)}(a) = f_x(a)$ for all $a \in \big[\overline{f}(x)\big)$ and $\overline{f}(x)$ is the least element in $\big[\overline{f}(x)\big)$. Hence, $f_{\overline{f}(x)}(\overline{f}(x)) = f_x(\overline{f}(x)) = \overline{f}(x)$ and thus $\overline{f}(x)$ is the least fixpoint of $f_{\overline{f}(x)}$. This gives $\overline{f}(\overline{f}(x)) = \mu(f_{\overline{f}(x)}) = \overline{f}(x)$.

Because $x \leq \overline{f}(x)$, we get $f(x) = f_x(x) \leq f_x(\overline{f}(x)) = \overline{f}(x)$, which implies that $\overline{f}$ is above f in $\mathcal{P}^P$. Suppose c is a closure operator which is above f, that is, $f(x) \leq c(x)$ for all $x \in P$. Thus, $f(c(x)) \leq c(c(x)) = c(x) \leq f(c(x))$ which implies $f(c(x)) = c(x)$ and especially $f_x(c(x)) = c(x)$. Then $c(x)$ is a fixpoint of f_x and hence $\overline{f}(x) \leq c(x)$. □

By our next lemma the map $f \mapsto \overline{f}$ is a closure operator.

Lemma 3.1.9. *If $\mathcal{P} = (P, \leq)$ is a complete join-semilattice, then the map*

$$\overline{}\colon \langle P \to P \rangle \to \langle P \to P \rangle, f \mapsto \overline{f}$$

is a closure operator.

Proof. It is clear that $f \leq \overline{f}$ and $\overline{\overline{f}} = \overline{f}$. Suppose $f \leq g$. Then $f(x) \leq g(x)$ for all $x \in P$. Let $x \in P$. Because $\overline{g}(x) \in [x)$, we have $\overline{g}(x) \leq f_x(\overline{g}(x)) = f(\overline{g}(x)) \leq g(\overline{g}(x)) = g_x(\overline{g}(x)) = \overline{g}(x)$. Thus, $f_x(\overline{g}(x)) = \overline{g}(x)$ and hence $\overline{g}(x)$ is a fixpoint of f_x, which implies $\overline{f}(x) \leq \overline{g}(x)$. Therefore $\overline{f} \leq \overline{g}$ holds. □

Next we present a proposition which generalizes the result of Ward by showing that $(\mathrm{Clo}(\mathcal{P}), \leq)$ is a complete lattice whenever $\mathcal{P}$ is a complete join-semilattice.

Proposition 3.1.10. *If $\mathcal{P} = (P, \leq)$ is a complete join-semilattice, then $(\mathrm{Clo}(\mathcal{P}), \leq)$ is a complete lattice in which*

$$\bigvee H = \overline{\textstyle\bigvee_{P^P} H} \quad \textit{and} \quad \bigwedge H = \textstyle\bigwedge_{P^P} H$$

for all $\emptyset \neq H \subseteq \mathrm{Clo}(\mathcal{P})$. Moreover, $1_P\colon x \mapsto x$ is the least element and $f_\top\colon x \mapsto \top$ is the greatest element of $(\mathrm{Clo}(\mathcal{P}), \leq)$.

Proof. We have seen that $(\mathrm{Clo}(\mathcal{P}), \leq)$ has the bottom element $x \mapsto x$. Since $\mathcal{P}$ is a complete join-semilattice, $\mathcal{P}$ has the top element $\top = \bigvee P$. This implies that $(\mathrm{Clo}(\mathcal{P}), \leq)$ has the top element $f_\top\colon x \mapsto \top$. If $\emptyset \neq H \subseteq \mathrm{Clo}(\mathcal{P})$, then $\bigvee_{P^P} H$ is extensive and order-preserving by Lemma 3.1.4. This implies by Proposition 3.1.8 that $\bigvee H = \overline{\bigvee_{P^P} H}$ in $(\mathrm{Clo}(\mathcal{P}), \leq)$.

Obviously, H has a lower bound $x \mapsto x$ in $\mathrm{Clo}(\mathcal{P})$. Because $\mathcal{P}$ and hence $\mathcal{P}^P$ are complete join-semilattices, $\bigwedge_{P^P} H$ exists by Lemma 2.2.3. By the proof of Proposition 3.1.5, $\bigwedge_{P^P} H$ is a closure operator and hence $\bigwedge H = \bigwedge_{P^P} H$ in $\mathrm{Clo}(\mathcal{P})$. □

Next we present two examples concerning closure operators.

Example 3.1.11. If we define the order $\leq$ on $P = \{1, 2, 3\}$ so that $1 < 3$ and $2 < 3$, and 1 and 2 are incomparable, then $\mathcal{P} = (P, \leq)$ is a finite join-semilattice and $\mathrm{Clo}(\mathcal{P})$ contains four elements:

$$\begin{aligned}
c_1&\colon \quad 1 \mapsto 1, 2 \mapsto 2, 3 \mapsto 3;\\
c_2&\colon \quad 1 \mapsto 1, 2 \mapsto 3, 3 \mapsto 3;\\
c_3&\colon \quad 1 \mapsto 3, 2 \mapsto 2, 3 \mapsto 3;\\
c_4&\colon \quad 1 \mapsto 3, 2 \mapsto 3, 3 \mapsto 3.
\end{aligned}$$

The Hasse diagram of $(\mathrm{Clo}(\mathcal{P}), \leq)$ is given in Figure 3.

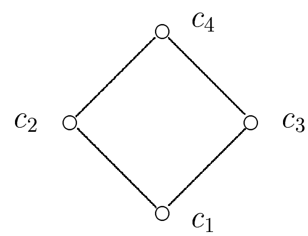


Figure 3.

Example 3.1.12. Let us consider the set $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$, in which the order relation $\leq$ is defined by

$$n \leq m \quad \iff \quad n \leq m \text{ holds in } \mathbb{N} \text{ or } m = \infty.$$

It is clear that $\mathcal{P} = (\mathbb{N}_\infty, \leq)$ is a complete lattice. The closure operators c_1 and c_2 are defined on $\mathcal{P}$ by

$$c_1(n) = \begin{cases} n+1 & \text{if } n (\in \mathbb{N}) \text{ is odd,} \\ n & \text{if } n (\in \mathbb{N}) \text{ is even,} \\ \infty & \text{if } n = \infty; \end{cases}$$

and

$$c_2(n) = \begin{cases} n+1 & \text{if } n (\in \mathbb{N}) \text{ is even,} \\ n & \text{if } n (\in \mathbb{N}) \text{ is odd,} \\ \infty & \text{if } n = \infty. \end{cases}$$

Now $f = c_1 \vee_{P^P} c_2$ is defined by

$$f(n) = \begin{cases} \infty & \text{if } n = \infty, \\ n+1 & \text{otherwise.} \end{cases}$$

The map f is not a closure operator and thus $\mathrm{Clo}(\mathcal{P})$ is not a join-subsemilattice of $\mathcal{P}^P$. For all $x \in \mathbb{N}_\infty$, the function $f_x\colon [x) \to [x)$ is defined by

$$f_x(n) = \begin{cases} \infty & \text{if } n = \infty, \\ n+1 & \text{otherwise.} \end{cases}$$

It is obvious that ∞ is the only fixpoint of f_x for all $x \in \mathbb{N}_\infty$. Thus, the map $\overline{f}\colon x \mapsto \infty$ is the join of c_1 and c_2 in $\mathrm{Clo}(\mathcal{P})$.

Next we consider a special type of maps $f \in \langle P \to P \rangle$ such that $\overline{f}\colon P \to P$ has a more constructive description. Let $(P, \leq)$ be an ordered set and $\emptyset \neq S \subseteq P$. Then S is said to be *directed* if $F^u \cap S \neq \emptyset$ for all finite $F \subseteq S$. The following lemma appears in [5], for example.

Lemma 3.1.13. *Let $(P, \leq)$ and $(Q, \leq)$ be ordered sets. If D is a directed subset of P and $f\colon P \to Q$ is an order-preserving map, then $f[D]$ is directed.* $\square$

An ordered set $(P, \leq)$ is a *CPO* (a *complete partially ordered set*) if (a) P has a bottom element and (b) $\bigvee D$ exists for every directed subset D of P. If we disregard (a), we say that $(P, \leq)$ is a *pre-CPO*. We use the special notation $\bigsqcup D$ in place of $\bigvee D$ when we want to emphasize that the set D is directed. It is obvious that each complete join-semilattice is a pre-CPO.

Let $(P, \leq)$ and $(Q, \leq)$ be pre-CPOs. Then $f\colon P \to Q$ is a *continuous* map if

$$f(\bigsqcup D) = \bigsqcup f[D]$$

for all directed $D(\subseteq P)$.

It is a well-known fact that each continuous map is order-preserving; namely, if $a \leq b$, then $\{a, b\}$ and $f[\{a, b\}]$ are directed, and hence $f(a) \leq f(a) \sqcup f(b) = f(a \sqcup b) = f(b)$.

Let $\mathcal{P} = (P, \leq)$ and $\mathcal{Q} = (Q, \leq)$ be pre-CPOs. We denote by $[P \to Q]$ the set of all continuous maps from $\mathcal{P}$ to $\mathcal{Q}$. It is now clear that $[P \to Q]$ has an order inherited from $\mathcal{Q}^P$, and if $\mathcal{Q}$ has a bottom element $\bot$, then $f_\bot\colon x \mapsto \bot$ is the bottom element of $[P \to Q]$.

It is known that $[P \to Q]$ is a pre-CPO and it is a CPO if $\mathcal{Q}$ is a CPO (see [5], for example). In the next lemma we present a version of this result for complete join-semilattices.

Lemma 3.1.14. *If $\mathcal{P} = (P, \leq)$ is a complete join-semilattice, then $[P \to P]$ is a complete join-semilattice with respect to the pointwise order.*

Proof. It suffices to show that $\varphi = \bigvee_{\mathcal{P}^P} \{\varphi_i \mid i \in I\}$ is continuous whenever $\emptyset \neq \{\varphi_i \mid i \in I\} \subseteq [P \to P]$. Let D be a directed subset of P. Then

$$\varphi(\bigsqcup D) = \bigvee_{i \in I} \varphi_i(\bigsqcup D) = \bigvee_{i \in I} (\bigsqcup_{x \in D} \varphi_i(x)),$$

because each φ_i is continuous. It is clear that $\varphi_i(x) \leq \varphi(x) \leq \bigsqcup_{x \in D} \varphi(x)$ for all $i \in I$ and $x \in D$ (note that $\{\varphi(x) \mid x \in D\}$ is directed, because φ is order-preserving). Thus, $\bigsqcup_{x \in D} \varphi_i(x) \leq \bigsqcup_{x \in D} \varphi(x)$ for all $i \in I$ and so, $\bigvee_{i \in I} (\bigsqcup_{x \in D} \varphi_i(x)) \leq \bigsqcup_{x \in D} \varphi(x)$. Hence,

$$\varphi(\bigsqcup D) = \bigvee_{i \in I} (\bigsqcup_{x \in D} \varphi_i(x)) \leq \bigsqcup_{x \in D} \varphi(x) = \bigsqcup \varphi[D].$$

Because $\varphi(x) \leq \varphi(\bigsqcup D)$ holds for any $x \in D$,

$$\bigsqcup \varphi[D] = \bigsqcup_{x \in D} \varphi(x) \leq \varphi(\bigsqcup D).$$

Now we have proved that $\varphi(\bigsqcup D) = \bigsqcup \varphi[D]$; it means that φ is continuous. □

The following well-known result can be found in [5], for example.

Theorem 3.1.15. (Kleene's Fixpoint Theorem) *Let $(P, \leq)$ be a CPO and let $f\colon P \to P$ be continuous. Then*

$$\mu(f) = \bigsqcup \{ f^i(\bot) \mid i \geq 0 \},$$

where $f^i(x)$ is defined by $f^0(x) = x$ and $f^{i+1}(x) = f(f^i(x))$ for all $i \geq 0$ and $x \in P$. □

By Kleene's Fixpoint Theorem and Proposition 3.1.8 we can write the following lemma, which describes the smallest closure operator above a continuous $f \in \langle P \to P \rangle$. Recall that every continuous map is order-preserving and therefore we may leave out the assumption "f is order-preserving" from our next lemma.

Lemma 3.1.16. *If $\mathcal{P} = (P, \leq)$ is a complete join-semilattice and $f\colon P \to P$ is a continuous and extensive map, then*

$$\overline{f}(x) = \bigsqcup \{ f^i(x) \mid i \geq 0 \}$$

for all $x \in P$.

Proof. Because $\mathcal{P}$ is a complete join-semilattice and $f \in \langle P \to P \rangle$, the function $\overline{f}\colon P \to P, x \mapsto \mu(f_x)$, is the smallest closure operator above f. Let $x \in P$. It is clear that $([x), \leq)$ is a complete lattice in which x is the bottom element, and that the map $f_x\colon [x) \to [x)$ is continuous. This implies that $\mu(f_x)$ can be obtained by using Kleene's Fixpoint Theorem. Since $f_x(a) = f(a)$ for all $a \in [x)$, we obtain

$$\overline{f}(x) = \mu(f_x) = \bigsqcup \{ (f_x)^i(x) \mid i \geq 0 \} = \bigsqcup \{ f^i(x) \mid i \geq 0 \}.$$

□

The previous lemma and Proposition 3.1.10 have the following corollary which describes the joins of subsets of continuous closure operators. Note that by Lemma 3.1.14, the map $\bigvee_{P^P} H$ is continuous for all $H \subseteq [P \to P]$, and by Lemma 3.1.4, $\bigvee_{P^P} H$ is extensive for all $H \subseteq \langle P \to P \rangle$.

Corollary 3.1.17. *Let $\mathcal{P} = (P, \leq)$ be a complete join-semilattice. If H is a nonempty subset of $\mathrm{Clo}(\mathcal{P})$ such that each $c \in H$ is continuous, then for the supremum of H in $(\mathrm{Clo}(\mathcal{P}), \leq)$,*

$$(\bigvee H)(x) = \bigsqcup\{(\bigvee_{P^P} H)^i(x) \mid i \geq 0\}$$

for all $x \in P$.

Example 3.1.18. It can be seen that the closure operators c_1 and c_2 defined in Example 3.1.12 are continuous. We form the join of c_1 and c_2 by applying Corollary 3.1.17. The join $c_1 \vee_{P^P} c_2$ is defined by

$$f(n) = \begin{cases} \infty & \text{if } n = \infty, \\ n+1 & \text{otherwise.} \end{cases}$$

It is easy to see that for all $i \in \mathbb{N}_0$,

$$f^i(n) = \begin{cases} \infty & \text{if } n = \infty, \\ n+i & \text{otherwise.} \end{cases}$$

Then $\bigsqcup\{f^i(n) \mid i \geq 0\} = \bigsqcup\{n, n+1, \ldots\} = \infty$ for all $n \in \mathbb{N}$ and $\bigsqcup\{f^i(\infty) \mid i \geq 0\} = \bigsqcup\{\infty\} = \infty$. Thus, the map $\mathbb{N}_\infty \to \mathbb{N}_\infty$, $x \mapsto \infty$ is the join of c_1 and c_2 in $(\mathrm{Clo}(\mathcal{P}), \leq)$, as we already saw in Example 3.1.12.

Not every closure operator in a complete join-semilattice is continuous as we shall see in the next example.

Example 3.1.19. Let us consider the set $\mathbb{N} \cup \{\top_1, \top_2\}$ in which an order relation $\leq$ is defined by

$$\begin{array}{lll} a \leq b & \text{iff} & a \leq b \text{ holds in } \mathbb{N} \\ & \text{or} & a \in \mathbb{N} \text{ and } b = \top_1 \\ & \text{or} & b = \top_2. \end{array}$$

Let c be the closure operator on the complete lattice $(\mathbb{N} \cup \{\top_1, \top_2\}, \leq)$ defined by

$$c(n) = \begin{cases} \top_2 & \text{if } n = \top_1 \text{ or } n = \top_2, \\ n & \text{if } n \in \mathbb{N}. \end{cases}$$

The map c is not continuous since for the directed subset $\mathbb{N}$ we get $c(\bigsqcup \mathbb{N}) = c(\top_1) = \top_2$ and $\bigsqcup c[\mathbb{N}] = \bigsqcup \mathbb{N} = \top_1$.

3.2 Complete Congruences on Semilattices

We continue now by considering complete congruences on semilattices. These are congruences Θ on a join-semilattice $\mathcal{P} = (P, \vee)$ such that each Θ-class has a greatest element. Complete congruence generalize congruences on finite semilattices, since for every congruence Θ on a finite semilattice $(P, \vee)$, the congruence class $x/\Theta = \{x_1, \ldots, x_n\}$ of any $x \in P$ has a greatest element $x_1 \vee \cdots \vee x_n$. We will show that complete congruences on a complete join-semilattice are exactly the equivalences which satisfy the complete $\vee$-compatibility property. The set of all complete congruences on $\mathcal{P}$ may be ordered by the set inclusion relation. We prove that this ordered set is isomorphic to the set of all closure operators on $(P, \leq)$. This implies that if $(P, \leq)$ is a complete join-semilattice, then the set of all complete congruences on $\mathcal{P}$ is a complete lattice with respect to the inclusion relation, but it is not necessarily a sublattice of $(\mathrm{Con}(\mathcal{P}), \subseteq)$. We show that if Θ is a complete congruence on $\mathcal{P}$, then the quotient semilattice $\mathcal{P}/\Theta$, ordered by $a/\Theta \leq b/\Theta$ if and only if $(a \vee b)/\Theta = b/\Theta$, is isomorphic to the set P_Θ of the greatest element of Θ-classes ordered by the same order as P. We also point out that if $(P, \leq)$ is additionally a lattice, a complete join-semilattice, a complete meet-semilattice, or a complete lattice, then so are $(P/\Theta, \leq)$ and $(P_\Theta, \leq)$. We end this section by describing the closure operator $\theta_c \colon \mathrm{Rel}(P) \to \mathrm{Rel}(P)$ which maps each $R \in \mathrm{Rel}(P)$ to the least complete congruence on the complete join-semilattice $(P, \leq)$ containing R.

Definition. A congruence Θ on a semilattice $(P, \vee)$ is a *complete congruence* if each congruence class x/Θ has a greatest element $c_\Theta(x)$.

Example 3.2.1. Let A be a set, $x \in A$, and let Θ be the congruence on the semilattice $(\wp(A), \cup)$, which has the congruence classes

$$\begin{aligned} C_1 &= \{X \subseteq A \mid x \in X\}; \\ C_2 &= \{X \subseteq A \mid x \notin X\}. \end{aligned}$$

The congruence Θ is complete, since C_1 has the greatest element A and C_2 has the greatest element $A - \{x\}$.

Note that Θ is also a congruence on the semilattice $(\wp(A), \cap)$ such that its congruence classes have a least element; C_1 has a least element $\{x\}$ and $\emptyset$ is the least element of C_2. In Section 3.5 we consider such congruences on a semilattice $(P, \wedge)$ that each congruence class has a least element.

In the next lemma we give some basic properties of complete congruences.

Lemma 3.2.2. *If Θ is a complete congruence on a semilattice $(P, \vee)$ and $x, y \in P$, then*
(a) $(x, c_\Theta(x)) \in \Theta$*;*
(b) $c_\Theta(x) = \bigvee x/\Theta$*;*
(c) $x \leq c_\Theta(x)$*;*
(d) $x \Theta y$ *if and only if* $c_\Theta(x) = c_\Theta(y)$*;*
(e) $c_\Theta(x) = c_\Theta(c_\Theta(x))$*;*
(f) $x \leq y$ *implies* $c_\Theta(x) \leq c_\Theta(y)$.

Proof. Statements (a), (b), (c), (d), and (e) are obvious. Because $x\Theta c_\Theta(x)$ and $y\Theta c_\Theta(y)$, also $(x \vee y)\Theta(c_\Theta(x) \vee c_\Theta(y))$ holds. Thus $c_\Theta(x) \vee c_\Theta(y) \leq c_\Theta(x \vee y)$. If $x \leq y$, then $c_\Theta(x) \leq c_\Theta(x) \vee c_\Theta(y) \leq c_\Theta(x \vee y) = c_\Theta(y)$, which proves (f). □

We denote by $\mathrm{Con_c}(\mathcal{P})$ the set of all complete congruences on a semilattice $\mathcal{P} = (P, \vee)$. It is obvious that $\mathrm{Con_c}(\mathcal{P}) \subseteq \mathrm{Con}(\mathcal{P})$, and $\mathrm{Con_c}(\mathcal{P}) = \mathrm{Con}(\mathcal{P})$ for every finite semilattice $\mathcal{P} = (P, \vee)$.

The next proposition generalizes some results presented in [35] for finite join-semilattices. In particular, we show that the correspondence between the closure operators and complete congruences on a semilattice is bijective. Statement (a) follows from Lemma 3.2.2. If we check the proof of the theorem of Novotný [35] corresponding to (b), we see that he uses only the "completeness" property, i.e., that every congruence class has a largest element, and not finiteness as such. Thus, the proof of (b) can be omitted. It is also proved in [35] that $\Theta \mapsto c_\Theta$ and $c \mapsto \Theta_c$ are mutually inverse bijections. We note that Day presents similar connections for complete semilattices in the sense of [6]

Proposition 3.2.3. *Let $\mathcal{P} = (P, \vee)$ be a semilattice.*

(a) *If Θ is a complete congruence on $\mathcal{P}$, then $c_\Theta\colon P \to P, x \mapsto c_\Theta(x)$, is a closure operator on $(P, \leq)$.*

(b) *If c is a closure operator on $(P, \leq)$, then its kernel Θ_c is a complete congruence on $\mathcal{P}$ such that the greatest element in the Θ_c-class of any $x \in P$ is* $c(x)$.

(c) *The mappings $\Theta \mapsto c_\Theta$ and $c \mapsto \Theta_c$ form a pair of mutually inverse order-isomorphisms between the ordered set of closure operators $(\mathrm{Clo}(\mathcal{P}), \leq)$ and the ordered set $(\mathrm{Con_c}(\mathcal{P}), \subseteq)$ of complete congruences on $\mathcal{P}$.*

Proof. (c) We show that the maps $\Theta \mapsto c_\Theta$ and $c \mapsto \Theta_c$ are order-preserving. Let $\Theta, \Omega \in \mathrm{Con_c}(\mathcal{P})$ and suppose that $\Theta \subseteq \Omega$. For any $x \in P$, $(x, c_\Theta(x)) \in \Theta \subseteq \Omega$ implies that $c_\Theta(x) \leq c_\Omega(x)$, and hence $c_\Theta \leq c_\Omega$ holds in $\mathrm{Clo}(\mathcal{P})$. Thus, $\Theta \mapsto c_\Theta$ is order-preserving. On the other hand, assume that $c \leq k$ in $\mathrm{Clo}(\mathcal{P})$. If $(x, y) \in \Theta_c$, then $c(x) = c(y)$ which implies $x \leq c(x) = c(y) \leq k(y)$ and furthermore, $k(x) \leq k(k(y)) = k(y)$. Similarly, $y \leq c(y) = c(x) \leq k(x)$ and $k(y) \leq k(k(x)) = k(x)$. Thus, $k(x) = k(y)$ which implies $(x, y) \in \Theta_k$. Hence, $\Theta_c \subseteq \Theta_k$ holds in $\mathrm{Con_c}(\mathcal{P})$. Thus, also $c \mapsto \Theta_c$ is order-preserving.

Because the order-preserving maps $\Theta \mapsto c_\Theta$ and $c \mapsto \Theta_c$ are by [35] mutually inverse bijections, they are also order-isomorphisms. □

Proposition 3.2.4. *Let $(P, \leq)$ be an ordered set.*

(a) *If $(P, \leq)$ is a lattice, then the set of all complete congruences on $\mathcal{P} = (P, \vee)$ is a meet-semilattice with respect to set inclusion such that for all $\Theta_1, \Theta_2 \in \mathrm{Con_c}(\mathcal{P})$,*

$$\Theta_1 \wedge \Theta_2 = \Theta_1 \cap \Theta_2.$$

The greatest element in the $(\Theta_1 \wedge \Theta_2)$-class of any $x \in P$ is $c_{\Theta_1}(x) \wedge c_{\Theta_2}(x)$.

(b) *If $(P, \leq)$ is a lattice which also is a complete meet-semilattice, then the set of all complete congruences on $\mathcal{P} = (P, \vee)$ is a complete meet-semilattice with respect to set inclusion such that*

$$\bigwedge H = \bigcap H$$

for all $\emptyset \neq H \subseteq \mathrm{Con_c}(\mathcal{P})$. The greatest element in the $\bigwedge H$-class of any $x \in P$ is $\bigwedge\{c_\Theta(x) \mid \Theta \in H\}$.

(c) *If $(P, \leq)$ is a complete join-semilattice, then the set of all complete congruences on $\mathcal{P} = (P, \vee)$ is a complete lattice with respect to set inclusion which has the bottom element Δ_P, the top element ∇_P, and for all $\emptyset \neq H \subseteq \mathrm{Con_c}(\mathcal{P})$,*

$$\bigwedge H = \bigcap H \quad \textit{and} \quad \bigvee H = \Theta_{\overline{f}},$$

where $f = \bigvee_{P^P}\{c_\Theta \mid \Theta \in H\}$. The greatest element in the $\bigwedge H$-class of any $x \in P$ is $\bigwedge\{c_\Theta(x) \mid \Theta \in H\}$ and the greatest element in the $\bigvee H$-class of any $x \in P$ is $\overline{f}(x)$.

Proof. (b) Let $(P, \leq)$ be a lattice, which is also a complete meet-semilattice and let $\emptyset \neq H \subseteq \mathrm{Con_c}(\mathcal{P})$, where $\mathcal{P} = (P, \vee)$. It is clear that $\bigcap H$ is a congruence on $(P, \vee)$ (see [3], for example). Let $x \in P$. For any $\Omega \in H$, $x \leq c_\Omega(x)$ implies that $x \leq \bigwedge\{c_\Theta(x) \mid \Theta \in H\} \leq c_\Omega(x)$ and hence $(x, \bigwedge\{c_\Theta(x) \mid \Theta \in H\}) \in \Omega$, because congruence classes are known to be convex. Hence, $(x, \bigwedge\{c_\Theta(x) \mid \Theta \in H\}) \in \bigcap H$. Suppose $(x,y) \in \bigcap H$. This implies $y \leq c_\Theta(x)$ for all $\Theta \in H$, and thus $y \leq \bigwedge\{c_\Theta(x) \mid \Theta \in H\}$. So, $\bigwedge\{c_\Theta(x) \mid \Theta \in H\}$ is the greatest element in the $\bigcap H$-class of x. The proof of (a) is analogous.

(c) Let $\mathcal{P}$ be a complete join-semilattice, $\emptyset \neq H \subseteq \mathrm{Con_c}(\mathcal{P})$, and $x \in P$. Because $x \leq c_\Theta(x)$ for all $\Theta \in H$, the set $\{c_\Theta(x) \mid \Theta \in H\}$ has a lower bound x in P. So, by Lemma 2.2.3, $\bigwedge\{c_\Theta(x) \mid \Theta \in H\}$ exists in P. It is clear that $\bigcap H$ is a congruence on $(P, \vee)$ and by the proof of (b), $\bigwedge\{c_\Theta(x) \mid \Theta \in H\}$ is the greatest element in the $\bigcap H$-class of x. Thus, $\bigcap H$ is a complete congruence and hence $\bigwedge H = \bigcap H$.

Because $\mathcal{P}$ is a complete join-semilattice, $(\mathrm{Clo}(\mathcal{P}), \leq)$ is a complete lattice by Proposition 3.1.10. Let us denote $f = \bigvee_{P^P}\{c_\Theta \mid \Theta \in H\}$. Then by Proposition 3.1.10, $\bigvee\{c_\Theta \mid \Theta \in H\} = \overline{f}$ in $(\mathrm{Clo}(\mathcal{P}), \leq)$. This implies by Proposition 3.2.3(c) that $\bigvee H = \Theta_{\overline{f}}$ in $(\mathrm{Con_c}(\mathcal{P}), \subseteq)$ and Proposition 3.2.3(b) implies that the greatest element in the $\bigvee H$-class of x is $\overline{f}(x)$. $\square$

By Example 3.1.6, $(\mathrm{Con_c}(\mathcal{P}), \leq)$ is not necessarily a join-semilattice, even if $(P, \leq)$ is a lattice and a complete meet-semilattice. Moreover, in cases when $\mathcal{P}$ is a complete join-semilattice, the complete lattice $(\mathrm{Con_c}(\mathcal{P}), \leq)$ is not always a sublattice of $(\mathrm{Con}(\mathcal{P}), \leq)$.

Example 3.2.5. Let us consider the set $\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$, where $\mathbb{R}$ is the set of real numbers. The order $\leq$ is defined on $\mathbb{R}_\infty$ so that

$$x \leq y \iff x \leq y \text{ holds in } \mathbb{R} \text{ or } y = \infty.$$

In addition, we denote $(x,y] = \{z \in \mathbb{R} \mid x < z \leq y\}$. Let us define two complete congruences Θ_1, Θ_2 on the semilattice $\mathcal{P} = (\mathbb{R}_\infty, \vee)$, where $m \vee n = \max\{m, n\}$, such that

$$\begin{aligned} \mathbb{R}_\infty/\Theta_1 &= \{\ldots(0,2],(2,4],(4,6],\ldots,(2k,2k+2],\ldots,\{\infty\}\}; \\ \mathbb{R}_\infty/\Theta_2 &= \{\ldots(-1,1],(1,3],(3,5],\ldots,(2k-1,2k+1],\ldots,\{\infty\}\}. \end{aligned}$$

If we denote by $c_1 \colon \mathbb{R}_\infty \to \mathbb{R}_\infty$ and $c_2 \colon \mathbb{R}_\infty \to \mathbb{R}_\infty$ the closure operators corresponding to Θ_1 and Θ_2, respectively, then it is easy to see that

$$c_1(x) = \begin{cases} \lceil x \rceil + 1 & \text{if } \lceil x \rceil \text{ is odd}, \\ \lceil x \rceil & \text{if } \lceil x \rceil \text{ is even}, \\ \infty & \text{if } x = \infty; \end{cases}$$

and

$$c_2(x) = \begin{cases} \lceil x \rceil + 1 & \text{if } \lceil x \rceil \text{ is even,} \\ \lceil x \rceil & \text{if } \lceil x \rceil \text{ is odd,} \\ \infty & \text{if } x = \infty, \end{cases}$$

where $\lceil x \rceil$ is the least integer greater than or equal to x. Now $f = c_1 \vee_{P^P} c_2$ is defined by

$$f(x) = \begin{cases} \infty & \text{if } x = \infty, \\ \lceil x \rceil + 1 & \text{otherwise.} \end{cases}$$

It is obvious that ∞ is the only fixpoint of f in $\mathbb{R}_\infty$. Hence, the map $c\colon \mathbb{R}_\infty \to \mathbb{R}_\infty, x \mapsto \infty$, is the join of c_1 and c_2 in $(\mathrm{Clo}(\mathcal{P}), \leq)$. We can easily see that the all relation $\mathbb{R}_\infty \times \mathbb{R}_\infty$ is the complete congruence corresponding to c. Thus, $\mathbb{R}_\infty \times \mathbb{R}_\infty$ is the join of Θ_1 and Θ_2 in $\mathrm{Con_c}(\mathcal{P})$.

Note that the join of Θ_1 and Θ_2 in $(\mathrm{Con}(\mathcal{P}), \leq)$ is $\mathbb{R} \times \mathbb{R} \cup \{(\infty, \infty)\}$, which has two congruence classes $\mathbb{R}$ and $\{\infty\}$. This congruence is not complete because the congruence class $\mathbb{R}$ does not have a greatest element.

Let Θ be a complete congruence on $(P, \vee)$. In the sequel we shall study the quotient semilattice $(P/\Theta, \vee)$ more closely. We denote by P_Θ the set of all greatest elements of Θ-classes: $P_\Theta = \{c_\Theta(x) \mid x \in P\}$. In the next lemma we list some simple properties of P_Θ.

Lemma 3.2.6. *Let Θ and Ω be two complete congruences on $(P, \vee)$ and $x \in P$.*

(a) *$P_\Theta = P_\Omega$ implies $\Theta = \Omega$;*

(b) *$c_\Theta(x) = \bigwedge_P \{z \in P_\Theta \mid x \leq z\}$;*

(c) *for all $z \in P_\Theta$, $x \leq z$ iff $c_\Theta(x) \leq z$.*

Proof. (a) Suppose $P_\Theta = P_\Omega$. Then by Lemma 3.1.1(b) $c_\Theta = c_\Omega$, which implies $\Theta = \Omega$ by Proposition 3.2.3(c). Equation (b) follows from Lemma 3.1.1(c).

(c) Let $z \in P_\Theta$. If $x \leq z$, then $c_\Theta(x) \leq c_\Theta(z) = z$. On the other hand, $c_\Theta(x) \leq z$ implies trivially $x \leq c_\Theta(x) \leq z$. □

If Θ is a complete congruence on a semilattice $(P, \vee)$, then $(P/\Theta, \vee)$ is a semilattice such that $a/\Theta \vee b/\Theta = (a \vee b)/\Theta$. Our next proposition shows that $(P/\Theta, \vee)$ and $(P_\Theta, \vee)$ are isomorphic. Note that the join in P_Θ is defined by $a \vee b = c_\Theta(a \vee_P b)$ for all $a, b \in P_\Theta$.

Proposition 3.2.7. *If $\mathcal{P} = (P, \vee)$ is a semilattice and Θ is a complete congruence on $\mathcal{P}$, then the map $\varphi\colon a/\Theta \mapsto c_\Theta(a)$ is an isomorphism between $(P/\Theta, \vee)$ and $(P_\Theta, \vee)$.*

Proof. Because P_Θ consists of the greatest elements of P/Θ-classes, φ is obviously a bijection. If $a, b \in P$, then

$$\begin{aligned}
\varphi(a/\Theta \vee_{P/\Theta} b/\Theta) &= \varphi(c_\Theta(a)/\Theta \vee_{P/\Theta} c_\Theta(b)/\Theta) \\
&= \varphi((c_\Theta(a) \vee_P c_\Theta(b))/\Theta) \\
&= c_\Theta(c_\Theta(a) \vee_P c_\Theta(b)) \\
&= c_\Theta(a) \vee_{P_\Theta} c_\Theta(b) \\
&= \varphi(a/\Theta) \vee_{P_\Theta} \varphi(a/\Theta).
\end{aligned}$$

So, φ is also a homomorphism. □

Recall that the order relation in P/Θ is defined by $a/\Theta \leq b/\Theta$ if and only if $(a \vee b)/\Theta = b/\Theta$. Proposition 3.2.7 implies that

$$a/\Theta \leq b/\Theta \iff c_\Theta(a) \leq c_\Theta(b). \tag{3.1}$$

Next we show that if $(P, \leq)$ is a lattice, then $(P/\Theta, \leq)$ is a lattice, and similar statements hold when $(P, \leq)$ is a complete join- or meet-semilattice.

Proposition 3.2.8. *Let Θ be a complete congruence on a semilattice $(P, \vee)$.*

(a) *If $(P, \leq)$ is a lattice, then $(P/\Theta, \leq)$ is a lattice in which*

$$\begin{aligned}
a/\Theta \vee b/\Theta &= (a \vee_P b)/\Theta; \\
a/\Theta \wedge b/\Theta &= (c_\Theta(a) \wedge_P c_\Theta(b))/\Theta
\end{aligned}$$

for all $a, b \in P$.

(b) *If $(P, \leq)$ is a complete join-semilattice, then $(P/\Theta, \leq)$ is a complete join-semilattice in which*

$$\bigvee\{x/\Theta \mid x \in S\} = (\textstyle\bigvee_P S)/\Theta$$

for all $\emptyset \neq S \subseteq P$.

(c) *If $(P, \leq)$ is a complete meet-semilattice, then $(P/\Theta, \leq)$ is a complete meet-semilattice in which*

$$\bigwedge\{x/\Theta \mid x \in S\} = (\textstyle\bigwedge_P\{c_\Theta(x) \mid x \in S\})/\Theta$$

for all $\emptyset \neq S \subseteq P$.

Proof. (a) Of course, the identity $a/\Theta \vee b/\Theta = (a \vee_P b)/\Theta$ holds for any congruence Θ on a join-semilattice. Suppose now that $(P, \leq)$ is also a lattice and consider any elements $a, b \in P$. Because $c_\Theta(a) \geq c_\Theta(a) \wedge_P c_\Theta(b)$, we obtain that $a/\Theta = c_\Theta(a)/\Theta \geq (c_\Theta(a) \wedge_P c_\Theta(b))/\Theta$. Similarly, we can show that $(c_\Theta(a) \wedge_P c_\Theta(b))/\Theta$ is a lower bound of b/Θ.

Suppose x/Θ is a lower bound for a/Θ and b/Θ. Then $x/\Theta \leq a/\Theta$ and $x/\Theta \leq b/\Theta$ imply $c_\Theta(x) \leq c_\Theta(a)$ and $c_\Theta(x) \leq c_\Theta(b)$ by (3.1). Thus, $c_\Theta(x) \leq c_\Theta(a) \wedge_P c_\Theta(b) \leq c_\Theta(c_\Theta(a) \wedge_P c_\Theta(b))$, which means that $x/\Theta \leq (c_\Theta(a) \wedge_P c_\Theta(b))/\Theta$.

(b) Suppose $(P, \leq)$ is a complete join-semilattice and $\emptyset \neq S \subseteq P$. Because $x \leq \bigvee S$ for all $x \in S$, we have $x/\Theta \leq (\bigvee S)/\Theta$ for all $x \in S$, that is, $(\bigvee S)/\Theta$ is an upper bound of $\{x/\Theta \mid x \in S\}$. If y/Θ is an upper bound of $\{x/\Theta \mid x \in S\}$, then $x/\Theta \leq y/\Theta$ and $c_\Theta(x) \leq c_\Theta(y)$ for all $x \in S$ by (3.1). Thus,

$$c_\Theta(y) \geq \bigvee_{P_\Theta} \{c_\Theta(x) \mid x \in S\} = c_\Theta(\bigvee_P \{c_\Theta(x) \mid x \in S\}),$$

which implies $y/\Theta \geq (\bigvee_P \{c_{\Theta(x)} \mid x \in S\})/\Theta$ by (3.1). Because $x \leq c_\Theta(x)$ for all $x \in S$, we get $\bigvee_P S \leq \bigvee_P \{c_\Theta(x) \mid x \in S\}$. Therefore,

$$(\bigvee_P S)/\Theta \leq (\bigvee_P \{c_\Theta(x) \mid x \in S\})/\Theta \leq y/\Theta.$$

Hence, $\bigvee \{x/\Theta \mid x \in S\} = (\bigvee_P S)/\Theta$.

(c) Suppose $(P, \leq)$ is a complete meet-semilattice and $\emptyset \neq S \subseteq P$. If $a \in S$, then $c_\Theta(a) \geq \bigwedge_P \{c_\Theta(x) \mid x \in S\}$, which implies $a/\Theta = c_\Theta(a)/\Theta \geq (\bigwedge_P \{c_\Theta(x) \mid x \in S\})/\Theta$.

Suppose that y/Θ is a lower bound for $\{x/\Theta \mid x \in S\}$. Then $y/\Theta \leq x/\Theta$ and $c_\Theta(y) \leq c_\Theta(x)$ for all $x \in S$. Thus, $c_\Theta(y) \leq \bigwedge_P \{c_\Theta(x) \mid x \in S\}) \leq c_\Theta(\bigwedge_P \{c_\Theta(x) \mid x \in S\})$, which is equivalent to $y/\Theta \leq (\bigwedge_P \{c_\Theta(x) \mid x \in S\})/\Theta$. □

It is clear that if $(P, \leq)$ is a complete lattice and Θ is a complete congruence on $(P, \vee)$, then P/Θ is a complete lattice in which the joins and the meets of nonempty subsets are formed as in the previous proposition. Moreover, P/Θ has the bottom element $\bot/\Theta$ and the top element $\top/\Theta$.

Definition. Let $\mathcal{P} = (P, \leq)$ be a complete join-semilattice and let $\Theta \in \mathrm{Rel}(P)$. We say that Θ has the *complete $\vee$-compatibility property* if for every nonempty index set I and any $x_i, y_i \in P$ $(i \in I)$,

$$x_i \Theta y_i \text{ for all } i \in I \text{ implies } (\bigvee_{i \in I} x_i) \Theta (\bigvee_{i \in I} y_i).$$

In the next proposition we show that in a complete join-semilattice the equivalences having the complete $\vee$-compatibility property are exactly the complete congruences.

Proposition 3.2.9. *Let $(P, \leq)$ be a complete join-semilattice and let Θ be an equivalence on P. Then Θ has the complete $\vee$-compatibility property if and only if Θ is a complete congruence.*

Proof. Suppose that an equivalence Θ has the complete $\vee$-compatibility property. Then obviously Θ is a congruence on $(P, \vee)$. Consider any $x \in P$ and let $x/\Theta = \{x_i \mid i \in I\}$ for some $I \neq \emptyset$. Because $x_i \Theta x$ for all $i \in I$, we have $\bigvee\{x_i \mid i \in I\}\Theta x$. This means that x/Θ has the greatest element $\bigvee\{x_i \mid i \in I\}$.

Conversely, if Θ is a complete congruence on $(P, \vee)$ and I is a nonempty index set such that $x_i \Theta y_i$ for all $i \in I$, then by Proposition 3.2.8(b),

$$\begin{aligned}(\textstyle\bigvee_P\{x_i \mid i \in I\})/\Theta &= \textstyle\bigvee_{P/\Theta}\{x_i/\Theta \mid i \in I\} \\ &= \textstyle\bigvee_{P/\Theta}\{y_i/\Theta \mid i \in I\} \\ &= (\textstyle\bigvee_P\{y_i \mid i \in I\})/\Theta.\end{aligned}$$

So, $(\bigvee\limits_{i \in I} x_i)\Theta(\bigvee\limits_{i \in I} y_i)$. □

Remark. Day [6] defines complete join-semilattices as ordered sets $\mathcal{P} = (P, \leq)$ in which $\bigvee S$ exists for every $S \subseteq P$; this means that $\mathcal{P}$ is actually a complete lattice. Moreover, an equivalence Θ on P is a congruence on a $\mathcal{P}$ according to Day's terminology, if $(\bigvee_{i \in I} x_i)\Theta(\bigvee_{i \in I} y_i)$ for every set $(x_i, y_i) \subseteq \Theta$, where I is an arbitrary index set.

By Proposition 3.2.9 it is easy to observe that in a complete lattice $\mathcal{P} = (P, \leq)$ a binary relation $\Theta \in \mathrm{Rel}(P)$ is a complete congruence on $(P, \vee)$ if and only if Θ is a congruence on $\mathcal{P}$ in the sense of Day. Of course, $\bot = \bigvee \emptyset$ is always congruent with itself. Thus, our concept of complete congruences is a generalization of Day's congruences, since it is applicable to all kinds of semilattices.

Consider a complete join-semilattice $\mathcal{P} = (P, \leq)$. At the end of this section we intend to describe for any $R \in \mathrm{Rel}(P)$ the smallest complete congruence on $\mathcal{P}$ containing R. The following proposition can be found in [3], for example.

Proposition 3.2.10. *For any algebra $\mathcal{A}$, there is an algebraic closure operator*

$$\theta \colon \mathrm{Rel}(A) \to \mathrm{Rel}(A), R \mapsto \bigcap\{\Theta \in \mathrm{Con}(\mathcal{A}) \mid R \subseteq \Theta\},$$

such that the closed elements of $\mathrm{Rel}(A)$ are precisely the congruences on $\mathcal{A}$. □

For an algebra $\mathcal{A} = (A, \Sigma)$ and an arbitrary $X \subseteq A$, let $\theta(X)$ denote the congruence generated by $X \times X$, that is, the smallest congruence such that all elements of X are in the same congruence class. The congruence $\theta(\{a, b\})$ will be denoted by $\theta(a, b)$ and it is called a *principal congruence*. It is known (see [3, 16], for example) that for all $X \subseteq A$,

$$\theta(X) = \bigvee \{\theta(a, b) \mid a, b \in X\}.$$

Let $\mathcal{P} = (\mathcal{P}, \leq)$ be a complete join-semilattice. Since by Proposition 3.2.4(c) $\bigcap \mathcal{R} \in \mathrm{Con_c}(\mathcal{P})$, for all $\mathcal{R} \subseteq \mathrm{Con_c}(\mathcal{P})$, also $\mathrm{Con_c}(\mathcal{P})$ is a closure system. Thus, there exists a closure operator θ_c on $P \times P$ such that the closed elements of $P \times P$ are exactly the complete congruences on $(P, \vee)$. Moreover, for any $X \subseteq P$, we denote by $\theta_c(X)$ the smallest complete congruence on $(P, \vee)$ such that all elements of X are in the same congruence class.

Example 3.2.11. Let us consider again the complete join-semilattice $(\mathbb{N}_\infty, \vee)$ defined in Example 3.1.12. For any finite subset S of $\mathbb{N}$,

$$\theta_c(S) = \Delta_{\mathbb{N}_\infty} \cup \{(x, y) \mid \min S \leq x, y \leq \max S\}.$$

It is easy to see that $\theta_c(\mathbb{N}) = \mathbb{N}_\infty \times \mathbb{N}_\infty$ and that

$$\bigcup \{\theta_c(F) \mid F \text{ is a finite subset of } \mathbb{N}\} = \mathbb{N} \times \mathbb{N} \cup \{(\infty, \infty)\}.$$

This shows that the closure operator $\theta_c \colon \mathrm{Rel}(\mathbb{N}) \to \mathrm{Rel}(\mathbb{N})$ is not algebraic.

Let $\mathcal{P} = (P, \leq)$ be a complete join-semilattice and let Θ be a congruence on $(P, \vee)$. Let us consider the function $\Phi^\Theta \colon P \to P, x \mapsto \bigvee x/\Theta$. Because $\mathcal{P}$ is a complete join-semilattice and $x/\Theta \neq \emptyset$, Φ^Θ is a well-defined map. Note that if Θ is a complete congruence on $(P, \vee)$, then $\Phi^\Theta = c_\Theta$, i.e., $\Phi^\Theta(x) = c_\Theta(x)$ for all $x \in P$.

Example 3.2.12. Let us consider the complete lattice $(\mathbb{N}_\infty, \leq)$ defined in Example 3.1.12. Let Θ be the congruence on $\mathcal{P} = (\mathbb{N}_\infty, \leq)$ which has the congruence classes $\mathbb{N}$ and $\{\infty\}$. For all $n \in \mathbb{N}$, $\bigvee n/\Theta = \bigvee \mathbb{N} = \infty$. Hence, the map $\Phi^\Theta \colon \mathbb{N}_\infty \to \mathbb{N}_\infty$ is defined by $x \mapsto \infty$ for all $x \in \mathbb{N}_\infty$.

The following lemma shows that $\Phi^\Theta \in \langle P \to P \rangle$.

Lemma 3.2.13. *If $\mathcal{P} = (P, \leq)$ is a complete join-semilattice and Θ is a congruence on $(P, \vee)$, then the map $\Phi^{\Theta} \colon P \to P$ is extensive and order-preserving.*

Proof. The fact $x \in x/\Theta$ implies that $x \leq \Phi^{\Theta}(x)$ for all $x \in P$, that is, Φ^{Θ} is extensive. Suppose $x \leq y$. If $z \in x/\Theta$, then $z \vee y \Theta x \vee y = y$, which implies $z \leq z \vee y \leq \bigvee y/\Theta$. Thus, $\bigvee x/\Theta \leq \bigvee y/\Theta$ and $\Phi^{\Theta}(x) \leq \Phi^{\Theta}(y)$. Hence, Φ^{Θ} is order-preserving. □

By the previous lemma the map $\Phi^{\Theta} \colon P \to P$ is extensive and order-preserving. By Proposition 3.1.8, the map $\overline{\Phi^{\Theta}}$ is a closure operator. In the next lemma we use this fact.

Proposition 3.2.14. *If $\mathcal{P} = (P, \leq)$ is a complete join-semilattice and $R \in \mathrm{Rel}(P)$, then $\theta_c(R)$ is the complete congruence induced by the closure operator $\overline{\Phi^{\theta(R)}}$.*

Proof. Let us denote the complete congruence induced by the closure operator $\overline{\Phi^{\theta(R)}}$ simply by Θ. First we show that $R \subseteq \Theta$. If $(x, y) \in R$, then $(x, y) \in \theta(R)$ and so $\Phi^{\theta(R)}(x) = \Phi^{\theta(R)}(y)$. Thus, also $\overline{\Phi^{\theta(R)}}(x) = \overline{\Phi^{\theta(R)}}(y)$ and hence $(x, y) \in \Theta$.

Suppose Ω is a complete congruence which contains R. Then $\theta(R) \subseteq \theta(\Omega) = \Omega$, which implies for all $x \in P$, $x/\theta(R) \subseteq x/\Omega$ and so $\Phi^{\theta(R)}(x) = \bigvee x/\theta(R) \leq \bigvee x/\Omega = c_{\Omega}(x)$. So, $\overline{\Phi^{\theta(R)}}(x) \leq \overline{c_{\Omega}}(x) = c_{\Omega}(x)$ for all $x \in P$. Thus, $\Theta \subseteq \Theta_{c_{\Omega}} = \Omega$. □

3.3 Complete Morphisms of Semilattices

In this section we study complete join- and meet-morphisms. A complete join-morphism is an order-preserving map which preserves every existing join. It is known that the kernel of a morphism from an algebra $\mathcal{A}$ to an algebra $\mathcal{B}$ is a congruence on $\mathcal{A}$. Here we show that in a complete join-semilattice the kernel of a complete join-morphism is a complete congruence. This means that in a complete join-semilattice, each complete join-morphism f induces a complete congruence Θ_f and a closure operator c_f. We shall see that if $(P, \leq)$ is a complete join-semilattice, $(Q, \leq)$ is an ordered set, and $f \colon P \to Q$ is a complete join-morphism, then $(f[P], \leq)$, $(P/\Theta_f, \leq)$, and $(P_f, \leq)$ are isomorphic complete join-semilattices; P_f is the set of the greatest elements of the Θ_f-classes. We also note that if $(P, \leq)$ is a complete lattice, then $(f[P], \leq)$, $(P/\Theta_f, \leq)$, and $(P_f, \leq)$ are complete lattices.

Definition. Let $(P, \leq)$ and $(Q, \leq)$ be ordered sets. A map $f\colon P \to Q$ is a *complete join-morphism* if whenever $S \subseteq P$ and $\bigvee S$ exists, then $\bigvee f[S]$ exists and $f(\bigvee S) = \bigvee f[S]$. The dual of a complete join-morphism is a *complete meet-morphism*. If f is both a complete join- and a complete meet-morphism, then it is a *complete morphism*.

The next obvious lemma connects complete morphisms to order-isomorphisms, order-preserving maps, and continuous maps.

Lemma 3.3.1. *Every order-isomorphism is a complete morphism and every complete join-morphism is order-preserving. Moreover, a complete join-morphism from a pre-CPO to a pre-CPO is continuous.*

Let $\mathcal{P} = (P, \leq)$ and $\mathcal{Q} = (Q, \leq)$ be ordered sets and let $f\colon P \to Q$ be a join-complete morphism. If $\mathcal{P}$ is a join-semilattice, then obviously $(f[P], \leq)$ is a join-semilattice and f is a homomorphism onto $(f[P], \leq)$. If $\mathcal{Q}$ is a join-semilattice, then f is a homomorphism $(P, \vee) \to (Q, \vee)$. In the next example we see that not every order-preserving map is necessarily a complete join-morphism.

Example 3.3.2. Let $P = \{1, \ldots, 7\}$ and $Q = \{a, \ldots, e\}$, and suppose $(P, \leq)$ and $(Q, \leq)$ are the ordered sets defined by the Hasse diagrams of Figure 4.

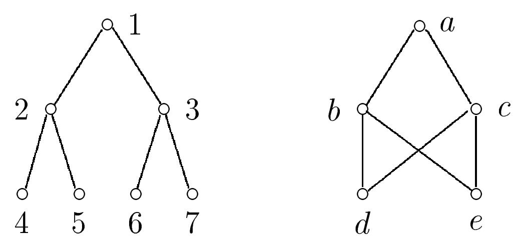


Figure 4.

Let us define an order-preserving map f from $(P, \leq)$ to $(Q, \leq)$ such that $1 \mapsto a$, $2 \mapsto b$, $3 \mapsto c$, $4 \mapsto d$, $5 \mapsto e$, $6 \mapsto d$, $7 \mapsto e$. Obviously, $(P, \leq)$ is a join-semilattice and so $x \vee y$ exists for all $x, y \in P$. But in $(Q, \leq)$ the join of $d = f(4)$ and $e = f(5)$ does not exists, which implies that f is not a complete join-morphism.

In the following proposition we present some properties of the kernel Θ_f of a join-complete morphism f.

Proposition 3.3.3. *Let $\mathcal{P} = (P, \leq)$ and $\mathcal{Q} = (Q, \leq)$ be ordered sets and let $f \colon P \to Q$ be a complete join-morphism.*

(a) *If $\mathcal{P}$ is a join-semilattice, then Θ_f is a congruence on $(P, \vee)$.*

(b) *If $\mathcal{P}$ is a complete join-semilattice, then Θ_f is a complete congruence on $(P, \vee)$.*

(c) *If $\mathcal{P}$ and $\mathcal{Q}$ are complete meet-semilattices, then for all $\emptyset \neq S \subseteq P$,*

$$f(\bigwedge S) \leq \bigwedge f[S].$$

Proof. Assertions (a) and (c) are obvious.

(b) Let $x \in P$. Because $(P, \leq)$ is a complete join-semilattice and x/Θ_f is nonempty, $\bigvee x/\Theta_f$ exists in P and

$$\begin{aligned} f(\bigvee x/\Theta_f) &= \bigvee f[x/\Theta_f] \\ &= \bigvee \{f(y) \mid y \in x/\Theta_f\} \\ &= \bigvee \{f(y) \mid f(y) = f(x)\} \\ &= f(x). \end{aligned}$$

Thus, $\bigvee x/\Theta_f \in x/\Theta_f$ and clearly $\bigvee x/\Theta_f$ is the greatest element in x/Θ_f. □

Let $(P, \vee)$ be a join-semilattice, $(Q, \leq)$ an ordered set, and let $f \colon P \to Q$ be a complete join-morphism. As we have noted, $(f[P], \leq)$ is a join-semilattice such that $f(a) \vee f(b) = f(a \vee b)$, and hence f is a homomorphism from $(P, \vee)$ onto $(f[P], \vee)$. By the Homomorphism Theorem $\varphi_1 \colon a/\Theta_f \mapsto f(a)$ is an isomorphism between $(P/\Theta_f, \vee)$ and $(f[P], \vee)$.

If $(P, \leq)$ is a complete join-semilattice, then Θ_f is a complete congruence on $(P, \vee)$. So, $(P/\Theta_f, \leq)$ is a complete join-semilattice by Proposition 3.2.8(b). It is obvious that also $(f[P], \leq)$ is a complete join-semilattice.

Let us denote by c_f the closure operator corresponding to the complete congruence Θ_f; that is, $c_f(x) = \bigvee x/\Theta_f$ for all $x \in P$. Furthermore, the set of c_f-closed elements (i.e., the set of the greatest elements of Θ_f-classes) is denoted by P_f. By Corollary 3.1.2(c), $(P_f, \leq)$ is a complete join-semilattice. Proposition 3.2.7 implies that the complete join-semilattices $(P/\Theta_f, \leq)$ and $(P_f, \leq)$ are isomorphic. The isomorphism is $\varphi_2 \colon a/\Theta_f \mapsto c_f(a)$.

By Lemma 2.3.1, the inverse of an isomorphism is an isomorphism and the composition of two isomorphisms is an isomorphism. Thus, the map $\varphi_3 \colon f(a) \mapsto c_f(a)$, which is the composition $\varphi_2 \circ \varphi_1^{-1}$, is an order-isomorphism between $(f[P], \leq)$ and $(P_f, \leq)$. Figure 5 illustrates the isomorphisms $\varphi_1, \varphi_2, \varphi_3$.

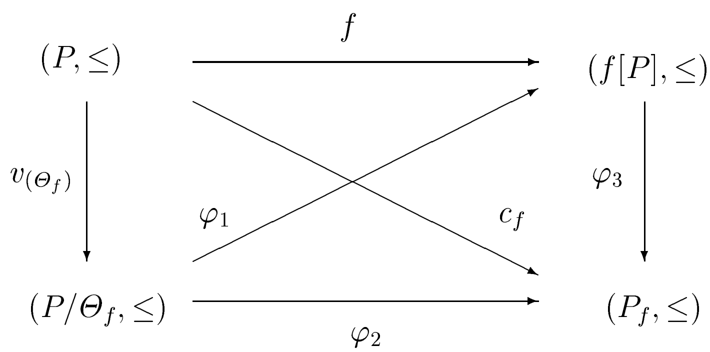


Figure 5.

Now we have proved the following proposition.

Proposition 3.3.4. *If $(P, \leq)$ is a complete join-semilattice, $(Q, \leq)$ is an ordered set, and $f\colon P \to Q$ is a complete join-morphism, then the complete join-semilattices $(f[P], \leq)$, $(P/\Theta_f, \leq)$, and $(P_f, \leq)$ are isomorphic.* □

Let $(P, \leq)$ be a complete lattice and let $(Q, \leq)$ be an ordered set. If $f\colon P \to Q$ is a complete join-morphism, then by Corollary 3.1.2 $(P_f, \leq)$ is a complete lattice in which

$$\bigvee\{c_f(x) \mid x \in S\} = c_f(\bigvee_P\{c_f(x) \mid x \in S\})$$

and

$$\bigwedge\{c_f(x) \mid x \in S\} = \bigwedge_P\{c_f(x) \mid x \in S\}$$

for all $S \subseteq P$. Secondly $(P/\Theta_f, \leq)$ is by Proposition 3.2.8 a complete lattice such that

$$\bigvee\{x/\Theta_f \mid x \in S\} = (\bigvee_P S)/\Theta_f$$

and

$$\bigwedge\{x/\Theta_f \mid x \in S\} = (\bigwedge_P\{c_f(x) \mid x \in S\})/\Theta_f$$

for all $S \subseteq P$. It is also obvious that $(f[P], \leq)$ is a complete lattice in which

$$\bigvee f[S] = f(\bigvee_P S)$$

for all $S \subseteq P$.

Next we describe the meets $\bigwedge f[S]$ in $(f[P], \leq)$.

Proposition 3.3.5. *Let $(P, \leq)$ be a complete lattice and let $(Q, \leq)$ be an ordered set. If $f\colon P \to Q$ is a complete join-morphism, then in the complete lattice $(f[P], \leq)$,*

$$\bigwedge f[S] = f(\textstyle\bigwedge_P c_f[S])$$

for all $S \subseteq P$.

Proof. Let $x \in S$. Because $\bigwedge_P c_f[S] \leq c_f(x)$, we get

$$\begin{aligned} f(\bigwedge c_f[S]) &\leq f(c_f(x)) \\ &= f(\bigvee\{y \in P \mid f(x) = f(y)\}) \\ &= \bigvee\{f(y) \mid f(x) = f(y)\} \\ &= f(x). \end{aligned}$$

Hence, $f(\bigwedge_P c_f[S])$ is a lower bound of $f[S]$. Assume $f(y)$ is a lower bound of $f[S]$. Then $f(y) \leq f(x)$ for all $x \in S$. The map $\varphi_3\colon f(a) \mapsto c_f(a)$ is an order-isomorphism between $(f[P], \leq)$ and $(P_f, \leq)$. Thus, $c_f(y) \leq c_f(x)$ for all $x \in S$. This implies $c_f(y) \leq \bigwedge_{P_f} c_f[S] = \bigwedge_P c_f[S]$. Therefore, $f(y) = f(c_f(y)) \leq f(\bigwedge_P c_f[S])$. Hence, $\bigwedge_{f[P]} f[S] = f(\bigwedge_P c_f[S])$. □

3.4 Dense Sets

Let $\mathcal{P} = (P, \leq)$ be an ordered set. In [35] Novotný associates with each subset S of P an equivalence Θ_S on P (see (3.2) for the definition), which is a congruence on $(P, \vee)$ whenever $\mathcal{P}$ is a join-semilattice. We will show that if $\mathcal{P}$ is a complete join-semilattice, then the congruence Θ_S is complete.

Consider a congruence Θ on a semilattice $(P, \vee)$. A subset $S(\subseteq P)$ is said to be Θ-dense if $\Theta_S = \Theta$. On the other hand, a subset S of an ordered set $(P, \leq)$ is said to be meet-dense in $(P, \leq)$ if for every element $x \in P$ there is a subset Q of S such that $x = \bigwedge_P Q$. We prove that if Θ is a complete congruence on $(P, \vee)$, then the Θ-dense subsets of P are exactly the meet-dense subsets of $(P_\Theta, \leq)$; recall that P_Θ is the set of the greatest elements of Θ-classes. We also show that every complete congruence on a complete join-semilattice is defined by at least two subsets. This implies that in a finite semilattice $(P, \vee)$ the number of congruence relations and closure operators is at most $2^{|P|-1}$. Furthermore, we prove that this upper bound is optimal. We conclude this section by some chain conditions which can be used for identifying dense sets.

Let $\mathcal{P} = (P, \leq)$ be an ordered set and $S \subseteq P$. Let us define an equivalence relation Θ_S on P by

$$\Theta_S = \{(x, y) \in P^2 \mid (\forall z \in S)\ x \leq z \iff y \leq z\}. \tag{3.2}$$

By the definition of Θ_S,

$$S \subseteq T \text{ implies } \Theta_T \subseteq \Theta_S \tag{3.3}$$

for all $S, T \subseteq P$. For any $S \subseteq P$ and $x \in P$, let

$$(\uparrow x)_S = \{z \in S \mid x \leq z\}.$$

Note that $(\uparrow x)_S = [x) \cap S$. In the following proposition we give some properties of Θ_S.

Proposition 3.4.1. *Let $\mathcal{P} = (P, \leq)$ be an ordered set and $S \subseteq P$.*

(a) *The map $S \mapsto \Theta_S$ is a complete join-morphism $(\wp(P), \subseteq) \to (\mathrm{Eq}(P), \supseteq)$.*

(b) *If $\mathcal{P}$ is a join-semilattice, then Θ_S is a congruence on $(P, \vee)$.*

(c) *If $\mathcal{P}$ is a complete join-semilattice, then Θ_S is a complete congruence on $(P, \vee)$ such that for each $x \in P$, the greatest element in x/Θ_S is $\bigwedge(\uparrow x)_S$.*

Proof. (a) We show that $\Theta_{(\bigcup \mathcal{H})} = \bigcap\{\Theta_S \mid S \in \mathcal{H}\}$ for all $\mathcal{H} \subseteq \wp(P)$. If $x, y \in P$, then

$$\begin{aligned}
(x, y) \in \Theta_{(\bigcup \mathcal{H})} &\iff (\forall z \in \textstyle\bigcup \mathcal{H})\ x \leq z \text{ iff } y \leq z \\
&\iff (\forall S \in \mathcal{H})(\forall z \in S)\ x \leq z \text{ iff } y \leq z \\
&\iff (\forall S \in \mathcal{H})\ (x, y) \in \Theta_S \\
&\iff (x, y) \in \textstyle\bigcap\{\Theta_S \mid S \in \mathcal{H}\}.
\end{aligned}$$

Assertion (b) was noted in [35]. It follows directly from the definitions.

(c) We show that the congruence Θ_S is complete. Let $x \in P$. It is clear that for every $S \subseteq P$, the set $(\uparrow x)_S$ is bounded from below by x. By Lemma 2.2.3, this implies that $\bigwedge(\uparrow x)_S$ exists in P. We denote $\bigwedge(\uparrow x)_S$ by $c_S(x)$. Next we show that $(x, c_S(x)) \in \Theta_S$. Let $z \in S$. If $x \leq z$, then $z \in (\uparrow x)_S$, which implies $c_S(x) \leq z$. Because $(\uparrow x)_S \subseteq (\uparrow x)_P$, we get $x = \bigwedge(\uparrow x)_P \leq \bigwedge(\uparrow x)_S = c_S(x)$. Thus, $c_S(x) \leq z$ implies $x \leq c_S(x) \leq z$ and so $(x, c_S(x)) \in \Theta_S$.

If $y \in x/\Theta_S$, then $(\uparrow x)_S = (\uparrow y)_S$ and $y \leq c_S(y) = c_S(x)$. This means that $c_S(x)$ is the greatest element in x/Θ_S. $\square$

If $(P,\vee)$ is not a complete join-semilattice, then Θ_S is not necessarily a complete congruence of $(P,\vee)$, as we see in the next example.

Example 3.4.2. Let us denote $F(\mathbb{N}) = \{X \subseteq \mathbb{N} \mid X \text{ is finite }\}$. Then $(F(\mathbb{N}),\subseteq)$ is a lattice, such that $X \vee Y = X \cup Y$ and $X \wedge Y = X \cap Y$. Furthermore, $(F(\mathbb{N}),\subseteq)$ is a complete meet-semilattice such that $\bigwedge H = \bigcap H$ for all $\emptyset \neq H \subseteq F(\mathbb{N})$. Note that $(F(\mathbb{N}),\subseteq)$ is not a complete join-semilattice since $\bigvee F(\mathbb{N})$, for instance, does not exist.

If $S = \{X\}$ for some $X \in F(\mathbb{N})$, then Θ_S has two congruence classes

$$\begin{aligned} C_1 &= \{Y \in F(\mathbb{N}) \mid Y \subseteq X\}; \\ C_2 &= \{Y \in F(\mathbb{N}) \mid Y \nsubseteq X\}. \end{aligned}$$

The class C_1 has the greatest element X, but obviously C_2 does not have a greatest element.

If $\mathcal{P} = (P,\leq)$ is a complete join-semilattice and $S \subseteq P$, then the map

$$c_S \colon P \to P, x \mapsto \bigwedge({\uparrow}x)_S,$$

is the closure operator corresponding to the complete congruence Θ_S. The set of c_S-closed elements is denoted by P_S. Because for all $z \in S$, $z \in ({\uparrow}z)_S$, we have $c_S(z) = \bigwedge({\uparrow}z)_S = z$ which implies $S \subseteq P_S$.

In our next lemma we present some properties of the map $S \mapsto \Theta_S$ in a complete join-semilattice $(P,\leq)$.

Proposition 3.4.3. *If $\mathcal{P} = (P,\leq)$ is a complete join-semilattice, then $S \mapsto \Theta_S$ is a complete join-morphism from $(\wp(P),\subseteq)$ onto $(\mathrm{Con_c}(\mathcal{P}),\supseteq)$. Its kernel is a complete congruence on $(\wp(P),\cup)$ such that the greatest element in the congruence class of S is P_S.*

Proof. By Proposition 3.4.1, $S \mapsto \Theta_S$ is a complete join-morphism from $(\wp(P) \subseteq)$ to $(\mathrm{Con_c}(\mathcal{P}),\supseteq)$.

Suppose $\Theta \in \mathrm{Con_c}(\mathcal{P})$. We claim that $\Theta = \Theta_{(P_\Theta)}$. Let $(x,y) \in \Theta$ and $z \in P_\Theta$. If $x \leq z$, then $y \leq c_\Theta(y) = c_\Theta(x) \leq c_\Theta(z) = z$. Similarly, $y \leq z$ implies $x \leq z$. Thus, $(x,y) \in \Theta_{(P_\Theta)}$ and $\Theta \subseteq \Theta_{(P_\Theta)}$ holds. Conversely, if $(x,y) \in \Theta_{(P_\Theta)}$, then for all $z \in P_\Theta$, $x \leq z \iff y \leq z$. In particular, $x \leq c_\Theta(x)$ implies $y \leq c_\Theta(x)$. Thus, $c_\Theta(y) \leq c_\Theta(c_\Theta(x)) = c_\Theta(x)$. Similarly, we can show that $c_\Theta(x) \leq c_\Theta(y)$. So, $c_\Theta(x) = c_\Theta(y)$ and this is equivalent to $(x,y) \in \Theta$.

Hence, also $\Theta \supseteq \Theta_{(P_\Theta)}$ and thus, $\Theta = \Theta_{(P_\Theta)}$. This means that $S \mapsto \Theta_S$ is onto $\mathrm{Con_c}(\mathcal{P})$.

Since $S \mapsto \Theta_S$ is a complete morphism, its kernel is a complete congruence on $(\wp(P), \cup)$ by Proposition 3.3.3. We just showed that $\Theta_S = \Theta_{P_{\Theta_S}}$ and since $P_{\Theta_S} = P_S$, we obtain $\Theta_{P_S} = \Theta_S$. If $\Theta_T = \Theta_S$, then $T \subseteq P_T = P_S$. Thus, P_S is the greatest subset of P which induces the same complete congruence as S. $\square$

We denote the kernel of the map $S \mapsto \Theta_S$ by κ. If we define the order relation $\leq$ in $\wp(P)/\kappa$ so that $B/\kappa \leq C/\kappa$ iff $(B \cup C)/\kappa = C/\kappa$, then $(\wp(P)/\kappa, \leq)$ is a complete lattice by Proposition 3.2.8. By our following proposition, this complete lattice is dually isomorphic to $(\mathrm{Con_c}(\mathcal{P}), \subseteq)$ and to $(\mathrm{Clo}(\mathcal{P}), \leq)$.

Proposition 3.4.4. *If $\mathcal{P} = (P, \leq)$ is a complete join-semilattice, then*

$$(\mathrm{Con_c}(\mathcal{P}), \subseteq) \cong (\mathrm{Clo}(\mathcal{P}), \leq) \cong (\wp(P)/\kappa, \geq).$$

Proof. The isomorphism $(\mathrm{Con_c}(\mathcal{P}), \subseteq) \cong (\mathrm{Clo}(\mathcal{P}), \leq)$ was shown in Proposition 3.2.3. Because $S \mapsto \Theta_S$ is a homomorphism from $(\wp(P), \cup)$ onto $(\mathrm{Con_c}(\mathcal{P}), \cap)$, $(\wp(P)/\kappa, \vee)$ and $(\mathrm{Con_c}(\mathcal{P}), \cap)$ are isomorphic by the Homomorphism Theorem. This implies clearly that $(\wp(P)/\kappa, \geq) \cong (\mathrm{Con_c}(\mathcal{P}), \subseteq)$. $\square$

We have seen that in a complete join-semilattice $(P, \leq)$ each subset $S \subseteq P$ defines a complete congruence Θ_S on $(P, \vee)$. Our next lemma shows that for any $S \subseteq P$, there exists a $T \neq S$ such that $\Theta_T = \Theta_S$.

Lemma 3.4.5. *If $\mathcal{P} = (P, \leq)$ is a complete join-semilattice, then $|S/\kappa| \geq 2$ for all $S \subseteq P$.*

Proof. Consider any $S \subseteq P$. If $\top$ is the greatest element of P, then $\top \in P_S$ as $c_S(\top) = \top$. Since $P_S - \{\top\} \subseteq P_S$, we get $\Theta_{(P_S)} \subseteq \Theta_{(P_S - \{\top\})}$ by (3.3). If $(x, y) \in \Theta_{(P_S - \{\top\})}$, then for all $z \in P_S - \{\top\}$, $x \leq z \iff y \leq z$. Trivially, $x \leq \top$ and $y \leq \top$ and hence $(x, y) \in \Theta_{(P_S)}$. Thus, $\Theta_{(P_S)} \supseteq \Theta_{(P_S - \{\top\})}$ and $\Theta_S = \Theta_{(P_S)} = \Theta_{(P_S - \{\top\})}$. $\square$

If $\mathcal{P} = (P, \vee)$ is a finite semilattice, then $(P, \leq)$ is a complete join-semilattice and each congruence on $\mathcal{P}$ is complete, and therefore the previous lemma has the following corollary.

Corollary 3.4.6. *If $\mathcal{P} = (P, \vee)$ is a finite semilattice, then*

$$|\mathrm{Clo}(\mathcal{P})| = |\mathrm{Con}(\mathcal{P})| \leq 2^{|P|-1}.$$

In the next example we show that the upper bound given in the previous corollary is the best possible.

Example 3.4.7. Let $P = \{x_1, \dots, x_{n-1}\} \cup \{\top\}$ and assume that the order $\leq$ is defined on P by

$$x \leq y \iff x = y \text{ or } y = \top.$$

Then $(P, \leq)$ is a join-semilattice such that $x \vee y = \top$ for all $x \neq y$.

If $S, T \subseteq \{x_1, \dots, x_{n-1}\}$ and $S \neq T$, then obviously $\Theta_S \neq \Theta_T$. This means that every subset of $\{x_1, \dots, x_{n-1}\}$ defines a different congruence on $\mathcal{P} = (P, \vee)$. Hence, $|\mathrm{Con}(\mathcal{P})| \geq 2^{|P|-1}$.

Novotný introduced in [35] the notion of dense sets for dependence spaces. Here we define dense sets for any congruence on a semilattice $(P, \vee)$. Note that we do not require that the corresponding join-semilattice $(P, \leq)$ is complete.

Definition. Let Θ be a congruence on a semilattice $(P, \vee)$. We say that $S(\subseteq P)$ is *Θ-dense* if $\Theta_S = \Theta$.

In the following lemma we present some simple properties of dense sets.

Lemma 3.4.8. *Let Θ be a complete congruence on a semilattice $(P, \vee)$. If S is Θ-dense and $x, y \in P$, then*

(a) $S \subseteq P_\Theta$;

(b) $c_\Theta(x) = \bigwedge_P (\uparrow x)_S$;

(c) $x/\Theta \leq y/\Theta \iff$ *for all* $z \in S$, $y \leq z$ *implies* $x \leq z$;

(d) $x \Theta y \iff (\uparrow x)_S = (\uparrow y)_S$;

(e) P_Θ *is Θ-dense.*

Proof. (a) Let $z \in S$. Since $(z, c_\Theta(z)) \in \Theta = \Theta_S$ and $z \leq z$, we obtain $c_\Theta(z) \leq z$. Hence, $c_\Theta(z) = z$ and $z \in P_\Theta$.

(b) Let $x \in P$. If $z \in (\uparrow x)_S$, then $x \leq z$ implies $c_\Theta(x) \leq c_\Theta(z) = z$ by (a). Thus, $c_\Theta(x)$ is a lower bound of $(\uparrow x)_S$. Let $y \in P$ be any lower bound of $(\uparrow x)_S$. If $x \leq z$ for some $z \in S$, then $z \in (\uparrow x)_S$ and hence $c_\Theta(x) \vee y \leq z$. On the other hand, if $c_\Theta(x) \vee y \leq z$ for some $z \in S$, then trivially $x \leq z$. This implies that $(x, c_\Theta(x) \vee y) \in \Theta_S = \Theta$. Hence, $y \leq c_\Theta(x) \vee y \leq c_\Theta(x)$ from which we get $c_\Theta(x) = \bigwedge (\uparrow x)_S$.

(c) Assume that $x/\Theta \leq y/\Theta$. By (3.1), this is equivalent to $c_\Theta(x) \leq c_\Theta(y)$. If $z \in S$ and $y \leq z$, then $x \leq c_\Theta(x) \leq c_\Theta(y) \leq c_\Theta(z) = z$. Conversely, if for all $z \in S$, $y \leq z$ implies $x \leq z$, then $(\uparrow y)_S \subseteq (\uparrow x)_S$ and thus $c_\Theta(x) = \bigwedge (\uparrow x)_S \leq \bigwedge (\uparrow y)_S = c_\Theta(y)$. Condition (d) is obvious by (c).

(e) Let $x, y \in P$. Since for all $z \in P_\Theta$, $x \leq z$ iff $c_\Theta(x) \leq z$, the assumption $(x, y) \in \Theta$ implies $c_\Theta(x) = c_\Theta(y)$ and $(x, y) \in \Theta_{P_\Theta}$. On the other hand, if $(x, y) \in \Theta_{P_\Theta}$, then $x \leq c_\Theta(x) \in P_\Theta$ and $y \leq c_\Theta(y) \in P_\Theta$ imply $x \leq c_\Theta(y)$ and $y \leq c_\Theta(x)$. Hence, $c_\Theta(x) \leq c_\Theta(y) \leq c_\Theta(x)$, which means that $(x, y) \in \Theta$. □

By Lemma 3.4.8(e), every complete congruence Θ on a semilattice $(P, \vee)$ has at least one Θ-dense set, namely P_Θ. Observe that not all congruences on semilattices have dense sets. For example, if A is an infinite set and Θ is the congruence on $(\wp(A), \cup)$, which has the congruence classes

$$\begin{aligned} C_1 &= \{B \subseteq A \mid B \text{ is finite}\}; \\ C_2 &= \{B \subseteq A \mid B \text{ is infinite}\}, \end{aligned}$$

then there exists no $\mathcal{H} \subseteq \wp(A)$ such that

$$\Theta_{\mathcal{H}} = \{(B, C) \mid (\forall X \in \mathcal{H})\ B \subseteq X \iff C \subseteq X\}$$

equals Θ.

Let $\mathcal{P} = (P, \leq)$ be an ordered set and let $S \subseteq P$. Then S is called *meet-dense* in $\mathcal{P}$ if for every element $x \in P$ there is a subset Q of S such that $x = \bigwedge_P Q$. The dual of meet-dense is *join-dense* (see [5], for example). Our next proposition connects Θ-dense sets to sets meet-dense in $(P_\Theta, \leq)$.

Proposition 3.4.9. *Suppose Θ is a complete congruence on a semilattice $(P, \vee)$. If $S \subseteq P$, then the following three conditions are equivalent:*
(a) *S is Θ-dense;*
(b) *S is meet-dense in $(P_\Theta, \leq)$;*
(c) *$z = \bigwedge_P (\uparrow z)_S$ for all $z \in P_\Theta$.*

Proof. It is proved in [5], that (b) and (c) are equivalent. Recall that by Lemma 3.1.1(e) all existing meets in P_Θ coincide with the meets formed in P. If S is Θ-dense, then by Lemma 3.4.8(b), $z = c_\Theta(z) = \bigwedge_P (\uparrow z)_S$, for all $z \in P_\Theta$. Hence, (a) implies (c).

Suppose S is a meet-dense subset of $(P_\Theta, \leq)$. Because $S \subseteq P_\Theta$ holds by the definition of meet-dense sets, and P_Θ is Θ-dense by Lemma 3.4.8(e), we obtain $\Theta = \Theta_{(P_\Theta)} \subseteq \Theta_S$ by (3.3). If $(x, y) \in \Theta_S$, then $(\uparrow x)_S = (\uparrow y)_S$. Since $S \subseteq P_\Theta$, $(\uparrow a)_S = (\uparrow c_\Theta(a))_S$ holds for all $a \in P$. Thus, we get by (c) that

$$c_\Theta(x) = \bigwedge (\uparrow c_\Theta(x))_S = \bigwedge (\uparrow x)_S = \bigwedge (\uparrow y)_S = \bigwedge (\uparrow c_\Theta(y))_S = c_\Theta(y).$$

Hence, $(x, y) \in \Theta$. So, $\Theta_S \subseteq \Theta$ and $\Theta = \Theta_S$, which means that S is Θ-dense. □

We end this section by presenting some results which help us to determine dense sets. If $\mathcal{P} = (P, \vee, \wedge)$ is a lattice, then an element $a \in P$ is *meet-irreducible* if $a = b \wedge c$ implies $a = b$ or $a = c$. We denote the set of all meet-irreducible elements $a \neq 1$ (in case $\mathcal{P}$ has a unit) of $\mathcal{P}$ by $\mathcal{M}(\mathcal{P})$. The *join-irreducible* elements and their set $\mathcal{J}(\mathcal{P})$ are defined dually. The following lemma can be found in [5], for example.

Lemma 3.4.10. *If $\mathcal{P} = (P, \vee, \wedge)$ is a lattice satisfying the* ACC, *then*

$$x = \bigwedge\{a \in \mathcal{M}(\mathcal{P}) \mid x \leq a\},$$

for all $x \in P$. □

Now we can present a proposition which characterizes Θ-dense sets for a complete congruence Θ on $(P, \vee)$ in the case $\mathcal{P} = (P, \leq)$ is a lattice and the quotient set P/Θ has no infinite chains. We denote the ordered set $(P_\Theta, \leq)$ by $\mathcal{P}_\Theta$.

Proposition 3.4.11. *If $(P, \leq)$ is a lattice and Θ is a complete congruence on $(P, \vee)$ such that $(P/\Theta, \leq)$ has no infinite chains, then $S \subseteq P$ is Θ-dense if and only if*

$$\mathcal{M}(\mathcal{P}_\Theta) \subseteq S \subseteq P_\Theta.$$

Proof. Because $(P, \leq)$ is a lattice, $(P/\Theta, \leq)$ and $(P_\Theta, \leq)$ are lattices by Proposition 3.2.8 and Corollary 3.1.2, respectively. By our assumption they do not contain any infinite chains. This implies by Lemma 2.2.9 that $(P/\Theta, \leq)$ and $(P_\Theta, \leq)$ are in fact complete lattices. By Lemma 3.4.10,

$$z = \bigwedge\{a \in \mathcal{M}(\mathcal{P}_\Theta) \mid z \leq a\}$$

for all $z \in P_\Theta$, which means that $\mathcal{M}(\mathcal{P}_\Theta)$ is a meet-dense in $\mathcal{P}_\Theta$ and so it is Θ-dense by Proposition 3.4.9. By Lemma 3.4.8(e), P_Θ is Θ-dense. Hence $\mathcal{M}(\mathcal{P}_\Theta) \subseteq S \subseteq P_\Theta$ implies that $\Theta = \Theta_{P_\Theta} \subseteq \Theta_S \subseteq \Theta_{\mathcal{M}(\mathcal{P}_\Theta)} = \Theta$.

Let S be an arbitrary Θ-dense set. Then $S \subseteq P_\Theta$ by Lemma 3.4.8(a). Because S is Θ-dense, it is meet-dense in $\mathcal{P}_\Theta$. Assume $x \in \mathcal{M}(\mathcal{P}_\Theta)$. Then there exists an $A \subseteq S$ such that $x = \bigwedge A$. If $A = \emptyset$, then $x = 1$, a contradiction! Hence, $A \neq \emptyset$. Because $\mathcal{P}_\Theta$ satisfies the DCC, there exists by the dual of Lemma 2.2.9 a finite $F \subseteq A$ such that $x = \bigwedge F$. Since x is meet-irreducible and $F(\subseteq P_\Theta)$ is finite and nonempty, this implies $x \in F$. The fact $F \subseteq A \subseteq S$ yields $x \in S$. Thus, $\mathcal{M}(\mathcal{P}_\Theta) \subseteq S$. □

By the next lemma, which appears in [5], Θ-dense sets can be found from the Hasse diagram of $\mathcal{P}_\Theta$.

Lemma 3.4.12. *Let $\mathcal{P} = (P, \leq)$ be a finite lattice. Then $a \in \mathcal{M}(\mathcal{P})$ if and only if a is covered by exactly one element of P.* □

We end this section by giving the following example.

Example 3.4.13. Let $(P, \leq)$ be the lattice defined by the Hasse diagram of Figure 6.

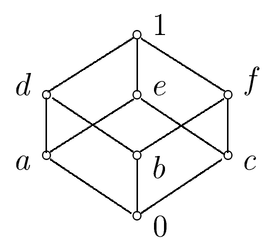


Figure 6.

Let Θ be the congruence of the semilattice $(P, \vee)$ with the congruence classes

$$\{0\}, \{a\}, \{c\}, \{b, d\}, \{e, f, 1\}.$$

The congruence Θ is naturally complete and $P_\Theta = \{0, a, c, d, 1\}$. The Hasse diagram of $\mathcal{P}_\Theta = (P_\Theta, \leq)$ is presented in Figure 7.

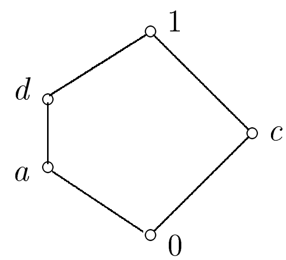


Figure 7.

Since $(P_\Theta, \leq)$ is finite, an element $x \neq 1$ is meet-irreducible in it if, and only if, x is covered by exactly one element. The least Θ-dense set $\mathcal{M}(\mathcal{P}_\Theta)$ is therefore $\{a, c, d\}$, the greatest Θ-dense is P_Θ, and $\{0, a, c, d\}$ is a third Θ-dense set, since $\mathcal{M}(\mathcal{P}_\Theta) \subseteq \{0, a, c, d\} \subseteq P_\Theta$.

3.5 Dual Galois Connections

In Section 6.1 we will show that for any $R \in \mathrm{Tol}(U)$, the maps ${}^{R}\colon \wp(U) \to \wp(U)$ and ${}_{R}\colon \wp(U) \to \wp(U)$, which assign to each subset of U its upper and lower R-approximations, respectively, form a Galois connection between $(\wp(U), \subseteq)$ and $(\wp(U), \supseteq)$.

In this work a Galois connection $({}^{\blacktriangleright}, {}^{\blacktriangleleft})$ between $(P, \leq)$ and $(P, \geq)$ is called a dual Galois connection on $(P, \leq)$. We show that if $\mathcal{P} = (P, \leq)$ is a complete lattice and $({}^{\blacktriangleright}, {}^{\blacktriangleleft})$ is a dual Galois connection on $\mathcal{P}$, then ${}^{\blacktriangleright}\colon P \to P$ is a complete join-morphism $\mathcal{P} \to \mathcal{P}$. Its kernel $\Theta_{\blacktriangleright}$ is a complete congruence on $(P, \vee)$ such that the greatest element in the $\Theta_{\blacktriangleright}$-class of any $x \in P$ is $x^{\blacktriangleright\blacktriangleleft}$. By duality, ${}^{\blacktriangleleft}\colon P \to P$ is a complete meet-morphism $\mathcal{P} \to \mathcal{P}$ and its kernel $\Theta_{\blacktriangleleft}$ is a congruence on $(P, \wedge)$ such that the $\Theta_{\blacktriangleleft}$-class of any $x \in P$ has a least element $x^{\blacktriangleleft\blacktriangleright}$.

The following definition of Galois connections can be found in [5, 12, 36, 51], for example

Definition. Let $\mathcal{P} = (P, \leq)$ and $\mathcal{Q} = (Q, \leq)$ be ordered sets. A pair $({}^{\blacktriangleright}, {}^{\blacktriangleleft})$ of maps ${}^{\blacktriangleright}\colon P \to Q$ and ${}^{\blacktriangleleft}\colon Q \to P$ (which we refer to as the *right map* and the *left map*, respectively) is called a *Galois connection* between $\mathcal{P}$ and $\mathcal{Q}$ if

(a) ${}^{\blacktriangleright}$ and ${}^{\blacktriangleleft}$ are order-reversing and

(b) $p \leq p^{\blacktriangleright\blacktriangleleft}$ for all $p \in P$, and $q \leq q^{\blacktriangleleft\blacktriangleright}$ for all $q \in Q$.

In the following proposition we present some basic properties of Galois connections. Statements (a)–(c) can be found in [36] and (d) in [5], for example.

Proposition 3.5.1. *Suppose* $({}^{\blacktriangleright}, {}^{\blacktriangleleft})$ *is a Galois connection between* $\mathcal{P} = (P, \leq)$ *and* $\mathcal{Q} = (Q, \leq)$.

(a) *For all* $p \in P$ *and* $q \in Q$, $p^{\blacktriangleright\blacktriangleleft\blacktriangleright} = p^{\blacktriangleright}$ *and* $q^{\blacktriangleleft\blacktriangleright\blacktriangleleft} = q^{\blacktriangleleft}$.

(b) *The maps* $c\colon P \to P$, $p \mapsto p^{\blacktriangleright\blacktriangleleft}$ *and* $k\colon Q \to Q$, $q \mapsto q^{\blacktriangleleft\blacktriangleright}$ *are closure operators on* $\mathcal{P}$ *and* $\mathcal{Q}$*, respectively.*

(c) *If* c *and* k *are the mappings defined in (b), then restricted to the sets of* c*-closed elements* P_c *and* k*-closed elements* Q_k*, respectively,* ${}^{\blacktriangleright}$ *and* ${}^{\blacktriangleleft}$ *yield a pair* ${}^{\blacktriangleright}\colon P_c \to Q_k$, ${}^{\blacktriangleleft}\colon Q_k \to P_c$ *of mutually inverse dual order-isomorphisms between* $(P_c, \leq)$ *and* $(Q_k, \leq)$.

(d) *The map* ${}^{\blacktriangleright}\colon P \to Q$ *is a complete join-morphism from* $(P, \leq)$ *to* $(Q, \geq)$ *and* ${}^{\blacktriangleleft}\colon Q \to P$ *is a complete join-morphism from* $(Q, \leq)$ *to* $(P, \geq)$. $\square$

We denote by $\Theta_{\blacktriangleright}$ and $\Theta_{\blacktriangleleft}$ the kernels of the maps ${}^{\blacktriangleright}\colon P \to Q$ and ${}^{\blacktriangleleft}\colon Q \to P$, respectively. Now we can write the following lemma.

Lemma 3.5.2. *Let* $(^{\blacktriangleright}, ^{\blacktriangleleft})$ *be a Galois connection between two complete join-semilattices* $\mathcal{P} = (P, \leq)$ *and* $\mathcal{Q} = (Q, \leq)$.

(a) *The relation* $\Theta_{\blacktriangleright}$ *is a complete congruence on* $(P, \vee)$ *such that the greatest element in the* $\Theta_{\blacktriangleright}$*-class of any* $p \in P$ *is* $p^{\blacktriangleright\blacktriangleleft}$.

(b) *The relation* $\Theta_{\blacktriangleleft}$ *is a complete congruence on* $(Q, \vee)$ *such that the greatest element in the* $\Theta_{\blacktriangleleft}$*-class of any* $q \in Q$ *is* $q^{\blacktriangleleft\blacktriangleright}$.

Proof. It follows directly from Propositions 3.3.3(b) and 3.5.1(d) that $\Theta_{\blacktriangleright}$ is a complete congruence on $(P, \vee)$. Let $p \in P$. By Proposition 3.5.1(a), $p^{\blacktriangleright\blacktriangleleft\blacktriangleright} = p^{\blacktriangleright}$, which implies $(p, p^{\blacktriangleright\blacktriangleleft}) \in \Theta_{\blacktriangleright}$. If $(p, x) \in \Theta_{\blacktriangleright}$, then $p^{\blacktriangleright} = x^{\blacktriangleright}$ and hence $x \leq x^{\blacktriangleright\blacktriangleleft} = p^{\blacktriangleright\blacktriangleleft}$. Thus, $p^{\blacktriangleright\blacktriangleleft}$ is the greatest element of $p/\Theta_{\blacktriangleright}$. Assertion (b) can be proved similarly. □

Before we introduce dual Galois connections, we consider shortly interior operators and congruences on semilattices $(P, \wedge)$ such that each congruence class has a least element.

Let $\mathcal{P} = (P, \leq)$ be an ordered set. If $i \colon P \to P$ is a closure operator on $\mathcal{P}^{\partial} = (P, \geq)$, it is an *interior operator* on $\mathcal{P}$. This means that $i \colon P \to P$ is an interior operator on $\mathcal{P}$ if and only if $i(x) \leq x$, $x \leq y$ implies $i(x) \leq i(y)$, and $i(i(x)) = i(x)$ for all $x, y \in P$. We denote by $\mathrm{Int}(\mathcal{P})$ the set of all interior operators on $\mathcal{P}$. The set $\mathrm{Int}(\mathcal{P})$ may be ordered with the pointwise order. Obviously, $1_P \colon x \mapsto x$ is the greatest element in $(\mathrm{Int}(\mathcal{P}), \leq)$, and if $\mathcal{P}$ has a bottom element, then $f_{\perp} \colon x \mapsto \perp$ is the bottom element of $(\mathrm{Int}(\mathcal{P}), \leq)$.

Let us consider a semilattice $\mathcal{P} = (P, \wedge)$. As before, we define the order $\leq$ on P by

$$a \leq b \text{ if and only if } a \wedge b = a.$$

A congruence Θ on $\mathcal{P}$ is *complete* if for any $x \in P$, the congruence class x/Θ has a least element $i_{\Theta}(x)$. The set of all complete congruences on $\mathcal{P}$ is denoted by $\mathrm{Con_i}(\mathcal{P})$. The letter "i" in $i_{\Theta}(x)$ and in $\mathrm{Con_i}(\mathcal{P})$ suggests the word "interior". Obviously, $\mathrm{Con_i}(\mathcal{P})$ may be ordered with the usual set inclusion. The ordered set $(\mathrm{Con_i}(\mathcal{P}), \subseteq)$ has Δ_P as the bottom element and if $\mathcal{P}$ has a bottom element, then ∇_P is the top element of $(\mathrm{Con_i}(\mathcal{P}), \subseteq)$.

The next proposition follows from Proposition 3.2.3.

Proposition 3.5.3. *Let* $\mathcal{P} = (P, \wedge)$ *be a semilattice.*

(a) *If* Θ *is a complete congruence on* $\mathcal{P}$, *then* $i_{\Theta} \colon P \to P, x \mapsto i_{\Theta}(x)$, *is an interior operator on* $(P, \leq)$.

(b) *If i is an interior operator on $(P, \leq)$, then its kernel Θ_i is a complete congruence on $\mathcal{P}$ such that the smallest element in the Θ_i-class of any $x \in P$ is $i(x)$.*

(c) *The mappings $\Theta \mapsto i_\Theta$ and $i \mapsto \Theta_i$ form a pair of mutually inverse dual order-isomorphisms between the ordered set of interior operators $(\mathrm{Int}(\mathcal{P}), \leq)$ and the ordered set $(\mathrm{Con_i}(\mathcal{P}), \subseteq)$ of complete congruences on $\mathcal{P}$.* □

Let $\mathcal{P} = (P, \leq)$ be a complete meet-semilattice. Then $(\mathrm{Int}(\mathcal{P}), \leq)$ and $(\mathrm{Con_i}(\mathcal{P}), \subseteq)$ are dually isomorphic complete lattices. Let $\mathcal{Q} = (Q, \leq)$ be an ordered set and let $f\colon P \to Q$ be a complete meet-morphism $\mathcal{P} \to \mathcal{Q}$. Then the kernel Θ_f of f is a complete congruence on $(P, \wedge)$ such that the least element of the Θ_f-class of any $x \in P$ is $\bigwedge x/\Theta_f$.

Definition. Let $\mathcal{P} = (P, \leq)$ be an ordered set. A Galois connection between $\mathcal{P}$ and $\mathcal{P}^\partial = (P, \geq)$ is called a *dual Galois connection* on $\mathcal{P}$.

So, $(^\blacktriangleright, ^\blacktriangleleft)$ is a dual Galois connection on $\mathcal{P} = (P, \leq)$ if and only if $^\blacktriangleright$ and $^\blacktriangleleft$ are order-preserving and $p^{\blacktriangleleft\blacktriangleright} \leq p \leq p^{\blacktriangleright\blacktriangleleft}$ for all $p \in P$. We conclude this chapter by presenting some results concerning dual Galois connections on a complete lattice.

Proposition 3.5.4. *Let $(^\blacktriangleright, ^\blacktriangleleft)$ be a dual Galois connection on a complete lattice $\mathcal{P} = (P, \leq)$.*

(a) *The map $c\colon P \to P$, $p \mapsto p^{\blacktriangleright\blacktriangleleft}$, is a closure operator on $\mathcal{P}$ and $k\colon P \to P$, $p \mapsto p^{\blacktriangleleft\blacktriangleright}$, is an interior operator on $\mathcal{P}$.*

(b) *If c and k are the mappings defined in (a), then restricted to the sets of c-closed elements P_c and k-closed elements P_k, respectively, $^\blacktriangleright$ and $^\blacktriangleleft$ yield a pair $^\blacktriangleright\colon P_c \to P_k$, $^\blacktriangleleft\colon P_k \to P_c$ of mutually inverse order-isomorphisms between $(P_c, \leq)$ and $(P_k, \leq)$.*

(c) *The map $^\blacktriangleright\colon P \to P$ is a complete join-morphism $\mathcal{P} \to \mathcal{P}$ and $^\blacktriangleleft\colon P \to P$ is a complete meet-morphism $\mathcal{P} \to \mathcal{P}$.*

(d) *The relation $\Theta_\blacktriangleright$ is a complete congruence on $(P, \vee)$ such that the greatest element in the $\Theta_\blacktriangleright$-class of any $p \in P$ is $p^{\blacktriangleright\blacktriangleleft}$.*

(e) *The relation $\Theta_\blacktriangleleft$ is a complete congruence on $(P, \wedge)$ such that the least element in the $\Theta_\blacktriangleleft$-class of any $p \in P$ is $p^{\blacktriangleleft\blacktriangleright}$.*

Proof. Claim (a) follows easily from Proposition 3.5.1(b).

(b) By Corollary 3.1.2, $(P_c, \leq)$ and $(P_k, \leq)$ are complete lattices and the maps $^\blacktriangleright\colon P_c \to P_k$ and $^\blacktriangleleft\colon P_k \to P_c$ are mutually inverse order-isomorphisms between these complete lattices by Proposition 3.5.1(c). Assertion (c) is obvious by Proposition 3.5.1(d) and statements (d) and (e) follow from Lemma 3.5.2. □

Chapter 4

Information Systems and Preimage Relations

4.1 Indiscernibility and Similarity in Information Systems

Here we consider informations systems introduced by Pawlak [41, 42]. An *information system* is a triple $\mathcal{S} = (U, A, \{V_a\}_{a \in A})$, where U is a set of *objects*, A is a set of *attributes*, and $\{V_a\}_{a \in A}$ is an indexed set of *value sets of attributes*. Each attribute $a \in A$ is a mapping $a \colon U \to V_a$.

Example 4.1.1. An information system $\mathcal{S}$ in which the sets U and A are finite can be represented by a table. The rows of the table are labeled by the objects and the columns by the attributes of the system $\mathcal{S}$. In the intersection of the row labeled by an object x and the column labeled by an attribute a we find the value $a(x)$.

Let us consider an information system $\mathcal{S} = (U, A, \{V_a\}_{a \in A})$, where the object set $U = \{1, 2, 3, 4\}$ consists of four persons called 1, 2, 3, and 4, respectively. The attribute set A has the attributes Age, Eyes, and Height. The corresponding value sets are $V_{\text{Age}} = \{\text{Young}, \text{Middle-aged}, \text{Old}\}$, $V_{\text{Eyes}} = \{\text{Blue}, \text{Brown}, \text{Green}\}$, $V_{\text{Height}} = \{\text{Short}, \text{Normal}, \text{Tall}\}$.

Let the values of attributes be defined as in Table 1.

	Age	Eyes	Height
1	Young	Blue	Short
2	Old	Brown	Normal
3	Middle-aged	Brown	Tall
4	Young	Green	Short

Table 1.

In [37] Orłowska and Pawlak defined nondeterministic information systems, in which attributes assign a nonempty subset of values to every object. In such systems it is possible to define several kinds of relations on the object set which are based on the attribute values. The following definition can be found in [38, 39, 40], for example.

Definition. A *nondeterministic information system* is a triple $\mathcal{S} = (U, A, \{V_a\}_{a\in A})$, where U is a nonempty set of *objects*, A is a nonempty set of *attributes*, and $\{V_a\}_{a\in A}$ is an indexed set of *value sets of attributes*. Each attribute is a function $a\colon U \to \wp(V_a) - \{\emptyset\}$.

There are two ways to interpret the knowledge represented by a nondeterministic information system. Let $\mathcal{S} = (U, A, \{V_a\}_{a\in A})$ be a nondeterministic information system and let $a \in A$ and $x \in U$.

1. If $\mathcal{S}$ is a *many-valued* information system, then $a(x)$ is the set of all values of the attribute a for the object x. This means that every $v \in a(x)$ is an actual a-value for x.

2. If $\mathcal{S}$ is an *approximate* information system, then the unique value of the attribute a for the object x is assumed to be in the set $a(x)$. Note that the complete ignorance is denoted by $a(x) = V_a$.

The use of "nondeterministic information system" is somewhat misleading when many-valued systems are considered. However, we accept this drawback since the use of this term is a standard practice in the literature.

An "ordinary" information system $\mathcal{S} = (U, A, \{V_a\}_{a\in A})$, where each $a(\in A)$ is a map $a\colon U \to V_a$, can be considered as a nondeterministic information system such that always $|a(x)| = 1$ and $a(x) = \{v\}$ is written $a(x) = v$.

Next we present two examples which illustrate the above classification. Both examples deal with a nondeterministic information system $\mathcal{S} = (U, A, \{V_a\}_{a\in A})$, where the object set $U = \{1, 2, 3, 4\}$ consists of four persons called 1, 2, 3, and 4, respectively.

Example 4.1.2. (Many-valued information system) Let $\mathcal{S}$ be the nondeterministic information system such that A has the attributes "Degrees" and "Knowledge of languages", $V_{\text{Degrees}} = \{\text{BA}, \text{BSc}, \text{MA}, \text{MSc}, \text{PhD}\}$, $V_{\text{Languages}} = \{\text{English}, \text{Finnish}\}$, and the values of the attributes are defined in Table 2.

	Degrees	Languages
1	{BSc}	{English}
2	{BSc, MSc, PhD}	{English}
3	{BA, MA}	{English, Finnish}
4	{BA,MA}	{English, Finnish}

Table 2.

For example, the person 1 is a Bachelor of Sciences who speaks only English.

Example 4.1.3. (Approximate information system) Let $\mathcal{S}$ be the nondeterministic information system in which $A = \{\text{Height, Weight}\}$, $V_{\text{Height}} = V_{\text{Weight}} = \mathbb{N}$, and let the values for the attributes be given in Table 3.

	Height (cm)	Weight (kg)
1	$\{184, \ldots, 187\}$	$\{80, \ldots, 85\}$
2	$\{170, \ldots, 178\}$	$\mathbb{N}$
3	$\{178, \ldots, 185\}$	$\{80, \ldots, 85\}$
4	$\{170, \ldots, 178\}$	$\mathbb{N}$

Table 3.

The values of attributes are now only approximations. We know, for instance, that the height of the person 1 is between 184 and 187 and her/his weight is between 80 and 85. Note that we know nothing about the weight of the persons 2 and 4.

Let $\mathcal{S} = (U, A, \{V_a\}_{a \in A})$ be a nondeterministic information system. The following definitions of strong and weak binary relations of indiscernibility and similarity can be found in e.g. [38, 39, 40]. Let $B \subseteq A$ and $x, y \in U$.

Strong indiscernibility: $(x, y) \in ind(B)$ iff $a(x) = a(y)$ for all $a \in B$.

Weak indiscernibility: $(x, y) \in wind(B)$ iff $a(x) = a(y)$ for some $a \in B$.

Strong similarity: $(x, y) \in sim(B)$ iff $a(x) \cap a(y) \neq \emptyset$ for all $a \in B$.

Weak similarity: $(x, y) \in wsim(B)$ iff $a(x) \cap a(y) \neq \emptyset$ for some $a \in B$.

Let $\mathcal{S}$ be a nondeterministic information system. Two objects are in relation $ind(B)$ whenever we cannot distinguish them by the values of the attributes in B.

If there exists at least one attribute a in B such that x and y are a-indiscernible (that is, $a(x) = a(y)$), then x and y are $wind(B)$-related. A pair (x, y) is in $sim(B)$, if the values of all attributes in B for x and y have at least one common value, and (x, y) is in $wsim(B)$ if x and y have at least one common a-value for some attribute a in B.

Example 4.1.4. (a) In the many-valued information system of Example 4.1.2, $(3, 4) \in ind(\{\text{Degrees}, \text{Languages}\})$, since 3 and 4 have earned the same degrees and they speak exactly the same languages. The persons 1 and 2 are in relation $ind(\{\text{Languages}\})$ and in relation $wind(\{\text{Degrees}, \text{Languages}\})$ because they both speak only English.

The persons 1 and 2 are in relation $sim(\{\text{Degrees}, \text{Languages}\})$ because they have a common degree and a common language. The relation $wsim(\{\text{Degrees}, \text{Languages}\}) = sim(\{\text{Languages}\})$ is the all relation of U, since all persons have a common language.

(b) In the approximate information system of Example 4.1.3, $(2, 3) \in ind(\{\text{Height}, \text{Weight}\})$ since the sets approximating the height and the weight of the persons 2 and 3 are the same. The persons 1 and 3 are in relation $ind(\{\text{Weight}\})$ and in relation $wind(\{\text{Height}, \text{Weight}\})$ because the sets approximating their weight are equal.

In an approximate information system strong similarity relations can be considered as indiscernibility relations. Namely, if $(x, y) \in sim(B)$ for some $B \subseteq A$, then we cannot certainly distinguish the objects x and y because it is possible that their actual values for every attribute in B are the same. The persons 1 and 3 are $sim(\{\text{Height}, \text{Weight}\})$-related because the subsets approximating their height and weight have common values. The relation $wsim(\{\text{Height}, \text{Weight}\}) = sim(\{\text{Weight}\})$ is the all relation of U since all objects have common possible values for the attribute Weight.

In the next lemma we present some obvious properties of indiscernibility and similarity relations. Most of them appear in [38, 39, 40].

Lemma 4.1.5. *If $\mathcal{S} = (U, A, \{V_a\}_{a \in A})$ is a nondeterministic information system, $\emptyset \neq B \subseteq A$, and $a \in A$, then*

(a) *$ind(B)$ is an equivalence;*

(b) *$wind(B)$, $sim(B)$, and $wsim(B)$ are tolerances;*

(c) *$ind(\{a\}) = wind(\{a\})$ and $sim(\{a\}) = wsim(\{a\})$;*

(d) $ind(B) \subseteq wind(B)$ *and* $sim(B) \subseteq wsim(B)$*;*
(e) $ind(B) \subseteq sim(B)$ *and* $wind(B) \subseteq wsim(B)$*;*
(f) $ind(\emptyset) = sim(\emptyset) = \nabla_U$ *and* $wind(\emptyset) = wsim(\emptyset) = \emptyset$. □

4.2 Preimage Relations

In the literature we can find several relations defined in the object set of an information system based on relationships between the values of attributes. It seems that these relations are similar in the following sense. Two objects are in a certain strong (resp. weak) relation with respect to an attribute set B if and only if their values of all (resp. some) attributes in B are in a specified relation. For example, objects x and y are in relation $sim(B)$ if and only if $a(x) \cap a(y) \neq \emptyset$ for all $a \in B$. In this section we introduce the general notion of preimage relations, which allows us to study the common features of strong and weak relations defined in information systems.

Let U and Y be nonempty sets, $R \in \mathrm{Rel}(Y)$, and let $f\colon U \to Y$ be a function. The *preimage relation* of R is defined by

$$f^{-1}(R) = \{(x, y) \in U^2 \mid f(x)Rf(y)\}.$$

So, two elements of U are in relation $f^{-1}(R)$ if and only if their images are in relation R. In particular, the preimage of the diagonal relation of Y is the kernel of the map f, that is to say, $f^{-1}(\Delta_Y) = \Theta_f$.

Our following obvious lemma shows that $f^{-1}(R)$ inherits many properties from R.

Lemma 4.2.1. *Let U and Y be nonempty sets, $f \in Y^U$ and $R \in \mathrm{Rel}(Y)$. If R is reflexive, irreflexive, symmetric, or transitive, then so is $f^{-1}(R)$.* □

It is also true that

$$f^{-1}(R^{\complement}) = (f^{-1}(R))^{\complement}.$$

Example 4.2.2. Let $\mathcal{S} = (U, A, \{V_a\}_{a \in A})$ be an approximate information system, in which $U = \{1, 2, 3, 4\}$, $A = \{\mathrm{Age}, \mathrm{Height}, \mathrm{Weight}\}$, and $V_a = \mathbb{N}$ for all $a \in A$. The values of the attributes are defined as in Table 4.

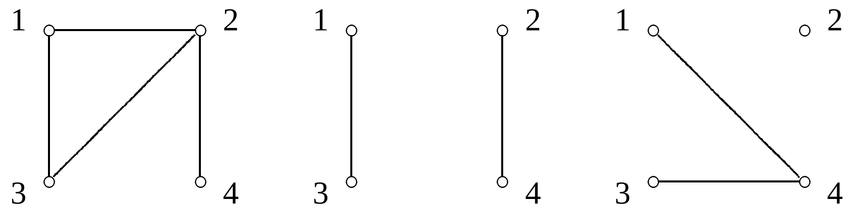
	Age (years)	Weight (kg)	Height (cm)
1	$\{22, \ldots, 26\}$	$\{48, \ldots, 54\}$	$\{154, \ldots, 157\}$
2	$\{26, \ldots, 33\}$	$\{73, \ldots, 78\}$	$\{170, \ldots, 175\}$
3	$\{24, \ldots, 29\}$	$\{51, \ldots, 58\}$	$\{159, \ldots, 162\}$
4	$\{31, \ldots, 37\}$	$\{75, \ldots, 82\}$	$\{157, \ldots, 161\}$

Table 4.

We denote $Y = \wp(\mathbb{N}) - \{\emptyset\}$. Let us define a binary relation SIM on Y by setting for all $W_1, W_2 \in Y$,

$$(W_1, W_2) \in SIM \iff W_1 \cap W_2 \neq \emptyset.$$

The relation SIM is obviously a tolerance. This implies that $a^{-1}(SIM)$ is a tolerance for all $a \in A$. Two objects are, for example, in relation $\mathrm{Age}^{-1}(SIM)$ if and only if their ages are possibly the same. The preimage relations of SIM with respect to the attributes Age, Weight, and Height are represented graphically by the following graphs.



$\mathrm{Age}^{-1}(SIM)$ $\mathrm{Weight}^{-1}(SIM)$ $\mathrm{Height}^{-1}(SIM)$

Next we shall extend the notion of preimage relation in a natural way. For any set of functions $B \subseteq Y^U$, the *strong* and the *weak preimage relations of* B are defined by

$$\begin{aligned} S_R(B) &= \{(x,y) \in U^2 \mid (\forall f \in B) f(x) R f(y)\}; \\ W_R(B) &= \{(x,y) \in U^2 \mid (\exists f \in B) f(x) R f(y)\}, \end{aligned}$$

respectively. In the next lemma we present some basic properties of strong and weak preimage relations.

Lemma 4.2.3. *Let U and Y be nonempty sets and $R \in \mathrm{Rel}(Y)$. If $\mathcal{H} \subseteq \wp(Y^U)$, $B, C \subseteq Y^U$, and $f \in Y^U$, then*

(a) $S_R(\{f\}) = W_R(\{f\}) = f^{-1}(R)$;

(b) $S_R(B) = \bigcap\{f^{-1}(R) \mid f \in B\}$ *and* $W_R(B) = \bigcup\{f^{-1}(R) \mid f \in B\}$;

(c) $S_R(\emptyset) = U \times U$ *and* $W_R(\emptyset) = \emptyset$;

(d) $S_R(\bigcup \mathcal{H}) = \bigcap\{S_R(B) \mid B \in \mathcal{H}\}$ *and* $W_R(\bigcup \mathcal{H}) = \bigcup\{W_R(B) \mid B \in \mathcal{H}\}$;

(e) $B \subseteq C$ *implies* $S_R(C) \subseteq S_R(B)$ *and* $W_R(B) \subseteq W_R(C)$;

(f) $S_R(B) \subseteq W_R(B)$ *whenever* $B \neq \emptyset$;

(g) $S_R(B)^{\complement} = W_{(R^{\complement})}(B)$ *and* $W_R(B)^{\complement} = S_{(R^{\complement})}(B)$.

Proof. Claims (a), (b), and (c) are obvious.

(d) For any $x, y \in U$,

$$\begin{aligned}(x,y) \in S_R(\bigcup \mathcal{H}) &\iff (\forall f \in \bigcup \mathcal{H})\ f(x) R f(y) \\ &\iff (\forall B \in \mathcal{H})(\forall f \in B)\ f(x) R f(y) \\ &\iff (\forall B \in \mathcal{H})\ (x,y) \in S_R(B) \\ &\iff (x,y) \in \bigcap \{S_R(B) \mid B \in \mathcal{H}\}.\end{aligned}$$

The other claim of (d) is proved similarly.

(e) If $B \subseteq C$, then by (d), $S_R(B) \cap S_R(C) = S_R(B \cup C) = S_R(C)$, which implies $S_R(C) \subseteq S_R(B)$. The proof for the other inclusion is analogous.

(f) is obvious.

(g) For any $x, y \in U$,

$$\begin{aligned}(x,y) \in S_R(B)^{\complement} &\iff (x,y) \notin S_R(B) \\ &\iff (\exists f \in B)\ (f(x), f(y)) \notin R \\ &\iff (\exists f \in B)\ (f(x), f(y)) \in R^{\complement} \\ &\iff (x,y) \in W_{(R^{\complement})}(B).\end{aligned}$$

The other equality can be proved in a similar way. □

Our following proposition, which extends Lemma 4.2.1, shows that also strong and weak preimage relations inherit many properties from the original relation.

Proposition 4.2.4. *Let U and Y be nonempty sets, $R \in \mathrm{Rel}(Y)$, and let $\emptyset \neq B \subseteq Y^U$ be a set of functions. If R is reflexive, irreflexive or symmetric, then so are $S_R(B)$ and $W_R(B)$. Moreover, if R is transitive, then $S_R(B)$ is transitive.* □

In the following we shall present some relations defined in the object set of a nondeterministic information system based on the values of attributes for objects. This kind of relations are in general called *information relations*. The following relations are defined in [38, 39, 40], for example. Let $\mathcal{S} = (U, A, \{V_a\}_{a \in A})$ be a nondeterministic information system and $B \subseteq A$.

Strong inclusion: $(x,y) \in inc(B)$ iff $a(x) \subseteq a(y)$ for all $a \in B$.

Weak inclusion: $(x,y) \in winc(B)$ iff $a(x) \subseteq a(y)$ for some $a \in B$.

Strong diversity: $(x, y) \in div(B)$ iff $a(x) \neq a(y)$ for all $a \in B$.

Weak diversity: $(x, y) \in wdiv(B)$ iff $a(x) \neq a(y)$ for some $a \in B$.

Strong orthogonality: $(x, y) \in ort(B)$ iff $a(x) \cap a(y) = \emptyset$ for all $a \in B$.

Weak orthogonality: $(x, y) \in wort(B))$ iff $a(x) \cap a(y) = \emptyset$ for some $a \in B$.

Strong negative similarity: $(x, y) \in nim(B)$ iff $a(x) \cap a(y)^{\complement} \neq \emptyset$ for all $a \in B$.

Weak negative similarity: $(x, y) \in wnim(B)$ iff $a(x) \cap a(y)^{\complement} \neq \emptyset$ for some $a \in B$.

For instance, two objects are weakly B-diverse if their values for all attributes in B are not the same, and two objects are strongly B-orthogonal if they have no common value for any attribute in B.

Information relations are preimage relations, as we see in the next example.

Example 4.2.5. Let $\mathcal{S} = (U, A, \{V_a\}_{a \in A})$ be a nondeterministic information system. Let us set $V = \bigcup_{a \in A} V_a$ and $Y = \wp(V) - \{\emptyset\}$. Now we can define the following relations on Y.

$$
\begin{array}{lcl}
(W_1, W_2) \in IND & \iff & W_1 = W_2; \\
(W_1, W_2) \in SIM & \iff & W_1 \cap W_2 \neq \emptyset; \\
(W_1, W_2) \in INC & \iff & W_1 \subseteq W_2; \\
(W_1, W_2) \in DIV & \iff & W_1 \neq W_2; \\
(W_1, W_2) \in ORT & \iff & W_1 \cap W_2 = \emptyset; \\
(W_1, W_2) \in NIM & \iff & W_1 \cap W_2{}^{\complement} \neq \emptyset.
\end{array}
$$

It is obvious that $IND^{\complement} = DIV$, $SIM^{\complement} = ORT$, and $INC^{\complement} = NIM$.

It is now easy to observe that information relations are preimage relations and hence we may apply results concerning preimage relations to information relations. Namely, for any subset $B(\subseteq A)$ of attributes,

$$
\begin{array}{lcl}
ind(B) = S_{IND}(B) & \text{and} & wind(B) = W_{IND}(B); \\
sim(B) = S_{SIM}(B) & \text{and} & wsim(B) = W_{SIM}(B); \\
inc(B) = S_{INC}(B) & \text{and} & winc(B) = W_{INC}(B);
\end{array}
$$

$$\begin{array}{lll} div(B) = S_{DIV}(B) & \text{and} & wdiv(B) = W_{DIV}(B); \\ ort(B) = S_{ORT}(B) & \text{and} & wort(B) = W_{ORT}(B); \\ nim(B) = S_{NIM}(B) & \text{and} & wnim(B) = W_{NIM}(B). \end{array}$$

By Lemma 4.2.3, $ind(B \cup C) = ind(B) \cap ind(C)$ and $wind(B \cup C) = wind(B) \cup wind(C)$ for all $B, C \subseteq A$. Moreover, $IND^{\complement} = DIV$ implies that $ind(B)^{\complement} = S_{IND}(B)^{\complement} = W_{(IND^{\complement})}(B) = W_{DIV}(B) = wdiv(B)$, for example.

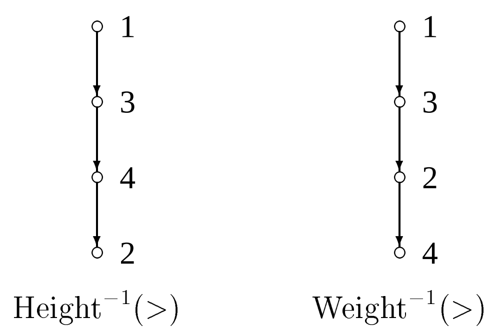
We can now easily define various relations in information systems as preimage relations as shown in the next example.

Example 4.2.6. Let $\mathcal{S} = (U, A, \{V_a\}_{a \in A})$ be an information system in which $U = \{1, 2, 3, 4\}$, $A = \{\text{Height}, \text{Weight}\}$, and $V_a = \mathbb{N}$ for all $a \in A$. The values of attributes are given in Table 5.

	Height (cm)	Weight (kg)
1	186	80
2	157	59
3	172	64
4	166	52

Table 5.

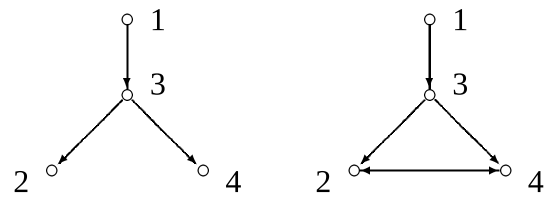
The attributes in A are functions $a \colon U \to \mathbb{N}$. Let us now consider the usual order relation $>$ on $\mathbb{N}$. The preimage relations of $>$ with respect to the attributes "Height" and "Weight" are represented graphically by the following graphs:



For all $B \subseteq A$,

$$\begin{array}{rcl} S_{>}(B) & = & \{(x, y) \in U^2 \mid (\forall a \in B)\ a(x) > a(y)\}; \\ W_{>}(B) & = & \{(x, y) \in U^2 \mid (\exists a \in B)\ a(x) > a(y)\}. \end{array}$$

Now $(x, y) \in S_>(A)$ if and only if x is taller and heavier than y, and $(x, y) \in W_>(A)$ if and only if x is taller or heavier than y.



$S_>(A)$ $W_>(A)$

4.3 Matrices of Preimage Relations

Skowron and Rauszer introduced discernibility matrices in [52]. They presented several results concerning cores, dependencies, and reducts defined in information systems by applying this notion. Here we introduce matrix representations of preimage relations as a generalization of discernibility matrices.

Let $U = \{x_i\}_{i \in I}$ and Y be nonempty sets, $R \in \mathrm{Rel}(Y)$, and let $A (\subseteq Y^U)$ be a set of functions. The *matrix* $M(R) = (c_{ij})$ *of preimage relations* of R with respect to A is defined so that

$$c_{ij} = \{f \in A \mid (x_i, x_j) \in f^{-1}(R)\},$$

for all $i, j \in I$. Thus, the entry c_{ij} consists of those functions $f \in A$ for which $f(x_i) R f(x_j)$. Obviously, the following lemma holds.

Lemma 4.3.1. *If $U = \{x_i\}_{i \in I}$ and Y are nonempty sets, $R \in \mathrm{Rel}(Y)$, $A \subseteq Y^U$, and $M(R) = (c_{ij})$ is the matrix of preimage relations of R with respect to A, then for all $B \subseteq A$ and $i, j \in I$,*

(a) $(x_i, x_j) \in S_R(B)$ *iff* $B \subseteq c_{ij}$;

(b) $(x_i, x_j) \in W_R(B)$ *iff* $B \cap c_{ij} \neq \emptyset$. □

In the previous section we saw that information relations are actually preimage relations. Therefore they can be represented by matrices.

Suppose $\mathcal{S} = (U, A, \{V_a\}_{a \in A})$ is a nondeterministic information system such that $U = \{x_i\}_{i \in I}$. Then $M(IND)_{\mathcal{S}} = (c_{ij})$ is the *indiscernibility matrix* of $\mathcal{S}$, if for all $i, j \in I$,

$$c_{ij} = \{a \in A \mid (x_i, x_j) \in a^{-1}(IND)\}.$$

Thus, the entry c_{ij} consists of those attributes $a \in A$ such that $a(x_i) = a(x_j)$.

Let us note that indiscernibility matrices defined here and the discernibility matrices defined by Skowron and Rauszer are not exactly the same. Namely, the entries of our indiscernibility matrices are the complements of the entries of their discernibility matrices.

In a similar manner we may define the *similarity matrix* $M(SIM)_{\mathcal{S}}$, the *matrix of inclusion* $M(INC)_{\mathcal{S}}$, the *diversity matrix* $M(DIV)_{\mathcal{S}}$, the *orthogonality matrix* $M(ORT)_{\mathcal{S}}$, and the *matrix of negative similarity* $M(NIM)_{\mathcal{S}}$ of a nondeterministic information system $\mathcal{S}$.

Example 4.3.2. Let $\mathcal{S} = (U, A, \{V_a\}_{a\in A})$ be the single-valued and incompletely defined information system described in Example 4.2.2. Let us denote $a = \mathrm{Age}$, $b = \mathrm{Height}$, and $c = \mathrm{Weight}$.

The similarity matrix $M(SIM)_{\mathcal{S}} = (c_{ij})_{4\times 4}$ of $\mathcal{S}$ is the following.

$$\begin{pmatrix} A & \{a\} & \{a,c\} & \{b\} \\ \{a\} & A & \{a\} & \{a,c\} \\ \{a,c\} & \{a\} & A & \{b\} \\ \{b\} & \{a,c\} & \{b\} & A \end{pmatrix}$$

It is clear that $(1,3) \in sim(\{\mathrm{Age}, \mathrm{Weight}\})$ because $\{a,c\} \subseteq c_{13} = \{a,c\}$. On the other hand, $(1,2) \notin wsim(\{\mathrm{Height}, \mathrm{Weight}\})$ because $\{b,c\} \cap c_{12} = \{b,c\} \cap \{a\} = \emptyset$.

We will return to the matrix representation of preimage relations in the next chapter when we consider dense families of dependence spaces induced by strong and weak preimage relations.

Chapter 5

Dependence Spaces

5.1 Closure Operators of Dependence Spaces

Many problems concerning information systems can be formulated in dependence spaces which are simpler algebraic structures. The study of dependence spaces started in [30], although the name "dependence space" was introduced later in [31]. Since then dependence spaces have been the subject of several papers (see [9, 18, 26, 27, 32], for example). In [35] Novotný presented an extensive study of dependence spaces, and in [34] he showed how the theory applies to contexts (in the sense of Wille [56]), relational systems, classificatory systems, information systems, and decision tables.

According to Novotný and Pawlak, a pair $\mathcal{D} = (A, \Theta)$ is a dependence space, if A is finite nonempty set and Θ is a congruence on the semilattice $(\wp(A), \cup)$. We have already seen that each congruence on a finite semilattice is complete, i.e., each congruence class has a greatest element. This implies that if A is finite, then each congruence on $(\wp(A), \cup)$ is complete. Therefore our following generalized definition of dependence spaces is justified.

Definition. If A is a set and Θ is a complete congruence on $(\wp(A), \cup)$, then the pair $\mathcal{D} = (A, \Theta)$ is called a *dependence space*. A dependence space $\mathcal{D} = (A, \Theta)$ is *finite* if A is a finite set.

For a dependence space $\mathcal{D} = (A, \Theta)$, the greatest element in the Θ-class of any $B \subseteq A$ is denoted by $\mathcal{C}_\mathcal{D}(B)$. In [26] the authors presented some properties of the map $\mathcal{C}_\mathcal{D} \colon \wp(A) \to \wp(A), B \mapsto \mathcal{C}_\mathcal{D}(B)$, for a finite dependence space $\mathcal{D} = (A, \Theta)$. Our next lemma, which follows from Lemma 3.2.2 and Proposition 3.2.3, shows that these properties hold also for dependence spaces defined more generally.

Lemma 5.1.1. *Let $\mathcal{D} = (A, \Theta)$ be a dependence space and $B, C \subseteq A$.*

(a) $\mathcal{C}_\mathcal{D}\colon \wp(A) \to \wp(A), B \mapsto \mathcal{C}_\mathcal{D}(B)$*, is a closure operator;*

(b) $\mathcal{C}_\mathcal{D}(B) = \bigcup B/\Theta$*;*

(c) $\mathcal{C}_\mathcal{D}(B) = \mathcal{C}_\mathcal{D}(C)$ *if and only if* $B \Theta C$*.* □

We denote by $\mathcal{L}_\mathcal{D}$ the set of the greatest elements of Θ-classes, that is to say,

$$\mathcal{L}_\mathcal{D} = \{\mathcal{C}_\mathcal{D}(B) \mid B \subseteq A\}.$$

Thus, $\mathcal{L}_\mathcal{D}$ is the closure system corresponding to the closure operator $\mathcal{C}_\mathcal{D}$ and so $(\mathcal{L}_\mathcal{D}, \subseteq)$ is a complete lattice such that

$$\bigvee \mathcal{H} = \mathcal{C}_\mathcal{D}(\bigcup \mathcal{H}) \qquad \text{and} \qquad \bigwedge \mathcal{H} = \bigcap \mathcal{H}$$

for all $\mathcal{H} \subseteq \mathcal{L}_\mathcal{D}$.

As we noted in Chapter 3, the set of all complete congruences on a semilattice $\mathcal{P} = (\wp(A), \cup)$ is a complete lattice with respect to the inclusion order, and it is isomorphic to the set of all closure operators $\wp(A) \to \wp(A)$ ordered by the pointwise order. This means that a dependence space could be equivalently defined as a pair $(A, \mathcal{C})$, where $\mathcal{C}\colon \wp(A) \to \wp(A)$ is a closure operator.

Since $(\mathrm{Con_c}(\mathcal{P}), \subseteq)$ is a complete lattice, the set of all dependence spaces on A

$$\{(A, \Theta) \mid \Theta \text{ is a complete congruence on } (\wp(A), \cup)\}$$

might be considered as a complete lattice, in which the order is defined by $(A, \Theta_1) \leq (A, \Theta_2) \iff \Theta_1 \subseteq \Theta_2$. In this complete lattice the join and the meet of $\{(A, \Theta_i) \mid i \in I\}$ are $(A, \bigvee_{i \in I} \Theta_i)$ and $(A, \bigwedge_{i \in I} \Theta_i)$, respectively. Note that if A is a finite, then by Corollary 3.4.6 the number of dependence spaces $\mathcal{D} = (A, \Theta)$ is at most $2^{2^{|A|-1}}$.

Let Θ be a congruence on a semilattice $(\wp(A), \cup)$. As before, we define the order $\leq$ on the quotient set $\wp(A)/\Theta$ by

$$B/\Theta \leq C/\Theta \text{ if and only if } (B \cup C)/\Theta = C/\Theta.$$

Our next proposition, which is presented in [18] for finite dependence spaces, follows from Propositions 3.2.7 and 3.2.8.

Proposition 5.1.2. *If $\mathcal{D} = (A, \Theta)$ is a dependence space, then $(\wp(A)/\Theta, \leq)$ is a complete lattice such that*

$$\begin{aligned} \bigvee\{B/\Theta \mid B \in \mathcal{H}\} &= (\bigcup \mathcal{H})/\Theta; \\ \bigwedge\{B/\Theta \mid B \in \mathcal{H}\} &= (\bigcap_{B \in \mathcal{H}} \mathcal{C}_{\mathcal{D}}(B))/\Theta, \end{aligned}$$

for all $\mathcal{H} \subseteq \wp(A)$. Moreover, the map $B/\Theta \mapsto \mathcal{C}_{\mathcal{D}}(B)$ is an isomorphism between the complete lattices $(\wp(A)/\Theta, \leq)$ and $(\mathcal{L}_{\mathcal{D}}, \subseteq)$. □

By the previous proposition it is clear that for all $B, C \subseteq A$,

$$B/\Theta \leq C/\Theta \iff \mathcal{C}_{\mathcal{D}}(B) \subseteq \mathcal{C}_{\mathcal{D}}(C). \tag{5.1}$$

Note that the congruence Θ satisfies the complete $\cup$-compatibility property, that is, if I is a nonempty index set such that $B_i \Theta C_i$ holds for all $i \in I$, then also $\left(\bigcup_{i \in I} B_i\right) \Theta \left(\bigcup_{i \in I} C_i\right)$ holds.

Example 5.1.3. Let $A = \{1, 2, 3, 4\}$ and let Θ be the congruence on $(\wp(A), \cup)$ whose congruence classes are $\{\emptyset\}$, $\{\{1\}\}$, $\{\{2\}\}$, $\{\{3\}\}$, $\{\{4\}, \{1, 2\}, \{1, 4\}, \{2, 4\}, \{1, 2, 4\}\}$, $\{\{1, 3\}\}$, $\{\{2, 3\}\}$ and $\{\{3, 4\}, \{1, 2, 3\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$. The closure lattice $(\mathcal{L}_{\mathcal{D}}, \subseteq)$ of the dependence space $\mathcal{D} = (A, \Theta)$ is presented in Figure 8.

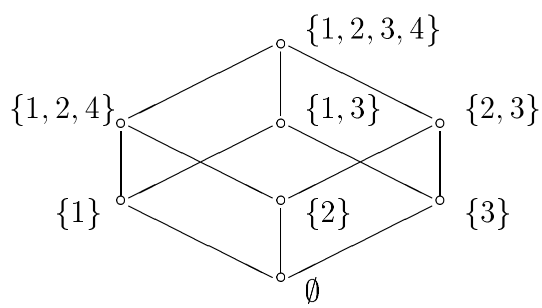


Figure 8.

Next we introduce some types of dependence spaces defined by applying general order-theoretical concepts.

Definition. A dependence space $\mathcal{D} = (A, \Theta)$ is *finitary* if $\wp(A)/\Theta$ is finite, and $\mathcal{D}$ *satisfies the* ACC (resp. the DCC), if $(\wp(A)/\Theta, \leq)$ satisfies the ACC (resp. the DCC). Furthermore, $\mathcal{D}$ has *no infinite chains* if $(\wp(A)/\Theta, \leq)$ has no infinite chains.

If $\mathcal{D} = (A, \Theta)$ is finite, then $\mathcal{D}$ is finitary, and if $\mathcal{D}$ is finitary, it has no infinite chains. By Lemma 2.2.8(b) it is clear that $\mathcal{D}$ satisfies the ACC and the DCC if and only if $\mathcal{D}$ has no infinite chains. Our next lemma presents a condition which we will need especially in Section 5.4.

Lemma 5.1.4. *A dependence space $\mathcal{D} = (A, \Theta)$ satisfies the* ACC *if and only if for all $B \subseteq A$, there exists a finite subset F of B such that $B \Theta F$.*

Proof. Suppose $\mathcal{D}$ satisfies the ACC and let $B \subseteq A$. Then, by Lemma 2.2.8(a), the nonempty subset

$$\mathcal{F} = \{F/\Theta \mid F \text{ is a finite subset of } B\}$$

of $\wp(A)/\Theta$ has a maximal element F/Θ for some finite $F \subseteq B$. Let $a \in B$. Then $(F \cup \{a\})/\Theta \geq F/\Theta$ and because $(F \cup \{a\})$ is a finite subset of B, this implies $(F \cup \{a\})/\Theta = F/\Theta$. So, $\{a\}/\Theta \vee F/\Theta = (F \cup \{a\})/\Theta = F/\Theta$, which means that $\{a\}/\Theta \leq F/\Theta$. By Proposition 5.1.2, $B/\Theta = (\bigcup\{\{a\} \mid a \in B\})/\Theta = \bigvee\{\{a\}/\Theta \mid a \in B\} \leq F/\Theta$. Since $F \subseteq B$, we have also $F/\Theta \leq B/\Theta$, and hence $F/\Theta = B/\Theta$.

On the other hand, suppose that for all $B \subseteq A$ there exists a finite subset F of B such that $B/\Theta = F/\Theta$. Consider any chain

$$B_1/\Theta \leq B_2/\Theta \leq B_3/\Theta \leq \cdots \tag{5.2}$$

in $\wp(A)/\Theta$. Let us denote $B = \bigcup_{i \geq 1} B_i$. Then there exists a finite subset $F = \{a_1, \ldots, a_n\}$ of B such that $F/\Theta = B/\Theta$.

For all $a_i \in F$, there exists a $j_i \geq 1$ such that $a_i \in B_{j_i}$. Because (5.2) is a chain, there exists a $k \geq 1$ such that $B_{j_i}/\Theta \leq B_k/\Theta$ for all $1 \leq i \leq n$. This implies that

$$\begin{aligned} B/\Theta &= F/\Theta \\ &= \{a_1\}/\Theta \vee \cdots \vee \{a_n\}/\Theta \\ &\leq B_{j_1}/\Theta \vee \cdots \vee B_{j_n}/\Theta \\ &\leq B_k/\Theta \\ &\leq B/\Theta. \end{aligned}$$

Therefore $B_k/\Theta = B/\Theta$ and so $B_k/\Theta = B_{k+1}/\Theta = \ldots$ Hence, $\wp(A)/\Theta$ satisfies the ACC. ☐

Our next proposition gives a condition under which the closure operator $\mathcal{C}_\mathcal{D}$ is algebraic.

Proposition 5.1.5. *If $\mathcal{D} = (A, \Theta)$ satisfies the ACC, then the closure operator $\mathcal{C}_\mathcal{D}$ is algebraic.*

Proof. Let $B \subseteq A$. It is clear that $\mathcal{C}_\mathcal{D}(B) \supseteq \bigcup\{\mathcal{C}_\mathcal{D}(F) \mid F$ is a finite subset of $B\}$. By Lemma 5.1.4, there exists a finite subset F of B such that $B\Theta F$. This implies $\mathcal{C}_\mathcal{D}(B) = \mathcal{C}_\mathcal{D}(F) \subseteq \bigcup\{\mathcal{C}_\mathcal{D}(F) \mid F$ is a finite subset of $B\}$, which means that $\mathcal{C}_\mathcal{D}$ is algebraic. □

It is known that a closure operator $\mathcal{C}\colon \wp(A) \to \wp(A)$ is algebraic if and only if $\mathcal{C}$ is continuous map from $(\wp(A), \subseteq)$ to $(\wp(A), \subseteq)$ (see [5], for example). Note that this implies that the join of algebraic closure operators can be obtained by applying Corollary 3.1.17. Moreover, in the complete lattice

$$(\{(A, \Theta) \mid \Theta \in \mathrm{Con_c}(\wp(A), \cup)\}, \leq),$$

the join of dependence spaces satisfying the ACC can be formed by using this result, because by Proposition 5.1.5 the corresponding closure operators are algebraic, and thus they are continuous.

We end this section by noting that in [33] M. Novotný generalized dependence spaces by defining them as pairs $(\mathcal{P}, E)$, where $\mathcal{P} = (P, \leq)$ is a complete lattice and E is an equivalence on P such that each equivalence class x/E has a greatest element $c_E(x)$. He also showed that the map $C_E\colon P \to P, x \mapsto c_E(x)$, is extensive and idempotent, but not necessarily order-preserving. Dependence spaces defined in [33] may be applied to algebraic linguistics, and reducts are connected with constructions of pure grammars of languages.

5.2 Dense Families of Dependence Spaces

Novotný introduced dense families of dependence spaces in [34]. In Section 3.4 we defined dense sets of any congruence Θ on a semilattice $(P, \vee)$. In this section we present some results concerning dense families of dependence spaces, which follow easily from the results we presented in Section 3.4.

By Proposition 3.4.1 each family $\mathcal{H} \subseteq \wp(A)$ defines a complete congruence

$$\Theta_\mathcal{H} = \{(B, C) \mid (\forall X \in \mathcal{H})\ B \subseteq X \iff C \subseteq X\}$$

on $(\wp(A), \cup)$ such that the greatest element in the $\Theta_{\mathcal{H}}$-class of B is

$$\bigcap\{X \in \mathcal{H} \mid B \subseteq X\}.$$

Definition. *Let $\mathcal{D} = (A, \Theta)$ be a dependence space. A family $\mathcal{H} \subseteq \wp(A)$ is* dense *in $\mathcal{D}$ if $\mathcal{H}$ is Θ-dense.*

The above definition means that a family $\mathcal{H} \subseteq \wp(A)$ is dense in $\mathcal{D} = (A, \Theta)$ if and only if the complete congruence $\Theta_{\mathcal{H}}$ defined by $\mathcal{H}$ equals Θ. We also note that our definition agrees with Novotný's definition of dense families.

Our next lemma, which follows from Proposition 3.4.11 and Lemma 3.4.12, helps us to recognize dense families of dependence spaces. Note that since $(\wp(A)/\Theta, \leq) \cong (\mathcal{L}_{\mathcal{D}}, \subseteq)$, $\mathcal{L}_{\mathcal{D}}$ is finite whenever $\mathcal{D}$ is finitary. Moreover, $\mathcal{L}_{\mathcal{D}}$ has no infinite chains, if $\mathcal{D}$ has no infinite chains.

Lemma 5.2.1. *Let $\mathcal{D} = (A, \Theta)$ be a dependence space.*

(a) *If $\mathcal{D}$ has no infinite chains, then $\mathcal{H} \subseteq \wp(A)$ is dense in $\mathcal{D}$ if and only if*

$$\mathcal{M}(\mathcal{L}_{\mathcal{D}}) \subseteq \mathcal{H} \subseteq \mathcal{L}_{\mathcal{D}}.$$

(b) *If $\mathcal{D}$ is finitary, then $\mathcal{M}(\mathcal{L}_{\mathcal{D}})$ consists of the elements of $\mathcal{L}_{\mathcal{D}}$ which are covered by exactly one element of $\mathcal{L}_{\mathcal{D}}$.* □

Example 5.2.2. Let us consider the dependence space $\mathcal{D} = (A, \Theta)$ of Example 5.1.3. The Hasse diagram of $\mathcal{L}_{\mathcal{D}}$ was given in Figure 8. Now $\mathcal{M}(\mathcal{L}_{\mathcal{D}}) = \{\{1, 3\}, \{2, 3\}, \{1, 2, 4\}\}$. Since $\mathcal{D}$ is finite, Lemma 5.2.1 applies to it, and hence the dense families of $\mathcal{D}$ are the 32 families $\mathcal{H}$ such that $\mathcal{M}(\mathcal{L}_{\mathcal{D}}) \subseteq \mathcal{H} \subseteq \mathcal{L}_{\mathcal{D}}$.

By applying dense families it is easy to decide whether two subsets are congruent, as we can see in our next proposition which ends this section.

Proposition 5.2.3. *If $\mathcal{D} = (A, \Theta)$ is a dependence space and $\mathcal{H}(\subseteq \wp(A))$ is dense in $\mathcal{D}$, then the following conditions are equivalent for all $B, C \subseteq A$:*

(a) *$B/\Theta \leq C/\Theta$;*

(b) *$\mathcal{C}_{\mathcal{D}}(B) \subseteq \mathcal{C}_{\mathcal{D}}(C)$;*

(c) *for all $X \in \mathcal{H}$, $C \subseteq X$ implies $B \subseteq X$;*

(d) *for all $X \in \mathcal{H}$, $B - X \neq \emptyset$ implies $C - X \neq \emptyset$.*

Proof. Conditions (a) and (b) are equivalent by (5.1), and (a) and (c) are equivalent by Lemma 3.4.8(c). Because $B \subseteq X$ is equivalent to $B - X = \emptyset$, also (c) and (d) are equivalent. □

5.3 Dependence Spaces of Preimage Relations

Let $\mathcal{S} = (U, A, \{V_a\}_{a \in A})$ be an information system such that each attribute $a \in A$ is a mapping $a: U \to V_a$ and A is finite. It is known [31] that the relation Θ_{ind} defined by

$$(B, C) \in \Theta_{ind} \iff ind(B) = ind(C)$$

is a congruence on the semilattice $(\wp(A), \cup)$. Thus, the pair (A, Θ_{ind}) forms a finite dependence space. In this section we generalize this observation by showing that also strong and weak preimage relations define dependence spaces. We also show how we can determine dense families of these dependence spaces by applying preimage matrices.

Let U and Y be nonempty sets, $R \in \mathrm{Rel}(Y)$, and let $A(\subseteq Y^U)$ be a set of functions. Let us now define two binary relations $\Theta_S(A, R)$ and $\Theta_W(A, R)$ on $\wp(A)$ so that, for all $B, C \subseteq A$,

$$\begin{aligned}(B, C) \in \Theta_S(A, R) \quad &\iff \quad S_R(B) = S_R(C);\\ (B, C) \in \Theta_W(A, R) \quad &\iff \quad W_R(B) = W_R(C).\end{aligned}$$

By Proposition 3.3.3(b) and Lemma 4.2.3(d) we can write the following proposition, since $\Theta_S(A, R)$ and $\Theta_W(A, R)$ are the kernels of the functions $S_R: \wp(A) \to \mathrm{Rel}(U), B \mapsto S_R(B)$, and $W_R: \wp(A) \to \mathrm{Rel}(U), B \mapsto W_R(B)$, respectively.

Proposition 5.3.1. *Let U and Y be nonempty sets, $R \in \mathrm{Rel}(Y)$, and $A(\subseteq Y^U)$ a set of functions.*

(a) *The map $S_R: \wp(A) \to \mathrm{Rel}(U), B \mapsto S_R(B)$ is a complete join-morphism from $(\wp(A), \subseteq)$ to $(\mathrm{Rel}(U), \supseteq)$ such that the greatest element in the $\Theta_S(A, R)$-class of any $B \subseteq A$ is $\bigcup B/\Theta_S(A, R)$.*

(b) *The map $W_R: \wp(A) \to \mathrm{Rel}(U), B \mapsto W_R(B)$ is a complete join-morphism from $(\wp(A), \subseteq)$ to $(\mathrm{Rel}(U), \subseteq)$ such that the greatest element in the $\Theta_W(A, R)$-class of any $B \subseteq A$ is $\bigcup B/\Theta_W(A, R)$.*

(c) *The pairs $(A, \Theta_S(A, R))$ and $(A, \Theta_W(A, R))$ are dependence spaces.* $\square$

The relations $\Theta_S(A, R)$ and $\Theta_W(A, R)$ are referred to as the *strong* and the *weak preimage congruences* of R with respect to the set of functions A.

Our next proposition can be viewed as a generalization of Theorem 6.5 in [31]. It follows easily from Propositions 3.3.4, 3.3.5 and Lemma 4.2.3(d). In this proposition the greatest element in the $\Theta_S(A, R)$-class of any $B \subseteq A$ is denoted by $\mathcal{C}^S(B)$, and the greatest element in the $\Theta_W(A,R)$-class of any $B \subseteq A$ is denoted by $\mathcal{C}^W(B)$.

Proposition 5.3.2. *Let U and Y be nonempty sets, $R \in \mathrm{Rel}(Y)$, and $A(\subseteq Y^U)$.*

(a) *The ordered set $(\{S_R(B) \mid B \subseteq A\}, \supseteq)$ is a complete lattice such that*

$$\bigvee_{i\in I} S_R(B_i) = S_R(\bigcup_{i\in I} B_i) = \bigcap_{i\in I} S_R(B_i)$$

and

$$\bigwedge_{i\in I} S_R(B_i) = S_R(\bigcap_{i\in I} \mathcal{C}^S(B_i)),$$

for all $\{B_i\}_{i\in I} \subseteq \wp(A)$. Moreover, the map $S_R(B) \mapsto B/\Theta_S(A,R)$ is an isomorphism between the complete lattices $(\{S_R(B) \mid B \subseteq A\}, \supseteq)$ and $(\wp(A)/\Theta_S(A,R), \leq)$.

(b) *The ordered set $(\{W_R(B) \mid B \subseteq A\}, \subseteq)$ is a complete lattice such that*

$$\bigvee_{i\in I} W_R(B_i) = W_R(\bigcup_{i\in I} B_i) = \bigcup_{i\in I} W_R(B_i)$$

and

$$\bigwedge_{i\in I} W_R(B_i) = W_R(\bigcap_{i\in I} \mathcal{C}^W(B_i)),$$

for all $\{B_i\}_{i\in I} \subseteq \wp(A)$. Moreover, the map $W_R(B) \mapsto B/\Theta_W(A,R)$ is an isomorphism between the complete lattices $(\{W_R(B) \mid B \subseteq A\}, \subseteq)$ and $(\wp(A)/\Theta_W(A,R), \leq)$. □

In the dependence space $(A, \Theta_S(A,R))$, we denote the relation $\Theta_S(A,R)$ by Θ_R^S and in the dependence space $(A, \Theta_W(A,R))$ the relation $\Theta_W(A,R)$ by Θ_R^W. Next we present a simple condition which guarantees that the dependence spaces (A, Θ_R^S) and (A, Θ_R^W) are finitary.

Lemma 5.3.3. *Let U and Y be nonempty sets, $R \in \mathrm{Rel}(Y)$, and $A(\subseteq Y^U)$. If U or A is finite, then the dependence spaces (A, Θ_R^S) and (A, Θ_R^W) are finitary.*

Proof. If A is finite, then (A, Θ_R^S) and (A, Θ_R^W) are trivially finitary. On the other hand, if U is finite, then $\{S_R(B) \mid B \subseteq A\} \subseteq \mathrm{Rel}(U)$ is finite. By Proposition 5.3.2(a), $\{S_R(B) \mid B \subseteq A\}, \supseteq) \cong (\wp(A)/\Theta_R^S, \leq)$ which implies that also $\wp(A)/\Theta_R^S$ is finite. The rest may be proved analogously. □

Let us consider a nondeterministic information system $\mathcal{S} = (U, A, \{V_a\}_{a \in A})$. As we have seen in Example 4.2.5, information relations are preimage relations, and hence they define dependence spaces.

In the sequel we shall denote the relation Θ_{IND}^{S} simply by Θ_{ind}. The relation Θ_{ind} will be called the *strong indiscernibility congruence* of $\mathcal{S}$. The dependence space (A, Θ_{ind}) is denoted simply by $\mathcal{D}_{ind}$. Similarly, the relation Θ_{IND}^{W} is called the *weak indiscernibility congruence* of $\mathcal{S}$ and is denoted by Θ_{wind}. We denote by $\mathcal{D}_{wind}$ the dependence space (A, Θ_{wind}). A similar notation is defined for the other information relations.

Lemma 5.3.3 implies that if U or A is finite, then the dependence spaces $\mathcal{D}_{ind}$, $\mathcal{D}_{sim}$, $\mathcal{D}_{inc}$, $\mathcal{D}_{div}$, $\mathcal{D}_{ort}$, $\mathcal{D}_{nim}$, $\mathcal{D}_{wind}$, $\mathcal{D}_{wsim}$, $\mathcal{D}_{winc}$, $\mathcal{D}_{wdiv}$, $\mathcal{D}_{wort}$, and $\mathcal{D}_{wnim}$ are finitary.

In the next example we see that even if in an information system $\mathcal{S} = (U, A, \{V_a\}_{a \in A})$ the set $\bigcup_{a \in A} V_a$ is finite, the dependence spaces $\mathcal{D}_{ind}$, $\mathcal{D}_{sim}$, $\mathcal{D}_{inc}$, $\mathcal{D}_{wdiv}$, $\mathcal{D}_{wort}$, and $\mathcal{D}_{wnim}$ may have infinite chains.

Example 5.3.4. Let $\mathcal{S} = (U, A, \{V_a\}_{a \in A})$ be an information system such that $U = \mathbb{N}$, $A = \{a_i \mid i \in \mathbb{N}\}$, and $V_a = \{0, 1\}$ for all $a \in A$.

For any $i \in \mathbb{N}$, the attribute a_i is defined by

$$a_i(n) = \begin{cases} 0 & \text{if } n \leq i, \\ 1 & \text{otherwise.} \end{cases}$$

The equivalence classes of $ind(\{a_i\})$ are $\{1, \ldots, i\}$ and $\{i+1, i+2, \ldots\}$. It is easy to see that

$$ind(\{a_1\}) \supset ind(\{a_1, a_2\}) \supset \cdots \supset ind(\{a_1, \ldots a_k\}) \supset \cdots$$

is an infinite chain in $\{ind(B) \mid B \subseteq A\}$. Because $(\{ind(B) \mid B \subseteq A\}, \supseteq) \cong (\wp(A)/\Theta_{ind}, \leq)$ by Proposition 5.3.2, we obtain that

$$\{a_1\}/\Theta_{ind} < \{a_1, a_2\}/\Theta_{ind} < \cdots < \{a_1, \ldots a_k\}/\Theta_{ind} < \cdots$$

is an infinite ascending chain in $(\wp(A)/\Theta_{ind}, \leq)$. Hence, the dependence space $\mathcal{D}_{ind} = (A, \Theta_{ind})$ does not satisfy the ACC.

On the other hand,

$$ind(\{a_1\}^{\complement}) \subset ind(\{a_1, a_2\}^{\complement}) \subset \cdots \subset ind(\{a_1, \ldots a_k\}^{\complement}) \subset \cdots$$

is also an infinite chain in $\{ind(B) \mid B \subseteq A\}$, and thus

$$\{a_1\}^{\complement}/\Theta_{ind} > \{a_1, a_2\}^{\complement}/\Theta_{ind} > \cdots > \{a_1, \ldots a_k\}^{\complement}/\Theta_{ind} > \cdots$$

is an infinite descending chain in $(\wp(A)/\Theta_{ind}, \leq)$. Thus, the dependence space $\mathcal{D}_{ind}$ does not satisfy the DCC either.

In this information system $ind(B) = sim(B) = inc(B)$ and $wdiv(B) = wort(B) = wnim(B) = ind(B)^{\complement}$ for all $B \subseteq A$. Thus, the dependence spaces $\mathcal{D}_{sim}$, $\mathcal{D}_{inc}$, $\mathcal{D}_{wdiv}$, $\mathcal{D}_{wort}$, and $\mathcal{D}_{wnim}$ do not satisfy the ACC or the DCC.

Similarly, the finiteness of $\bigcup_{a \in A} V_a$ does not guarantee that the dependence spaces $\mathcal{D}_{wind}$, $\mathcal{D}_{wsim}$, $\mathcal{D}_{winc}$, $\mathcal{D}_{div}$, $\mathcal{D}_{ort}$, and $\mathcal{D}_{nim}$ do not contain infinite chains.

Example 5.3.5. Let $\mathcal{S} = (U, A, \{V_a\}_{a \in A})$ be an information system such that $U = \mathbb{N} \cup \{\top\}$, $A = \{a_i \mid i \in \mathbb{N}\}$, and $V_a = \{0, 1\}$ for all $a \in A$.

For each $i \in \mathbb{N}$, the attribute a_i is defined by

$$a_i(n) = \begin{cases} 0 & \text{if } n = i \text{ or } n = \top, \\ 1 & \text{otherwise.} \end{cases}$$

The equivalence classes of $wind(\{a_i\})$ are $\{i, \top\}$ and $\mathbb{N} - \{i\}$. Obviously,

$$wind(\{a_1\}) \subset wind(\{a_1, a_2\}) \subset \cdots \subset wind(\{a_1, \ldots a_k\}) \subset \cdots$$

is an infinite chain in $\{wind(B) \mid B \subseteq A\}$. Because $(\{wind(B) \mid B \subseteq A\}, \subseteq) \cong (\wp(A)/\Theta_{wind}, \leq)$,

$$\{a_1\}/\Theta_{wind} < \{a_1, a_2\}/\Theta_{wind} < \cdots < \{a_1, \ldots a_k\}/\Theta_{wind} < \cdots$$

is an infinite ascending chain in $(\wp(A)/\Theta_{wind}, \leq)$.

Similarly,

$$wind(\{a_1\}^{\complement}) \supset wind(\{a_1, a_2\}^{\complement}) \supset \cdots \supset wind(\{a_1, \ldots a_k\}^{\complement}) \supset \cdots$$

is also an infinite chain in $\{wind(B) \mid B \subseteq A\}$, and thus

$$\{a_1\}^{\complement}/\Theta_{wind} > \{a_1, a_2\}^{\complement}/\Theta_{wind} > \cdots > \{a_1, \ldots a_k\}^{\complement}/\Theta_{wind} > \cdots$$

is an infinite descending chain in $(\wp(A)/\Theta_{wind}, \leq)$. Therefore, the dependence space $\mathcal{D}_{wind} = (A, \Theta_{wind})$ does not satisfy the ACC or the DCC.

Because in this information system $wind(B) = wsim(B) = winc(B)$ and $div(B) = ort(B) = nim(B) = wind(B)^{\complement}$ for all $B \subseteq A$, the dependence spaces $\mathcal{D}_{wsim}$, $\mathcal{D}_{winc}$, $\mathcal{D}_{div}$, $\mathcal{D}_{ort}$, and $\mathcal{D}_{nim}$ do not satisfy the ACC or the DCC, either.

We conclude this section by presenting a proposition which shows how matrices of preimage relations define dense families of dependence spaces.

Proposition 5.3.6. *Let $U = \{x_i\}_{i \in I}$ and Y be nonempty sets, $R \in \mathrm{Rel}(Y)$, $A \subseteq Y^U$, and let $M(R) = (c_{ij})$ be the matrix of preimage relations of R with respect to the set A.*

(a) *The family $\{c_{ij} \mid i, j \in I\}$ is dense in (A, Θ_R^S).*
(b) *The family $\{c_{ij}{}^{\complement} \mid i, j \in I\}$ is dense in (A, Θ_R^W).*

Proof. (a) Let us denote $\mathcal{H} = \{c_{ij} \mid i, j \in I\}$. We have to show that $\Theta_{\mathcal{H}} = \Theta_R^S$. If $(B, C) \in \Theta_R^S$, then by Lemma 4.3.1 for all $i, j \in I$, $B \subseteq c_{ij}$ iff $(x_i, x_j) \in S_R(B)$ iff $(x_i, x_j) \in S_R(C)$ iff $C \subseteq c_{ij}$, which implies $(B, C) \in \Theta_{\mathcal{H}}$. Hence, $\Theta_R^S \subseteq \Theta_{\mathcal{H}}$.

If $(B, C) \in \Theta_{\mathcal{H}}$, then for all $i, j \in I$, $(x_i, x_j) \in S_R(B)$ iff $B \subseteq c_{ij}$ iff $C \subseteq c_{ij}$ iff $(x_i, x_j) \in S_R(C)$, which implies $S_R(B) = S_R(C)$. Thus, $\Theta_{\mathcal{H}} \subseteq \Theta_R^S$ and so $\Theta_{\mathcal{H}} = \Theta_R^S$.

(b) Let us write $\mathcal{K} = \{c_{ij}{}^{\complement} \mid i, j \in I\}$. If $(B, C) \in \Theta_R^W$, then for all $i, j \in I$, $B \subseteq c_{ij}{}^{\complement}$ iff $B \cap c_{ij} = \emptyset$ iff $(x_i, x_j) \notin W_R(B)$ iff $(x_i, x_j) \notin W_R(C)$ iff $C \cap c_{ij} = \emptyset$ iff $C \subseteq c_{ij}{}^{\complement}$, which implies $(B, C) \in \Theta_{\mathcal{K}}$. Hence, $\Theta_R^W \subseteq \Theta_{\mathcal{K}}$.

If $(B, C) \in \Theta_{\mathcal{K}}$, then for all $i, j \in I$, $(x_i, x_j) \in W_R(B)$ iff $B \cap c_{ij} \neq \emptyset$ iff $B \nsubseteq c_{ij}{}^{\complement}$ iff $C \nsubseteq c_{ij}{}^{\complement}$ iff $C \cap c_{ij} \neq \emptyset$ iff $(x_i, x_j) \in W_R(C)$, which implies $W_R(B) = W_R(C)$. So, also $\Theta_{\mathcal{K}} \subseteq \Theta_R^W$ and hence $\Theta_{\mathcal{K}} = \Theta_R^W$. □

The next example shows how we may obtain dense families in dependence spaces defined by information systems by Proposition 5.3.6.

Example 5.3.7. Let $\mathcal{S} = (U, A, \{V_a\}_{a \in A})$ be the nondeterministic information system presented in Example 4.2.2. Let us denote $a = \mathrm{Age}$, $b = \mathrm{Height}$, and $c = \mathrm{Weight}$. The strong and the weak similarity relations of each subset of A are the following:

$$
\begin{aligned}
sim(\emptyset) &= \nabla_U; \\
sim(\{a\}) &= \Delta_U \cup \{(1,2),(2,1),(1,3),(3,1),(2,3),(3,2),(2,4),(4,2)\}; \\
sim(\{b\}) &= \Delta_U \cup \{(1,4),(4,1),(3,4),(4,3)\}; \\
sim(\{c\}) &= sim(\{a,c\}) = \Delta_U \cup \{(1,3),(3,1),(2,4),(4,2)\}; \\
sim(\{a,b\}) &= sim(\{b,c\}) = sim(A) = \Delta_U; \\
wsim(\emptyset) &= \emptyset; \\
wsim(\{a\}) &= wsim(\{a,c\}) = sim(\{a\});
\end{aligned}
$$

$$\begin{aligned} wsim(\{b\}) &= sim(\{b\}); \\ wsim(\{c\}) &= sim(\{c\}); \\ wsim(\{a,b\}) &= wsim(A) = \nabla_U; \\ wsim(\{b,c\}) &= \Delta_U \cup \{(1,3),(3,1),(1,4),(4,1),(2,4),(4,2),(3,4),(4,3)\}. \end{aligned}$$

So, the strong similarity congruence Θ_{sim} has the congruence classes $\{\emptyset\}$, $\{\{a\}\}$, $\{\{b\}\}$, $\{\{c\},\{a,c\}\}$, and $\{\{a,b\},\{b,c\},A\}$. The similarity matrix $M(SIM)_{\mathcal{S}} = (c_{ij})_{4\times 4}$ of $\mathcal{S}$ is presented in Example 4.3.2. By Proposition 5.3.6, the family

$$\mathcal{H} = \{c_{ij} \mid 1 \le i,j \le 4\} = \{\{a\}, \{b\}, \{a,c\}, A\}$$

is dense in the dependence space (A, Θ_{sim}). For example, $\{c\} \subseteq X$ if and only if $\{a,c\} \subseteq X$ holds for all $X \in \mathcal{H}$ because $(\{c\},\{a,c\}) \in \Theta_{sim}$.

On the other hand, the weak similarity congruence Θ_{wsim} has the congruence classes $\{\emptyset\}$, $\{\{a\},\{a,c\}\}$, $\{\{b\}\}\}$, $\{\{c\}\}$, $\{\{a,b\},A\}$, $\{\{b,c\}\}$. Now the family

$$\mathcal{K} = \{c_{ij}{}^{\complement} \mid 1 \le i,j \le 4\} = \{\emptyset, \{b\}, \{a,c\}, \{b,c\}\}$$

is dense in the dependence space (A, Θ_{wsim}). Thus, $(\{a\},\{a,c\}) \in \Theta_{wsim}$ implies that for all $X \in \mathcal{K}$, $\{a\} \subseteq X$ if and only if $\{a,c\} \subseteq X$.

5.4 Independent Sets and Reducts

In the literature there are many articles which concern independent sets and reducts in information systems (see e.g. [9, 44, 50]). In this section we review cores, independent sets and reducts defined in dependence spaces. We compare the independence defined in dependence spaces with some notions of independence studied in universal algebra. Our main result of this section gives a characterization of the reducts of a given subset of a dependence space in terms of dense families.

Let $\mathcal{D} = (A, \Theta)$ be a dependence space. A subset $B(\subseteq A)$ is called *independent* in $\mathcal{D}$ if B is minimal with respect to the inclusion relation in its Θ-class; otherwise it is *dependent*. We denote the set of independent subsets in $\mathcal{D}$ by $IND_{\mathcal{D}}$.

The following lemma was stated in [30] for finite dependence spaces.

Lemma 5.4.1. *If $\mathcal{D} = (A, \Theta)$ is a dependence space, then $B \in IND_{\mathcal{D}}$ if and only if $(B, B - \{a\}) \notin \Theta$ for all $a \in B$.*

Proof. If $B \in IND_{\mathcal{D}}$, then obviously $(B, B - \{a\}) \notin \Theta$ for all $a \in B$. Conversely, if $B \notin IND_{\mathcal{D}}$, then there exists a $C \in B/\Theta$ such that $C \subset B$. If $a \in B - C$, then $C \subseteq B - \{a\} \subseteq B$, which implies $(B, B - \{a\}) \in \Theta$, because each Θ-class is convex. □

It is now clear that every subset of an independent set is independent; in particular, the empty set is independent. In the next example we consider independent subsets of dependence spaces defined by information systems.

Example 5.4.2. Let $\mathcal{S} = (U, A, \{V_a\}_{a \in A})$ be a nondeterministic information system. Let us consider the dependence space $\mathcal{D}_{sim} = (A, \Theta_{sim})$. For any $B \subseteq A$,

$$\begin{aligned} B \notin IND_{\mathcal{D}_{sim}} &\iff (B, B - \{a\}) \in \Theta_{sim} \text{ for some } a \in B \\ &\iff sim(B) = sim(B - \{a\}) \text{ for some } a \in B. \end{aligned}$$

Thus, a subset B is independent in $\mathcal{D}_{sim}$ if we cannot omit any attribute from B without changing the original strong similarity relation. Analogous statements hold for dependence spaces defined by other information relations. In the dependence space $\mathcal{D}_{sim}$ considered in Example 5.3.7 the sets $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, and $\{b, c\}$ are independent.

Next we present some equivalent conditions which can be used for determining independent sets.

Lemma 5.4.3. *If $\mathcal{D} = (A, \Theta)$ is a dependence space in which $\mathcal{H}(\subseteq \wp(A))$ is dense, then the following conditions are equivalent for all $B \subseteq A$ and $a \in B$:*

(a) $(B, B - \{a\}) \notin \Theta$*;*
(b) $B - X = \{a\}$ *for some* $X \in \mathcal{H}$*;*
(c) $a \notin \mathcal{C}_{\mathcal{D}}(B - \{a\})$*;*
(d) $\{a\}/\Theta \not\leq (B - \{a\})/\Theta$*.*

Proof. Let $B \subseteq A$ and $a \in B$.

(a) ⇒ (b). If $(B, B - \{a\}) \notin \Theta$, then $B/\Theta \not\leq (B - \{a\})/\Theta$, which implies by Proposition 5.2.3 that there exists an $X \in \mathcal{H}$ such that $B - X \neq \emptyset$ and $(B - \{a\}) - X = \emptyset$. This means that $B - X = \{a\}$ for some $X \in \mathcal{H}$

(b) ⇒ (c). Suppose $B - X = \{a\}$ for some $X \in \mathcal{H}$. This implies that $(B - \{a\}) - X = \emptyset$, that is, $B - \{a\} \subseteq X$. Because $\mathcal{C}_{\mathcal{D}}(B - \{a\}) = \bigcap\{X \in \mathcal{H} \mid B - \{a\} \subseteq X\}$ and $a \notin X$ we obtain $a \notin \mathcal{C}_{\mathcal{D}}(B - \{a\})$.

(c) ⇒ (d). If $a \notin \mathcal{C}_{\mathcal{D}}(B - \{a\})$, then $\mathcal{C}_{\mathcal{D}}(\{a\}) \not\subseteq \mathcal{C}_{\mathcal{D}}(B - \{a\})$, which is equivalent to $\{a\}/\Theta \not\leq (B - \{a\})/\Theta$ by (5.1).

(d) ⇒ (a). If $\{a\}/\Theta \not\leq (B - \{a\})/\Theta$, then $B/\Theta = (\{a\} \cup (B - \{a\}))/\Theta = \{a\}/\Theta \vee (B - \{a\})/\Theta \neq (B - \{a\})/\Theta$. □

Remark. This notion of independence is actually equivalent to a general notion of independence with respect to a closure operator. Let $\mathcal{C}$ be a closure operator on a set A. A set $B \subseteq A$ is said to be *$\mathcal{C}$-independent* if $a \notin \mathcal{C}(B - \{a\})$ for every $a \in B$ (see [14], for example). If $\mathcal{D} = (A, \Theta)$ is a dependence space and $B \subseteq A$, then by Lemmas 5.4.1 and 5.4.3, $B \in IND_{\mathcal{D}} \iff (B, B - \{a\}) \notin \Theta$ for all $a \in B \iff a \notin \mathcal{C}_{\mathcal{D}}(B - \{a\})$ for all $a \in B \iff \mathcal{C}_{\mathcal{D}}$-independent.

As we have already noted, a dependence space could also be defined as a pair $\mathcal{D} = (A, \mathcal{C})$, where $\mathcal{C} \colon \wp(\mathcal{A}) \to \wp(\mathcal{A})$ is a closure operator. By our remark, the set $IND_{\mathcal{D}}$ may be defined in this structure by the means of a general notion of independence appearing in the literature.

In the literature there can be found several notions of dependence (see [14], for example). Here we consider abstract dependences studied in universal algebra. As noted in [45], the dependence in information systems in which the set of attributes is finite is an abstract dependence. Let A be a set. An *abstract dependence* on A is a family $\boldsymbol{D}$ of subsets of A such that $X \in \boldsymbol{D}$ if and only if some finite nonempty subset F of X belongs to $\boldsymbol{D}$. A subset X of A is said to be *$\boldsymbol{D}$-dependent* if $X \in \boldsymbol{D}$, and it is called *$\boldsymbol{D}$-independent* otherwise (see [4, 14], for example).

Now the following lemma holds. Assertion (a) is mentioned in [14] without proof. Statement (b) is verified by modifying the proof of the well-known Exchange Lemma (see [4]).

Lemma 5.4.4. *Let $\mathcal{C} \colon \wp(A) \to \wp(A)$ be an algebraic closure operator.*

(a) *The set of $\mathcal{C}$-dependent sets is an abstract dependence on A.*

(b) *Each set $B(\subseteq A)$ contains a maximal $\mathcal{C}$-independent subset.*

Proof. (a) Suppose B is $\mathcal{C}$-dependent. It means that $a \in \mathcal{C}(B - \{a\})$ for some $a \in B$. Since $\mathcal{C}$ is algebraic, there exists a finite subset F of $B - \{a\}$ such that $a \in \mathcal{C}(F)$. Since $a \notin F$, this means that $F \cup \{a\}$ is a finite $\mathcal{C}$-dependent subset of B.

(b) Consider any $B \subseteq A$. Let us denote

$$\mathcal{I} = \{Y \subseteq B \mid Y \text{ is } \mathcal{C}\text{-independent}\}.$$

Obviously, $\mathcal{I}$ is a nonempty since $\emptyset \in \mathcal{I}$. Let $\{Y_i\}_{i \in I}$ be a nonempty chain in $\mathcal{I}$. It is clear that $Y = \bigcup_{i \in I} Y_i$ is a subset of B. Assume that Y is $\mathcal{C}$-dependent. Then by (a), there exists a finite $\mathcal{C}$-dependent subset F of Y. Because $\{Y_i\}_{i \in I}$ is a chain, there exists $k \in I$ such that $F \subseteq Y_k$. Since F is a finite $\mathcal{C}$-dependent subset of Y_k, also Y_k is $\mathcal{C}$-dependent, a contradiction! Hence, Y belongs to $\mathcal{I}$. By Zorn's Lemma this implies that $\mathcal{I}$ has a maximal element. $\square$

Example 5.4.5. Let $\mathcal{C}\colon \wp(\mathbb{N}) \to \wp(\mathbb{N})$ be a closure operator such that $\mathcal{C}(B) = B$ if B is finite and $\mathcal{C}(B) = \mathbb{N}$ otherwise. It is easy to see that $\mathcal{C}$ is not algebraic.

The set $\mathbb{N}$ is $\mathcal{C}$-dependent, but it does not have any finite $\mathcal{C}$-dependent subsets, because all finite subsets of $\mathbb{N}$ are $\mathcal{C}$-independent. So, the set of all $\mathcal{C}$-dependent subsets is not an abstract dependence on $\mathbb{N}$. Furthermore, $\mathbb{N}$ does not have a maximal $\mathcal{C}$-independent subset.

From Proposition 5.1.5 and Lemma 5.4.4 it follows that when $\mathcal{D} = (A, \Theta)$ satisfies the ACC, the set of all dependent subsets in $\mathcal{D}$ is an abstract dependence on A and each subset of A has a maximal independent subset.

The following definitions can be found in [4], for example. Let $\boldsymbol{D}$ be an abstract dependence on A and $X \subseteq A$. An element $a (\in A)$ is said to be *dependent on* X if $a \in X$ or there exists an independent subset Y of X such that $Y \cup \{a\}$ is dependent. The *span* $\langle X \rangle$ of X is the set of all elements of A dependent on X. The dependence $\boldsymbol{D}$ is said to be *transitive* if $\langle\langle X \rangle\rangle = \langle X \rangle$ for every $X \subseteq A$.

We just mentioned that if $\mathcal{D} = (A, \Theta)$ is a dependence space which satisfies the ACC, then the set of all dependent subsets in $\mathcal{D}$ is an abstract dependence on A. In the next example we show that this abstract dependence is not necessarily transitive even if $\mathcal{D}$ is finite.

Example 5.4.6. In the dependence space of Example 5.1.3, $\langle\{1\}\rangle = \{1, 4\}$ and $\langle\langle\{1\}\rangle\rangle = \{1, 2, 4\}$.

The notion of reducts is important in the theory of information systems (see [50], for example). Here we study reducts in the more general setting of dependence spaces. Let $\mathcal{D} = (A, \Theta)$ be a dependence space. For any $B \subseteq A$, a subset $C \subseteq B$ is called a *reduct* of B, if $B \Theta C$ and $C \in IND_{\mathcal{D}}$. The set of all reducts of B is denoted by $RED_{\mathcal{D}}(B)$.

Lemma 5.4.7. *Let $\mathcal{D} = (A, \Theta)$ be a dependence space and $B \subseteq A$. Each reduct of B is a maximal independent subset of B.*

Proof. If $C \in RED_{\mathcal{D}}(B)$, then $C \in IND_{\mathcal{D}}$ and $C \subseteq B$. Suppose that $C \subset D \subseteq B$ for some $D \in IND_{\mathcal{D}}$. The fact $B \Theta C$ implies $C \Theta D$ because each Θ-class is convex. So, $D \notin IND_{\mathcal{D}}$, a contradiction! $\square$

In the next example we see that every maximal independent subset in a dependence space is not necessarily a reduct of that set.

Example 5.4.8. Consider the dependence space defined in Example 5.1.3. The set $\{1\}$ is a maximal independent subset of $\{1, 4\}$, but $\{1\}$ is not a reduct of $\{1, 4\}$.

Note that results closely related to Lemma 5.4.7 and Example 5.4.8 are presented in [9] for information systems.

It is possible that some subsets of a dependence space do not have reducts even if the closure operator $\mathcal{C}_\mathcal{D}$ is algebraic, as we see in the next example.

Example 5.4.9. Let us define a closure operator $\mathcal{C}\colon \wp(\mathbb{N}) \to \wp(\mathbb{N})$ by setting $\mathcal{C}(B) = \{n \in \mathbb{N} \mid n \leq \max B\}$ if B is finite, and $\mathcal{C}(B) = \mathbb{N}$ otherwise.

Suppose $B \subseteq \mathbb{N}$ is infinite. If $a \in \mathcal{C}(B) = \mathbb{N}$, then there exists a $b \in B$ such that $a \leq b$ since B is infinite. So, $a \in \mathcal{C}(\{b\})$ and thus $\mathcal{C}$ is algebraic.

Let us now consider the dependence space $\mathcal{D} = (\mathbb{N}, \Theta_\mathcal{C})$ where $\Theta_\mathcal{C}$ is the kernel of $\mathcal{C}$. The $\Theta_\mathcal{C}$-class of $\mathbb{N}$ consists of all infinite subsets of $\mathbb{N}$. Clearly, this congruence class has no minimal elements and so $\mathbb{N}$ has no reducts. Note that the maximal independent subsets of $\mathbb{N}$ are the sets $\{n\}$, $n \in \mathbb{N}$.

Next we intend to find conditions which guarantee the existence of reducts. Suppose $\mathcal{D} = (A, \Theta)$ is a dependence space and $B \subseteq A$. An element $a \in B$ is said to be *indispensable* for B if $(B, B - \{a\}) \notin \Theta$. The indispensable elements of B form the *core* of B, which is denoted by $CORE_\mathcal{D}(B)$. It is clear that $B \in IND_\mathcal{D}$ if and only if $B = CORE_\mathcal{D}(B)$. By Lemma 5.4.3, if $\mathcal{H}$ is dense in $\mathcal{D}$, then $a \in CORE_\mathcal{D}(B)$ if and only if $B - X = \{a\}$ for some $X \in \mathcal{H}$.

Example 5.4.10. Let us consider the dependence space $\mathcal{D} = (A, \Theta)$ defined in Example 5.1.3. We know that the family $\{\{1, 3\}, \{2, 3\}, \{1, 2, 4\}\}$ is dense in $\mathcal{D}$. Now $CORE_\mathcal{D}(A) = \{3\}$, since $A - \{1, 3\} = \{2, 4\}$, $A - \{2, 3\} = \{1, 4\}$, and $A - \{1, 2, 4\} = \{3\}$.

Our next proposition is a generalization of theorem appearing in [30].

Proposition 5.4.11. *Let $\mathcal{D} = (A, \Theta)$ be a dependence space and $B \subseteq A$.*

(a) *If every subset of B has a reduct, then $CORE_\mathcal{D}(B) = \bigcap RED_\mathcal{D}(B)$.*

(b) *If $CORE_\mathcal{D}(B) = \bigcap RED_\mathcal{D}(B)$, then B has reducts.*

Proof. (a) Suppose each subset of $B (\subseteq A)$ has a reduct. Assume $a \in CORE_\mathcal{D}(B)$ and $a \notin C$ for some $C \in RED_\mathcal{D}(B)$. Since $C \subseteq B - \{a\} \subseteq B$ and $B \Theta C$, we obtain $B \Theta B - \{a\}$, a contradiction! So, $CORE_\mathcal{D}(B) \subseteq \bigcap RED_\mathcal{D}(B)$.

If $a \in \bigcap RED_{\mathcal{D}}(B)$ and $a \notin CORE_{\mathcal{D}}(B)$, then $B\Theta B - \{a\}$. By our assumption, $B - \{a\}$ has a reduct C. It is clear that C is also a reduct of B. Because $a \notin C$, we obtain $a \notin \bigcap RED_{\mathcal{D}}(B)$, a contradiction! So, also $CORE_{\mathcal{D}}(B) \supseteq \bigcap RED_{\mathcal{D}}(B)$.

(b) Assume that $CORE_{\mathcal{D}}(B) = \bigcap RED_{\mathcal{D}}(B)$ and suppose that B has no reducts. Then $\bigcap RED_{\mathcal{D}}(B) = \{a \in B \mid a$ belongs to all reducts of $B\} = B$, which implies $CORE_{\mathcal{D}}(B) = B$. Hence, $B \in IND_{\mathcal{D}}$ and $B \in RED_{\mathcal{D}}(B)$, a contraction! □

Next we present a proposition which guarantees that each subset of a dependence space has a finite reduct.

Proposition 5.4.12. *A dependence space $\mathcal{D} = (A, \Theta)$ satisfies the* ACC *if and only if each subset of A has a finite reduct.*

Proof. Suppose that $\mathcal{D} = (A, \Theta)$ satisfies the ACC and consider any $B \subseteq A$. By Proposition 5.1.4 B has a finite subset F such that $B\Theta F$. Since F is finite, we may assume that it is a minimal subset of B with this property, and then F is obviously a finite reduct of B. The other direction is obvious by Proposition 5.1.4. □

This proposition has the following corollary.

Corollary 5.4.13. *If a dependence space $\mathcal{D} = (A, \Theta)$ satisfies the* ACC *and $B \subseteq A$, then all reducts of B are finite.*

Next we present a proposition which characterizes the reducts of a given subset by applying dense families

Proposition 5.4.14. *Let $\mathcal{H}(\subseteq \wp(A))$ be a dense family in a dependence space $\mathcal{D} = (A, \Theta)$. If $B \subseteq A$, then $C \in RED_{\mathcal{D}}(B)$ if and only if C is a minimal set with respect to the property of containing an element from each nonempty difference $B - X$, where $X \in \mathcal{H}$.*

Proof. Suppose that $C \in RED_{\mathcal{D}}(B)$. Then $C \subseteq B$, $B\Theta C$, and especially $B/\Theta \leq C/\Theta$. Thus, by Proposition 5.2.3, $C \cap (B - X) = (B \cap C) - X = C - X \neq \emptyset$ for all $X \in \mathcal{H}$ such that $B - X \neq \emptyset$. Assume that there exists a $D \subset C$ which contains an element from each nonempty difference $B - X$ where $X \in \mathcal{H}$. This implies $D - X = (D \cap B) - X = D \cap (B - X) \neq \emptyset$ for all $X \in \mathcal{H}$ which satisfy $B - X \neq \emptyset$. By Proposition 5.2.3 we obtain $B/\Theta \leq D/\Theta$. Since

$D \subset B$, also $D/\Theta \leq B/\Theta$ holds. So, $B\Theta D$ which implies that C is not a reduct of B, a contradiction!

Conversely, let C be a minimal subset of A with respect to the property of containing an element from each nonempty difference $B - X$, $X \in \mathcal{H}$. First we show that C is a subset of B. If $C \not\subseteq B$, then $B \cap C \subset C$ and $(B \cap C) \cap (B - X) = C \cap (B - X) \neq \emptyset$ whenever $B - X \neq \emptyset$, a contradiction! Thus, $C \subseteq B$. Since $C - X = C \cap (B - X) \neq \emptyset$ for all $X \in \mathcal{H}$ such that $B - X \neq \emptyset$, we obtain by Proposition 5.2.3 that $B\Theta C$. Assume C is dependent. Then there is a $D \subset C$ such that $C\Theta D$. Since Θ is transitive, we obtain $B\Theta D$ and hence for all $X \in \mathcal{H}$ such that $B - X \neq \emptyset$, $D \cap (B - X) = D - X \neq \emptyset$, a contradiction! So, C is also independent. □

Let us consider a nondeterministic information system $\mathcal{S} = (U, A, \{V_a\}_{a \in A})$. Then C is a reduct of $B(\subseteq A)$ in a dependence space $\mathcal{D}_{sim} = (A, \Theta_{sim})$ if and only if C is a minimal subset of B which defines the same strong similarity relation as B. On the other hand, in a dependence space $\mathcal{D}_{wsim} = (A, \Theta_{wsim})$ a set C is a reduct of B if and only if C is a minimal subset of B which defines the same weak similarity relation as B. Similar statements apply to dependence spaces induced by other information relations.

Example 5.4.15. In the dependence space $\mathcal{D}_{sim} = (A, \Theta_{sim})$ of Example 5.3.7, the family

$$\mathcal{H} = \{\{a\}, \{b\}, \{a, c\}, A\}$$

is dense by Proposition 5.3.6(a). We determine the reducts of the set A by using this fact. The differences $A - X$, where $X \in \mathcal{H}$ are

$$A - \{a\} = \{b, c\}, A - \{b\} = \{a, c\}, A - \{a, c\} = \{b\}, A - A = \emptyset.$$

Clearly $\{a, b\}$ and $\{b, c\}$ are the minimal sets which contain at least one element from each of the three nonempty differences. By Proposition 5.4.14 this implies that $\{a, b\}$ and $\{b, c\}$ are the reducts of A in (A, Θ_{sim}).

By Proposition 5.3.6(b) the family

$$\mathcal{K} = \{\emptyset, \{b\}, \{a, c\}, \{b, c\}\}$$

is dense in the dependence space $\mathcal{D}_{wsim}$ defined in Example 5.3.7. The differences

$$A - \emptyset = A, A - \{b\} = \{a, c\}, A - \{a, c\} = \{b\}, A - \{b, c\} = \{a\}$$

are all nonempty. Obviously, $\{a, b\}$ is the only minimal set which contains an element from all of these differences. Thus, $\{a, b\}$ is the only reduct of A in $\mathcal{D}_{wsim}$.

5.5 Dependency Relations

J. Novotný and M. Novotný [26] started the study of dependency relations defined in dependence spaces. Here we adopt their definition of dependency relations and generalize some of their results. Moreover, we introduce a method based on dense families which for a given dependency $B \to C$ finds all minimal subsets D of B such that $D \to C$.

Let $\mathcal{D} = (A, \Theta)$ be a dependence space. A subset $C (\subseteq A)$ is said to be *dependent on* $B (\subseteq A)$ in $\mathcal{D}$, which will be denoted by $B \to C$ $(\mathcal{D})$, if $\mathcal{C}_{\mathcal{D}}(C) \subseteq \mathcal{C}_{\mathcal{D}}(B)$. The relation $\to$ $(\mathcal{D})$ is called the *dependency relation of* $\mathcal{D}$. Usually we write simply $B \to C$ instead of $B \to C$ $(\mathcal{D})$ if there is no danger of confusion.

In the next lemma, which follows from Proposition 5.2.3, we present some equivalent definitions of dependency relations.

Lemma 5.5.1. *If $\mathcal{H} (\subseteq \wp(A))$ is dense in the dependence space $\mathcal{D} = (A, \Theta)$, then the following conditions are equivalent for all $B, C \subseteq A$:*
(a) $B \to C$*;*
(b) $C/\Theta \leq B/\Theta$*;*
(c) *for all $X \in \mathcal{H}$, $C - X \neq \emptyset$ implies $B - X \neq \emptyset$.* □

Example 5.5.2. Let $\mathcal{S} = (U, A, \{V_a\}_{a \in A})$ be an information system such that each attribute $a \in A$ is a map $a \colon U \to V_a$. Let us consider the dependence space $\mathcal{D}_{ind} = (A, \Theta_{ind})$. For all $B, C \subseteq A$,

$$\begin{aligned} B \to C \ (\mathcal{D}_{ind}) \quad &\iff \quad C/\Theta_{ind} \leq B/\Theta_{ind} \\ &\iff \quad ind(B) \subseteq ind(C) \end{aligned}$$

by Proposition 5.3.2 and Lemma 5.5.1. Thus, if $B \to C$ $(\mathcal{D}_{ind})$ and two objects have the same values for all attributes in B, then they have the same values for all attributes in C. Hence, the dependency $B \to C$ $(\mathcal{D}_{ind})$ means that the values of the attributes in C are determined by the values of the attributes in B.

Note that $C \subseteq B$ implies $B \to C$. If we denote by $\leftarrow$ the inverse of $\to$, then $\Theta = \to \cap \leftarrow$. Note also that if $B \to C$ when $C \subseteq B$, then $\mathcal{C}_{\mathcal{D}}(B) = \mathcal{C}_{\mathcal{D}}(C)$ and $B \Theta C$. The following lemma, which is given in [26] for finite dependence spaces, expresses the reducts by the means of dependency relation.

Proposition 5.5.3. *If $\mathcal{D} = (A, \Theta)$ is a dependence space, then $C (\subseteq A)$ is a reduct of $B (\subseteq A)$ if and only if C is a minimal subset of B such that $C \to B$.*

Proof. Assume first that C is a reduct of B. Then $C \to B$ follows from $\mathcal{C}_{\mathcal{D}}(B) = \mathcal{C}_{\mathcal{D}}(C)$. Moreover if $D \to B$ for since $D \subset C$, then $\mathcal{C}_{\mathcal{D}}(B) \subseteq \mathcal{C}_{\mathcal{D}}(D) \subseteq \mathcal{C}_{\mathcal{D}}(C) = \mathcal{C}_{\mathcal{D}}(B)$ would imply $B\Theta D$ contradicting our assumption that C is a reduct of B. $\square$

Let A be a set. We say that a relation $\dashrightarrow$ on $\wp(A)$ is a *dependency relation* on A if there exists a dependence space $\mathcal{D} = (A, \Theta)$ such that $\dashrightarrow$ is its dependency relation. In [26], J. Novotný and M. Novotný characterized the dependency relations of a finite set A. Here we generalize their result.

Proposition 5.5.4. *Let A be any set. A relation $\dashrightarrow$ on $\wp(A)$ is a dependency relation on A if and only if*

(a) $\dashrightarrow$ *is reflexive and transitive,*

(b) $\dashrightarrow$ *is completely $\cup$-compatible, and*

(c) *for all $B, C, X \subseteq A$,*

$$B \dashrightarrow C \text{ and } B \subseteq X \text{ imply } X \dashrightarrow C.$$

Proof. Let $\to$ be a dependency relation of $\mathcal{D} = (A, \Theta)$. Obviously, $B \to B$ holds for all $B \subseteq A$. If $B \to C$ and $C \to D$, then $\mathcal{C}_{\mathcal{D}}(D) \subseteq \mathcal{C}_{\mathcal{D}}(C) \subseteq \mathcal{C}_{\mathcal{D}}(B)$; that is, $B \to D$. Hence, $\to$ is reflexive and transitive.

Let B_i, C_i ($i \in I, I \neq \emptyset$) be subsets of A such that $B_i \to C_i$ for all $i \in I$. Then $C_i \subseteq \mathcal{C}_{\mathcal{D}}(C_i) \subseteq \mathcal{C}_{\mathcal{D}}(\bigcup_{i\in I} B_i)$ for all $i \in I$, which implies $\mathcal{C}_{\mathcal{D}}(\bigcup_{i\in I} C_i) \subseteq \mathcal{C}_{\mathcal{D}}(\bigcup_{i\in I} B_i)$ and $(\bigcup_{i\in I} B_i) \to (\bigcup_{i\in I} C_i)$. Thus, $\to$ is completely $\cup$-compatible. If $B \to C$ and $B \subseteq X$, then $\mathcal{C}_{\mathcal{D}}(C) \subseteq \mathcal{C}_{\mathcal{D}}(B) \subseteq \mathcal{C}_{\mathcal{D}}(X)$ and hence $X \to C$.

On the other hand, let $\dashrightarrow$ be reflexive and transitive binary relation on $\wp(A)$ which is completely $\cup$-compatible, and such that $B \dashrightarrow C$, $B \subseteq X$ imply $X \dashrightarrow C$ for all $B, C, X \subseteq A$. Let us denote by Θ the intersection of $\dashrightarrow$ and $\dashleftarrow$, where $\dashleftarrow$ is the inverse of $\dashrightarrow$. First we show that Θ is a complete congruence on $(\wp(A), \cup)$. It can be easily seen that Θ is an equivalence. Suppose $B_1 \Theta C_1$ and $B_2 \Theta C_2$. Then $B_1 \dashrightarrow C_1$, $B_2 \dashrightarrow C_2$, $C_1 \dashrightarrow B_1$ and $C_2 \dashrightarrow B_2$. This implies $(B_1 \cup B_2) \dashrightarrow (C_1 \cup C_2)$ and $(C_1 \cup C_2) \dashrightarrow (B_1 \cup B_2)$ since $\dashrightarrow$ is completely $\cup$-compatible. This means that $(B_1 \cup B_2)\Theta(C_1 \cup C_2)$ and thus Θ is a congruence on $(\wp(A), \cup)$. Let $B \subseteq A$ and suppose $B/\Theta = \{B_i\}_{i\in I}$. Because $B \dashrightarrow B_i$ and $B_i \dashrightarrow B$ for all $i \in I$, we get $B \dashrightarrow (\bigcup_{i\in I} B_i)$ and $(\bigcup_{i\in I} B_i) \dashrightarrow B$ since $\dashrightarrow$ is completely $\cup$-compatible. Hence, $B\Theta(\bigcup_{i\in I} B_i)$ and thus the congruence Θ is complete.

Finally, we show that $\dashrightarrow$ is the dependency relation of $\mathcal{D} = (A, \Theta)$. If $B \to C$ $(\mathcal{D})$, then $\mathcal{C}_{\mathcal{D}}(C) \subseteq \mathcal{C}_{\mathcal{D}}(B)$. By (c), $\mathcal{C}_{\mathcal{D}}(C) \dashrightarrow \mathcal{C}_{\mathcal{D}}(C)$ implies $\mathcal{C}_{\mathcal{D}}(B) \dashrightarrow$

$\mathcal{C}_{\mathcal{D}}(C)$. Because $(B, \mathcal{C}_{\mathcal{D}}(B)) \in \Theta \subseteq \dashrightarrow$ and $(\mathcal{C}_{\mathcal{D}}(C), C) \in \Theta \subseteq \dashrightarrow$, we obtain $B \dashrightarrow C$ by the transitivity of $\dashrightarrow$. On the other hand, if $B \dashrightarrow C$, then $(\mathcal{C}_{\mathcal{D}}(B), B) \in \Theta \subseteq \dashrightarrow$ and $(C, \mathcal{C}_{\mathcal{D}}(C)) \in \Theta \subseteq \dashrightarrow$ imply $\mathcal{C}_{\mathcal{D}}(B) \dashrightarrow \mathcal{C}_{\mathcal{D}}(C)$ by the transitivity of $\dashrightarrow$. The complete $\cup$-compatibility of $\dashrightarrow$ implies $\mathcal{C}_{\mathcal{D}}(B) \dashrightarrow \mathcal{C}_{\mathcal{D}}(B) \cup \mathcal{C}_{\mathcal{D}}(C)$. It is obvious that $\mathcal{C}_{\mathcal{D}}(B) \cup \mathcal{C}_{\mathcal{D}}(C) \dashrightarrow \mathcal{C}_{\mathcal{D}}(B)$. Thus, $\mathcal{C}_{\mathcal{D}}(B) \Theta \mathcal{C}_{\mathcal{D}}(B) \cup \mathcal{C}_{\mathcal{D}}(C)$ which implies $\mathcal{C}_{\mathcal{D}}(C) \subseteq \mathcal{C}_{\mathcal{D}}(B) \cup \mathcal{C}_{\mathcal{D}}(C) \subseteq \mathcal{C}_{\mathcal{D}}(\mathcal{C}_{\mathcal{D}}(B)) = \mathcal{C}_{\mathcal{D}}(B)$, that is, $B \to C$ $(\mathcal{D})$. □

In information systems it is important to find for a dependency $B \to C$ all or some minimal subsets D of B such that $D \to C$ holds. In the sequel we characterize these subsets, but first we give some conditions which guarantee that such a D exists.

Example 5.5.5. Let us consider the dependence $\mathcal{D} = (\mathbb{N}, \Theta_{\mathcal{C}})$ defined in Example 5.4.9. It is clear that $\mathbb{N} \to \mathbb{N}$ $(\mathcal{D})$. But as we have seen, the set $\mathbb{N}$ has no reducts in $\mathcal{D}$. This implies by Proposition 5.5.3 that there exists no minimal subset D of $\mathbb{N}$ such that $D \to \mathbb{N}$. Note that the closure operator $\mathcal{C}$ is algebraic.

Next we present a proposition which guarantees that for each dependency $B \to C$ there exists a finite minimal subset F of B such that $F \to C$. This result is akin to Proposition 5.4.12.

Proposition 5.5.6. *Let $\mathcal{D} = (A, \Theta)$ be a dependence space. For any dependency $B \to C$ there exists a finite subset F of B such that $F \to C$ if and only if $\mathcal{D}$ satisfies the ACC.*

Proof. Suppose that $\mathcal{D} = (A, \Theta)$ satisfies the ACC and let $B, C \subseteq A$ be such that $B \to C$. Then by Proposition 5.1.4, there exists a finite subset G of B such that $B \Theta G$, and since G is finite, it has a minimal subset F such that $F \to B$.

On the other hand, suppose that for any dependency $B \to C$ there exists a finite subset F of B such that $F \to C$. Let $B \subseteq A$. Because $B \to B$ holds trivially, then by our assumption there exists a finite subset F of B such that $F \to B$, which is equivalent to $B/\Theta \leq F/\Theta$. This implies $B \Theta F$ since F is a subset of B. By Proposition 5.1.4 we obtain that $\mathcal{D}$ satisfies the ACC. □

Next we present a proposition which characterizes in terms of dense families the minimal subsets D of B which satisfy $D \to B$ for a dependency $B \to C$. Note that this proposition is related to Proposition 5.4.14 which characterizes the reducts of given set.

Proposition 5.5.7. *Let $\mathcal{H}(\subseteq \wp(A)$ be a dense family in a dependence space $\mathcal{D} = (A, \Theta)$. If $B \to C$, then D is a minimal subset of B which satisfies $D \to C$ if and only if D is a minimal set with respect to the property of containing an element from each difference $B - X$, where $X \in \mathcal{H}$ and satisfies $C - X \neq \emptyset$.*

Proof. Suppose that $B \to C$ and let D be a minimal subset of B such that $D \to C$. Because $D \subseteq B$, the assumption $D \to C$ implies by Lemma 5.5.1 that $D \cap (B - X) = (D \cap B) - X = D - X \neq \emptyset$ for all $X \in \mathcal{H}$ such that $C - X \neq \emptyset$. Assume that there exists an $E \subset D$ which satisfies $E \cap (B - X) \neq \emptyset$ for all $X \in \mathcal{H}$ such that $C - X \neq \emptyset$. But $E \subseteq B$ implies that $E - X = E \cap (B - X) \neq \emptyset$ for all $X \in \mathcal{H}$ which satisfy $C - X \neq \emptyset$. Thus $E \to C$, a contradiction!

Conversely, assume $B \to C$ and suppose that D is a minimal set which contains an element from each difference $B - X$ where $X \in \mathcal{H}$ satisfies $C - X \neq \emptyset$. If $D \not\subseteq B$, then $D \cap B \subset D$ and $(D \cap B) \cap (B - X) = D \cap (B - X) \neq \emptyset$ for all $X \in \mathcal{H}$ such that $C - X \neq \emptyset$, a contradiction! Hence, $D \subseteq B$. This implies $D - X = D \cap (B - X) \neq \emptyset$ for all $X \in \mathcal{H}$ which satisfy $C - X \neq \emptyset$. This means $D \to C$. Suppose there exists an $E \subset D$ such that $E \to C$. Then $E \subset D \subseteq B$ implies $E - X = E \cap (B - X) \neq \emptyset$ whenever $C - X \neq \emptyset$, a contradiction! □

Example 5.5.8. Let $\mathcal{S} = (U, A, \{V_a\}_{a \in A})$ be an information system such that $U = \{1, \ldots, 5\}$, $A = \{a, \ldots, f\}$, $V_a = V_b = \{0, 1, 2\}$, $V_c = \{0, 1, 2, 3\}$, $V_d = V_e = V_f = \{0, 1\}$ and the values of the attributes are defined as in Table 6.

	a	b	c	d	e	f
1	0	1	2	0	1	0
2	0	2	3	1	1	1
3	1	0	3	1	1	1
4	2	0	1	0	0	1
5	2	0	0	0	0	1

Table 6.

Let $B = \{a, b, c, d\}$ and $C = \{e, f\}$. It is easy to see that the values of C are determined by the values of B. This means that $B \to C$ holds in $\mathcal{D}_{ind}$. Next we intend to find all minimal subsets D of B which satisfy $D \to C$ $(\mathcal{D}_{ind})$.

The indiscernibility matrix $M(IND)_{\mathcal{S}} = (c_{ij})_{5 \times 5}$ of $\mathcal{S}$ is the following:

$$\begin{pmatrix} A & \{a,e\} & \{e\} & \{d\} & \{d\} \\ \{a,e\} & A & \{c,d,e,f\} & \{f\} & \{f\} \\ \{e\} & \{c,d,e,f\} & A & \{b,f\} & \{b,f\} \\ \{d\} & \{f\} & \{b,f\} & A & \{a,b,d,e,f\} \\ \{d\} & \{f\} & \{b,f\} & \{a,b,d,e,f\} & A \end{pmatrix}$$

By Proposition 5.3.6, the family

$$\begin{aligned} \mathcal{H} &= \{c_{ij} \mid 1 \leq i,j \leq 5\} \\ &= \{\{a,e\},\{a,b,d,e,f\},\{b,f\},\{c,d,e,f\},\{d\},\{e\},\{f\},A\} \end{aligned}$$

is dense in $\mathcal{D}_{ind}$. The differences $C - X$, $X \in \mathcal{H}$ are nonempty for $X = \{a,e\}$, $\{b,f\}$, $\{d\}$, $\{e\}$, $\{f\}$. The corresponding differences $B - X$ are the following:

- $\{a,b,c,d\} - \{a,e\} = \{b,c,d\}$;
- $\{a,b,c,d\} - \{b,f\} = \{a,c,d\}$;
- $\{a,b,c,d\} - \{d\} = \{a,b,c\}$;
- $\{a,b,c,d\} - \{e\} = \{a,b,c,d\}$;
- $\{a,b,c,d\} - \{f\} = \{a,b,c,d\}$.

Next we must find all such minimal sets which contain an element from all of the above differences. Because $\{a,b,c\}$, $\{a,c,d\}$ and $\{b,c,d\}$ are the minimal differences, it suffices to consider them only. It is easy to see that $\{c\}$, $\{a,b\}$, $\{a,d\}$, $\{b,d\}$ are the minimal sets which contain an element from all of these differences. So, $\{c\}$, $\{a,b\}$, and $\{a,d\}$, and $\{b,d\}$ are the minimal subsets D of $\{a,b,c,d\}$ which satisfy $D \to \{e,f\}$.

Note that the sets $\{c\}$, $\{a,b\}$, $\{a,d\}$, and $\{b,d\}$ are not reducts of B in $\mathcal{D}_{ind}$, since they are not Θ_{ind}-equivalent to B. In fact, $RED_{\mathcal{D}_{ind}}(\{a,b,c,d\}) = \{\{a,c\},\{b,c\}\}$.

We conclude this section by discussing briefly a couple of notions closely related to the dependency concept considered above. In the theory of relational databases (cf. [11], for example) the concept of a functional dependency between sets of attributes is of fundamental importance. Let A be a set. A *functional dependency* over A is an ordered pair $B \to C$, where $B, C \subseteq A$. An *Armstrong system* on A is a set $\mathcal{F} \subseteq \wp(A) \times \wp(A)$ which satisfies the following (modified) *Armstrong Axioms* (see [6], for example):

(A1) $B \supseteq C$ implies $B \to C \in \mathcal{F}$;

(A2) $B \to C \in \mathcal{F}$ and $C \to D \in \mathcal{F}$ imply $B \to D \in \mathcal{F}$;

(A3) $\mathcal{F}$ is completely $\cup$-compatible.

It can be easily seen that (A1)–(A3) are equivalent to conditions (a)–(c) of Proposition 5.5.4. Hence, a relation $\dashrightarrow$ is a dependency relation on A if and only if it is an Armstrong system on A.

It is easy to see that the set $\mathrm{ASys}(A)$ of all Armstrong systems on a set A forms a complete lattice with respect to the inclusion order. Moreover, Day [6] observed the following correspondes between Armstrong systems on A and the closure operators on A. Each Armstrong system $\mathcal{F}$ on A defines a closure operator $\mathcal{C}_{\mathcal{F}} \colon \wp(A) \to \wp(A)$ if we set

$$\mathcal{C}_{\mathcal{F}}(B) = \bigcup \{ C \subseteq A \mid B \to C \in \mathcal{F} \} \tag{5.3}$$

for all $B \subseteq A$. On the other hand, each closure operator $\mathcal{C} \colon \wp(A) \to \wp(A)$ defines an Armstrong system $\mathcal{F}_{\mathcal{C}}$ on A by the rule

$$B \to C \in \mathcal{F}_{\mathcal{C}} \quad \text{if and only if} \quad \mathcal{C}(C) \subseteq \mathcal{C}(B).$$

Furthermore, the maps $\mathcal{F} \mapsto \mathcal{C}_{\mathcal{F}}$ and $\mathcal{C} \mapsto \mathcal{F}_{\mathcal{C}}$ form a pair of mutually inverse order-isomorphisms between $(\mathrm{ASys}(A), \subseteq)$ and $(\mathrm{Clo}(\wp(A)), \leq)$.

We also discuss shortly knowledge structures (see e.g. [8, 22]). Let A be a finite set of problems. A *knowledge state* is the set of problems a subject is capable of solving. A *knowledge structure* is a pair $(A, \mathcal{K})$, where $\mathcal{K} \subseteq \wp(A)$. Intuitively speaking, a knowledge structure consists of the different knowledge states that can occur within the members of a population. Let us denote by $\mathrm{Knowl}(A)$ the set of all knowledge structures on A. A knowledge structure which is closed under unions is called a *knowledge space*.

A relation $R \in \mathrm{Rel}(\wp(A))$ is called an *entail relation* for A if for all $B, C \subseteq A$,

(E1) $B \supseteq C$ implies $(B, C) \in R$;

(E2) R is transitive;

(E3) $(B, C_i) \in R$ for all i in an index set I implies $(B, \bigcup_{i \in I} C_i) \in R$.

It is easy to see that (A1)–(A3) and (E1)–(E3) are equivalent. Thus, a relation $\dashrightarrow$ is a dependency relation on A if and only if $\dashrightarrow$ is an Armstrong system on A if and only if $\dashrightarrow$ is an entail relation for A, as noted in [8].

Koppen and Doignon [22] have shown that the following correspondences hold between knowledge structures and binary relations. Every knowledge structure $(A, \mathcal{K})$ defines an entail relation $R_{\mathcal{K}}$ for A by the rule

(5.4) $(B, C) \in R_{\mathcal{K}} \iff$ for all $X \in \mathcal{K}$, $C \cap X \neq \emptyset$ implies $B \cap X \neq \emptyset$.

An interpretation of $(B, C) \in R_{\mathcal{K}}$ is, for example, that if a student masters some questions in C if he/she also masters some questions in B; or equivalently, if a student does not master any question in B, he/she does not master any question in C.

On the other hand, every binary relation R on $\wp(A)$ defines a knowledge space $(A, \mathcal{K}_R)$, where $\mathcal{K}_R$ is defined by

$$\mathcal{K}_R = \{X \subseteq A \mid \text{ for all } (B, C) \in R, C \cap X \neq \emptyset \text{ implies } B \cap X \neq \emptyset\}.$$

Moreover, the maps $R \mapsto (A, \mathcal{K}_R)$ and $(A, \mathcal{K}) \mapsto R_{\mathcal{K}}$ form a Galois connection between $(\mathrm{Rel}(\wp(A)), \subseteq)$ and $(\mathrm{Knowl}(A), \subseteq)$.

Let us also note that Düntsch and Gediga showed in [9] that if $(^{\blacktriangleright}, ^{\blacktriangleleft})$ is a Galois connection between $(\wp(A), \subseteq)$ and $(\wp(B), \subseteq)$, then $R \in \mathrm{Rel}(\wp(A))$ defined by

$$(X, Y) \in R \iff X^{\blacktriangleright} \subseteq Y^{\blacktriangleright}$$

is a dependence relation on A.

We end this chapter by noting the following. Consider an entail relation R for A. Since R is a dependency relation on A, the relation $\Theta_R = R \cap R^{-1}$ is by Proposition 5.5.4 a complete congruence on $(\wp(A), \cup)$, and this means that the pair (A, Θ_R) is a dependence space, which has the dependency relation R. By (5.3), the map

$$\wp(A) \to \wp(A), B \mapsto \bigcup\{X \mid (B, X) \in R\}$$

is the closure operator of this dependence space.

If $(A, \mathcal{K})$ is a knowledge structure, then obviously the relation $\Theta_{\mathcal{K}} = R_{\mathcal{K}} \cap R_{\mathcal{K}}^{-1}$ is a complete congruence on $(\wp(A), \cup)$ and the pair $(A, \Theta_{\mathcal{K}})$ is a dependence space. By Lemma 5.5.1 and (5.4),

$$\begin{aligned} C/\Theta_{\mathcal{K}} \leq B/\Theta_{\mathcal{K}} &\iff (B, C) \in R_{\mathcal{K}} \\ &\iff (\forall X \in \mathcal{K})(C \cap X \neq \emptyset \Longrightarrow B \cap X \neq \emptyset) \\ &\iff (\forall X \in \mathcal{K})(B \cap X = \emptyset \Longrightarrow C \cap X = \emptyset) \\ &\iff (\forall X \in \mathcal{K})(B \subseteq X^{\complement} \Longrightarrow C \subseteq X^{\complement}); \end{aligned}$$

this implies that the family $\{X^{\complement} \mid X \in \mathcal{K}\}$ is dense in $(A, \Theta_{\mathcal{K}})$.

Chapter 6

Rough Sets

6.1 Approximations and Definable Sets

Knowledge about objects may be represented as binary relations. For instance, if we classify all human beings into two disjoint sets consisting of women and men, respectively, then this classification determines an equivalence E such that xEy whenever x and y are of the same gender.

In rough set theory it is usually assumed that the knowledge about objects is restricted by some indiscernibility relation (see [43, 45], for example). Indiscernibility relations are equivalences which are interpreted so that two objects are equivalent if we cannot distinguish them by using our information. This means that the objects of the given universe U can be classified by the knowledge represented by an indiscernibility relation $E(\in \mathrm{Eq}(U))$ into three classes with respect to any subset $X(\subseteq U)$:

1. the objects, which surely are in X;

2. the objects, which are surely not in X;

3. the objects, which possibly are in X.

The objects in class 1 form the lower E-approximation of X, and the objects of type 1 and 3 form together its upper E-approximation. The E-boundary of X consists of objects in class 3. Some subsets of U are identical to both of their approximations and they are called E-definable.

This chapter can be viewed as a generalization of the theory concerning approximations and rough sets defined by equivalences (cf. [13, 17, 28, 29, 43, 45, 46, 54]). Here we assume that the knowledge about objects is given by a similarity relation. We suppose that similarity relations are tolerances. This requirement is quite natural. Namely, each object is obviously similar to itself, and if x is similar to y, then x and y are in some sense alike and so also y must be similar to x. Recall that indiscernibility and similarity relations defined in nondeterministic information systems are at least tolerances.

However, it should be noted that there are cases in which similarity is not necessarily a symmetric relation. For example in [23, p. 40] it is argued that

> "the statement yRx which means 'y *is similar to* x' is directional; it has a *subject* y and a *referent* x and it is not equivalent in general to the statement 'x *is similar to* y' as argued by Tversky. For example, in the following statement: '*a son resembles his father*' the son is the subject and the father is the referent; the inverse statement usually makes much less sense."

We may also observe that the inclusion relations defined in nondeterministic information systems may be viewed as directional similarity relations (or even as directional indiscernibility relations). Namely, if $a(x) \subseteq a(y)$, then we cannot distinguish x from y only by considering the a-values of x.

According to Pawlak's [43] definition, rough sets are $\equiv_E$-classes of some equivalence $E \in \mathrm{Eq}(U)$. The idea of rough sets is that if subsets of U are observed through the knowledge represented by E, then the sets in the same $\equiv_E$-class look the same; $X \equiv_E Y$ means that exactly the same elements belong certainly to X and to Y, and exactly the same elements belong possibly to X and to Y. In Section 6.3 we generalize Pawlak's notion by defining rough sets in terms of tolerances.

First we study approximations defined by tolerance relations. B. Konikowska [20, 21] and J.A. Pomykała [49] considered approximation operations defined by strong similarity relations of nondeterministic information systems. Also J. Nieminen [25] has studied approximations induced by tolerances but his definition is not the same as ours. Furthermore, J.A. Pomykała [47, 48] and W. Żakowski [57] have investigated approximations defined by covers which can be applied to tolerances. Recall that for any tolerance $R \in \mathrm{Tol}(U)$ and any $x \in U$, $x/R = \{y \in U \mid xRy\}$.

Definition. Let U be a set of *objects* and let R be a tolerance on U referred to as the *similarity relation*. The *lower* R*-approximation* of a set $X (\subseteq U)$ is

$$X_R = \{x \in U \mid x/R \subseteq X\};$$

its *upper R-approximation* is

$$X^R = \{x \in U \mid x/R \cap X \neq \emptyset\}.$$

The set $B_R(X) = X^R - X_R$ is called the *R-boundary* of X.

The lower R-approximation of X consists of elements which surely belong to X in view of the knowledge provided by R. The upper R-approximation of X is formed of elements which possibly are in X in light of the knowledge R. Obviously, the R-boundary is the area of uncertainty.

In the next proposition we give some basic properties of approximations.

Proposition 6.1.1. *If $R \in \mathrm{Tol}(U)$ and $X \subseteq U$, then*

(a) *$\emptyset_R = \emptyset^R = \emptyset$ and $U_R = U^R = U$;*
(b) *$X_R \subseteq X \subseteq X^R$;*
(c) *$(X_R)^{\complement} = (X^{\complement})^R$ and $(X^R)^{\complement} = (X^{\complement})_R$;*
(d) *$B_R(X) = B_R(X^{\complement})$;*
(e) *the pair $({}^R, {}_R)$ is a dual Galois connection on $(\wp(U), \subseteq)$;*
(f) *$X^R = \bigcup\{x/R \mid x \in X\}$;*
(g) *$(X_R)^R = \bigcup\{x/R \mid x/R \subseteq X\}$ and $(X^R)^R = \bigcup\{x/R \mid x/R \cap X \neq \emptyset\}$.*

Proof. Assertions (a), (b), and (c) can be found in [20, 21, 49], and (d) and (f) are obvious. Claim (g) follows from (f).

(e) It follows immediately from the definitions that the maps ${}_R\colon X \mapsto X_R$ and ${}^R\colon X \mapsto X^R$ are order-preserving.

If $x \in (X_R)^R$, then $x/R \cap X_R \neq \emptyset$ which implies that there exists a $y \in X_R$ such that xRy. Therefore, $x \in X$ and so $(X_R)^R \subseteq X$. Let us denote $Y = X^{\complement}$. Then $(Y_R)^R \subseteq Y$ implies

$$X = Y^{\complement} \subseteq ((Y_R)^R)^{\complement} = ((Y^{\complement})^R)_R = (X^R)_R.$$

□

By the previous proposition, the R-boundary of a set is equal to the R-boundary of its complement. It means simply that if we cannot decide when an object x is in X, then we obviously cannot decide whether x belongs to $X^{\complement}$ either. Moreover, $({}^R, {}_R)$ is a dual Galois connection on $(\wp(U), \subseteq)$. This fact implies by Propositions 3.3.3, 3.5.1, and 3.5.4 our following lemma.

Lemma 6.1.2. *Let $R \in \mathrm{Tol}(U)$, $X \subseteq U$, and $\mathcal{H} \subseteq \wp(U)$.*

(a) *$((X^R)_R)^R = X^R$ and $((X_R)^R)_R = X_R$.*
(b) *$(\bigcup \mathcal{H})^R = \bigcup\{X^R \mid X \in \mathcal{H}\}$ and $(\bigcap \mathcal{H})_R = \bigcap\{X_R \mid X \in \mathcal{H}\}$.*
(c) *$(\bigcap \mathcal{H})^R \subseteq \bigcap\{X^R \mid X \in \mathcal{H}\}$ and $(\bigcup \mathcal{H})_R \supseteq \bigcup\{X_R \mid X \in \mathcal{H}\}$.* □

Note that the map $X \mapsto X^R$ is a complete join-morphism $(\wp(U), \subseteq) \to (\wp(U), \subseteq)$ and $X \mapsto X_R$ is a complete meet-morphism $(\wp(U), \subseteq) \to (\wp(U), \subseteq)$.

Let $\mathcal{S} = (U, A, \{V_a\}_{a \in A})$ be a nondeterministic information system and let $\emptyset \neq B \subseteq A$. Then by Lemma 4.1.5, the relations $ind(B)$, $wind(B)$, $sim(B)$, and $wsim(B)$ are tolerances.

Example 6.1.3. Let us consider the nondeterministic information system $\mathcal{S} = (U, A, \{V_a\}_{a \in A})$ given in Example 4.2.2. If we denote $R = sim(\{\text{Height}\}) = \text{Height}^{-1}(SIM)$, then R is a similarity relation on U such that xRy if and only if x and y are approximately of the same height. By Example 4.2.2,

$$1/R = \{1, 4\}, 2/R = \{2\}, 3/R = \{3, 4\}, 4/R = \{1, 3, 4\}.$$

If $X = \{2, 3\}$, then $X_R = \{2\}$ and $X^R = \{2, 3, 4\}$. Now $B_R(X) = \{3, 4\}$ and $4 \in B_R(X)$, for example, because both in X and in $X^{\complement}$ there is an object which is similar to 4.

Note that the inclusions in Lemma 6.1.2(c) can be proper. Here, for example,

$$(\{3\} \cap \{4\})^R = \emptyset^R = \emptyset, \text{ but } \{3\}^R \cap \{4\}^R = \{3, 4\} \cap \{1, 3, 4\} = \{3, 4\}.$$

Similarly,

$$(\{3\} \cup \{4\})_R = \{3, 4\}_R = \{3\}, \text{ but } \{3\}_R \cup \{4\}_R = \emptyset \cup \emptyset = \emptyset.$$

Next we compare approximations defined by different similarity relations. The following lemma means that the approximations of a set X get closer to X, if the knowledge is more precise.

Lemma 6.1.4. *If $R, S \in \text{Tol}(U)$ are such that $R \subseteq S$, then $X_R \supseteq X_S$, $X^R \subseteq X^S$, and $B_R(X) \subseteq B_S(X)$ for any $X \subseteq U$.*

Proof. If $x \in X_S$, then $x/R \subseteq x/S \subseteq X$, i.e., $x \in X_R$. If $x \in X^R$, then $x/S \cap X \supseteq x/R \cap X \neq \emptyset$, which means that $x \in X^S$. Finally, $B_R(X) = X^R - X_R \subseteq X^S - X_S = B_S(X)$. $\square$

Let $^{\blacktriangleright}$ and $^{\blacktriangleleft}$ be maps on $\wp(U)$. We say that $(^{\blacktriangleright}, ^{\blacktriangleleft})$ is a *pair of approximation maps*, if there exists an $R \in \text{Tol}(U)$ such that $X^{\blacktriangleright} = X^R$ and $X^{\blacktriangleleft} = X_R$ for all $X \in \wp(U)$. Our next proposition characterizes the pairs of approximation maps.

Proposition 6.1.5. *Let* $^{\blacktriangleright}$ *and* $^{\blacktriangleleft}$ *be two maps on* $\wp(U)$*. Then* $(^{\blacktriangleright},^{\blacktriangleleft})$ *is a pair of approximation maps if and only if*

(a) $(^{\blacktriangleright},^{\blacktriangleleft})$ *is a dual Galois connection on* $(\wp(U),\subseteq)$*,*
(b) $y \in \{x\}^{\blacktriangleright}$ *implies* $x \in \{y\}^{\blacktriangleright}$ *for all* $x, y \in U$*,*
(c) $X \subseteq X^{\blacktriangleright}$ *for all* $X \subseteq U$*, and*
(d) $(X^{\complement})^{\blacktriangleright} = (X^{\blacktriangleleft})^{\complement}$ *for all* $X \subseteq U$.

Proof. Let $R \in \mathrm{Tol}(U)$. Then $(^{R},_{R})$ is a dual Galois connection on $(\wp(U),\subseteq)$ by Proposition 6.1.1(e). Condition (b) is satisfied since $y \in \{x\}^R$ if and only if xRy. Conditions (c) and (d) hold by Proposition 6.1.1.

On the other hand, let $(^{\blacktriangleright},^{\blacktriangleleft})$ be a dual Galois connection on $(\wp(U),\subseteq)$ which satisfies (b)–(d). Let us define a binary relation R on U by xRy iff $y \in \{x\}^{\blacktriangleright}$. By (c), $x \in \{x\} \subseteq \{x\}^{\blacktriangleright}$, which implies xRx. If xRy, then $y \in \{x\}^{\blacktriangleright}$, which implies by (b) that $x \in \{y\}^{\blacktriangleright}$. Thus, yRx and hence R is a tolerance.

Let $X \subseteq U$. Because by Proposition 3.5.4, $^{\blacktriangleright}\colon \wp(U) \to \wp(U)$ is a complete join-morphism $(\wp(U),\subseteq) \to (\wp(U),\subseteq)$, we obtain $(\bigcup_{x\in X}\{x\})^{\blacktriangleright} = \bigcup_{x\in X}\{x\}^{\blacktriangleright}$ for any $X \subseteq U$. Hence,

$$X^{\blacktriangleright} = (\textstyle\bigcup_{x\in X}\{x\})^{\blacktriangleright} = \bigcup_{x\in X}\{x\}^{\blacktriangleright} = \bigcup\{x/R \mid x \in X\} = X^R.$$

By (d),

$$X^{\blacktriangleleft} = ((X^{\blacktriangleleft})^{\complement})^{\complement} = ((X^{\complement})^{\blacktriangleright})^{\complement} = ((X^{\complement})^{R})^{\complement} = ((X_R)^{\complement})^{\complement} = X_R.$$

□

We conclude this section by considering R-definable sets.

Definition. Let U be a set and $R \in \mathrm{Tol}(U)$. A set $X(\subseteq U)$ is R-*definable* if $X_R = X^R$.

We denote by $\mathrm{Def}(R)$ the set of all R-definable sets. It is obvious that a set X is R-definable if and only if its R-boundary $B_R(X)$ is empty. This means that for any object $x \in U$, we can with certainty decide whether $x \in X$ by using the knowledge provided by R. To show that a set is definable requires only half as much work as the definition suggests.

Lemma 6.1.6. *For any* $R \in \mathrm{Tol}(U)$ *and* $X \subseteq U$*, the following three conditions are equivalent:*

(a) $X \in \mathrm{Def}(R)$*;*
(b) $X_R = X$*;*
(c) $X = X^R$.

Proof. It is obvious that (a) implies (b) and (c). Suppose $X_R = X$. We show that $X = X^R$. Trivially, $X \subseteq X^R$. If $x \in X^R$, then $x/R \cap X_R \neq \emptyset$, which implies $x \in X$ and hence $X^R \subseteq X$.

Assume that $X = X^R$. Clearly, $X_R \subseteq X$. If $x \in X$, then xRy implies $y \in X^R = X$, and hence $x \in X_R$ and $X \subseteq X_R$.

Now we have shown that (b) and (c) are equivalent conditions and from this it also follows that both of them imply (a). □

Next we characterize the R-definable sets in terms of sets saturated by an equivalence relation. We say that $X (\subseteq U)$ is *saturated* by $E \in \mathrm{Eq}(U)$, if X is the union of some equivalence classes of E or $X = \emptyset$. The set of all sets saturated by E is denoted by $\mathrm{Sat}(E)$.

A family $\mathcal{F} (\subseteq \wp(U))$ is called a *complete field of sets* if $\emptyset, U \in \mathcal{F}$, $X^{\complement} \in \mathcal{F}$ for all $X \in \mathcal{F}$ and $\bigcup \mathcal{H}, \bigcap \mathcal{H} \in \mathcal{F}$ for all $\mathcal{H} \subseteq \mathcal{F}$. Now the following lemma holds (see [54], for example).

Lemma 6.1.7. *Let* $E \in \mathrm{Eq}(U)$ *and* $X \subseteq U$.

(a) $X \in \mathrm{Sat}(E)$ *if and only if for all* $x \in X$, xEy *implies* $y \in X$.

(b) $\mathrm{Sat}(E)$ *is a complete field of sets.* □

Next we give a proposition which characterizes definable sets.

Proposition 6.1.8. *If* $R \in \mathrm{Tol}(U)$, *then*

$$\mathrm{Def}(R) = \mathrm{Sat}(R^E).$$

Proof. Suppose $X \in \mathrm{Def}(R)$. Then $X^R = X$. Let $x \in X$. If $(x, y) \in R^E$, then there exists a sequence $c_0, \ldots, c_n$ such that $x = c_0$, $y = c_n$, and $c_i R c_{i+1}$ or $c_{i+1} R c_i$ for all $0 \leq i \leq n-1$. Because R is symmetric, this means that $c_i R c_{i+1}$ for all $0 \leq i \leq n-1$. Since $X^R = X$, it is easy to show by induction that every c_i, $0 \leq i \leq n$, belongs to X. In particular, $y \in X$, which implies by Lemma 6.1.7 that X is saturated by R^E.

Conversely, suppose that X is saturated by R^E. Obviously, $X \subseteq X^R$. Let $y \in X^R$. Then there exists an $x \in X$ such that xRy and hence $(x, y) \in R^E$. This implies by Lemma 6.1.7 that $y \in X$. Now we have proved $X = X^R$, which implies by Lemma 6.1.6 that $X \in \mathrm{Def}(R)$. □

Corollary 6.1.9. *If* U *is a set and* $R \in \mathrm{Rel}(U)$, *then* $\mathrm{Def}(R)$ *is a complete field of sets.*

Our next lemma is a generalization of a result presented in [46].

Lemma 6.1.10. *Let $R \in \mathrm{Tol}(U)$ and $X, Y \subseteq U$. If X is R-definable, then*

$$(X \cup Y)_R = X_R \cup Y_R \text{ and } (X \cap Y)^R = X^R \cap Y^R.$$

Proof. It is obvious that $X_R \cup Y_R \subseteq (X \cup Y)_R$. Let $x \in (X \cup Y)_R$, i.e., $x/R \subseteq X \cup Y$. If $x/R \cap X \neq \emptyset$, then $x/R \subseteq X$ and $x \in X_R$ because X is R-definable. If $x/R \cap X = \emptyset$, then $x/R \subseteq Y$ and $x \in Y_R$. Hence, in both cases $x \in X_R \cup Y_R$.

It is also clear that $(X \cap Y)^R \subseteq X^R \cap Y^R$. Let $x \in X^R \cap Y^R$. Then $x/R \cap X \neq \emptyset$ and $x/R \cap Y \neq \emptyset$. Since X is R-definable, $x/R \subseteq X$, and $x/R \cap (X \cap Y) = (x/R \cap X) \cap Y = x/R \cap Y \neq \emptyset$. So, $x \in (X \cap Y)^R$. □

Let $E \in \mathrm{Eq}(U)$ be an equivalence relation. By Proposition 6.1.8, the E-definable sets are the unions of some (or none) E-classes. We note that this is actually Pawlak's original definition of E-definable sets [43]. We also mention that the sets X^E and X_E are E-definable for all $X \subseteq U$.

Example 6.1.11. Let R be the tolerance on $\{1, 2, 3, 4\}$ considered in Example 6.1.3. It is easy to see that R^E has the equivalence classes $\{2\}$ and $\{1, 3, 4\}$. So, $\mathrm{Def}(R) = \{\emptyset, \{2\}, \{1, 3, 4\}, U\}$.

Note that R-approximations are not necessarily R-definable. For instance, $\{1\}^R = \{1, 4\}$ and $\{1, 4\}_R = \{1\}$.

6.2 Rough Equalities

In this section we characterize the three types of rough equality relations defined by tolerances. Novotný and Pawlak [28, 29] have characterized the rough equalities defined by equivalences on a finite set of objects, and Steinby [54] generalized these characterizations by omitting the assumption of finiteness. In [25] Nieminen presented a characterization of rough equalities defined by tolerances, but his notion of rough equalities differ essentially from ours.

Let $\mathbb{E}$ be a family of subsets of a set U such that $\bigcup \mathbb{E} = U$. In [47, 48] J. A. Pomykała associates with any such family $\mathbb{E}$ five pairs of approximation operators $\overline{\mathbb{E}}_i$ and $\underline{\mathbb{E}}_i$ ($i = 0, \ldots, 4$) on U. We shall show how the operators $\overline{\mathbb{E}}_i$ and $\underline{\mathbb{E}}_i$, where $0 \leq i \leq 2$, relate to our work.

First we define different types of equalities based on approximations. For equivalence relations the corresponding notion were defined in [28, 29].

Definition. Let $R \in \mathrm{Tol}(U)$. We define in $\wp(U)$ the *lower R-equality* $\approx_R$, the *upper R-equality* $\approx^R$, and the *R-equality* $\equiv_R$ by the following conditions:

$$\begin{aligned} X \approx_R Y &\iff X_R = Y_R; \\ X \approx^R Y &\iff X^R = Y^R; \\ X \equiv_R Y &\iff X_R = Y_R \text{ and } X^R = Y^R. \end{aligned}$$

Obviously, $X \approx_R Y$ means that the same objects belong for certain to X and to Y. Similarly, $X \approx^R Y$ whenever the same objects are possibly in X and in Y. The relation $\equiv_R$ is the intersection of $\approx_R$ and $\approx^R$.

Next we study more closely the relations $\approx_R$ and $\approx^R$. Let U be any set and $\mathcal{H} \subseteq \wp(U)$. Recall that the relation $\Theta_{\mathcal{H}}$ is defined on $\wp(U)$ by the condition:

$$(X, Y) \in \Theta_{\mathcal{H}} \text{ if and only if } X \subseteq Z \iff Y \subseteq Z \text{ for all } Z \in \mathcal{H}.$$

A relation $\Omega_{\mathcal{H}}$ on $\wp(U)$ is defined by the condition:

$$(X, Y) \in \Omega_{\mathcal{H}} \text{ if and only if } Z \subseteq X \iff Z \subseteq Y \text{ for all } Z \in \mathcal{H}.$$

Note that the relation can $\Omega_{\mathcal{H}}$ can be considered as a dual form of $\Theta_{\mathcal{H}}$. The next lemma, which is obvious by Proposition 3.4.1(c), shows that the relations $\Theta_{\mathcal{H}}$ and $\Omega_{\mathcal{H}}$ are complete congruences on $(\wp(U), \cup)$ and $(\wp(U), \cap)$, respectively.

Lemma 6.2.1. *Let $\mathcal{H} \subseteq \wp(U)$.*

(a) *$\Theta_{\mathcal{H}}$ is a complete congruence on $(\wp(U), \cup)$ such that the greatest element in the $\Theta_{\mathcal{H}}$-class of any $X (\subseteq U)$ is $\bigcap\{Z \in \mathcal{H} \mid X \subseteq Z\}$.*

(b) *$\Omega_{\mathcal{H}}$ is a complete congruence on $(\wp(U), \cap)$ such that the least element in the $\Omega_{\mathcal{H}}$-class of any $X (\subseteq U)$ is $\bigcup\{Z \in \mathcal{H} \mid Z \subseteq X\}$.* □

Let Θ be a congruence on $(\wp(U), \cup)$. As in Section 3.4, we say that $\mathcal{H} (\subseteq \wp(U))$ is *Θ-dense,* if $\Theta_{\mathcal{H}} = \Theta$. Similarly, if Ω is a congruence on $(\wp(U), \cap)$, then $\mathcal{H}$ is *Ω-dense,* if $\Omega_{\mathcal{H}} = \Omega$.

If Θ is a complete congruence on $(\wp(U), \cup)$, we denote by $\mathfrak{G}(\Theta)$ the set of the greatest elements of Θ-classes. Note that $\mathfrak{G}(\Theta)$ means the same as the notation P_Θ, where $P = \wp(U)$, introduced in Section 3.2. It is clear that $\mathfrak{G}(\Theta)$ is Θ-dense and that it is a closure system. On the other hand, if Ω is a complete congruence on $(\wp(U), \cap)$, we denote by $\mathfrak{L}(\Omega)$ the set of the least elements of Ω-classes. Obviously, $\mathfrak{L}(\Omega)$ is Ω-dense and it is an interior system.

Proposition 6.2.2. *If* $R \in \mathrm{Tol}(U)$ *and* $X \subseteq U$*, then*

(a) $\approx^R$ *is a complete congruence on* $(\wp(U), \cup)$ *such that the greatest element in the* $\approx^R$*-class of any* $X(\subseteq U)$ *is* $(X^R)_R$*;*

(b) $\approx_R$ *is a complete congruence on* $(\wp(U), \cap)$ *such that the least element in the* $\approx_R$*-class of any* $X(\subseteq U)$ *is* $(X_R)^R$*;*

(c) $X \in \mathfrak{G}(\approx^R)$ *iff* $X^{\complement} \in \mathfrak{L}(\approx_R)$ *iff* $X = Y_R$ *for some* $Y \subseteq U$*;*

(d) $X \in \mathfrak{L}(\approx_R)$ *iff* $X^{\complement} \in \mathfrak{G}(\approx^R)$ *iff* $X = Y^R$ *for some* $Y \subseteq U$*;*

(e) $(\mathfrak{G}(\approx^R), \subseteq) \cong (\mathfrak{L}(\approx_R), \subseteq)$*;*

(f) $\{(x/R)^{\complement} \mid x \in U\}$ *is* $\approx^R$*-dense;*

(g) $\{x/R \mid x \in U\}$ *is* $\approx_R$*-dense.*

Proof. Assertions (a) and (b) follow directly from Proposition 3.5.4.

(c) For all $X \subseteq U$,

$$\begin{aligned} X \in \mathfrak{G}(\approx^R) \quad &\iff \quad X = (X^R)_R \iff X^{\complement} = ((X^R)_R)^{\complement} \\ &\iff \quad X^{\complement} = ((X^{\complement})_R)^R \iff X^{\complement} \in \mathfrak{L}(\approx_R). \end{aligned}$$

Furthermore, $X \in \mathfrak{G}(\approx^R)$ implies that $X = Y_R$ for $Y = X^R$. On the other hand, if $X = Y_R$ for some $Y \subseteq U$, then by Lemma 6.1.2, $(X^R)_R = ((Y_R)^R)_R = Y_R = X$, and so $X \in \mathfrak{G}(\approx^R)$. Condition (d) can be proved dually and (e) follows from Lemma 3.5.4(b).

(f) Let us denote $\mathcal{H} = \{(x/R)^{\complement} \mid x \in U\}$. Suppose $X \approx^R Y$. Then $X \subseteq (x/R)^{\complement}$ iff $x/R \cap X = \emptyset$ iff $x \notin X^R$ iff $x \notin Y^R$ iff $x/R \cap Y = \emptyset$ iff $Y \subseteq (x/R)^{\complement}$. Hence, $(X, Y) \in \Theta_{\mathcal{H}}$. Conversely, if $(X, Y) \in \Theta_{\mathcal{H}}$, then $x \in X^R$ iff $x/R \cap X \neq \emptyset$ iff $X \nsubseteq (x/R)^{\complement}$ iff $Y \nsubseteq (x/R)^{\complement}$ iff $x/R \cap Y \neq \emptyset$ iff $x \in Y^R$. Thus, $X \approx^R Y$ and we have proved that $\Theta_{\mathcal{H}} = \approx^R$.

(g) We write $\mathcal{K} = \{x/R \mid x \in U\}$. If $X \approx_R Y$, then $x/R \subseteq X$ iff $x \in X_R$ iff $x \in Y_R$ iff $x/R \subseteq Y$. Hence, $(X, Y) \in \Omega_{\mathcal{K}}$. On the other hand, if $(X, Y) \in \Omega_{\mathcal{K}}$, then $x \in X_R$ iff $x/R \subseteq X$ iff $x/R \subseteq Y$ iff $x \in Y_R$. Hence, $X \approx_R Y$ and clearly $\Omega_{\mathcal{K}} = \approx_R$. □

It is possible that different tolerances define the same lower and upper equality, as shown by the next example.

Example 6.2.3. Let $U = \{a, b, c, d\}$ and let R and S be tolerances on U such that

$$a/R = \{a, b, d\}, b/R = \{a, b, c\}, c/R = \{b, c, d\}, d/R = \{a, c, d\};$$

$$a/S = \{a, b, c\}, b/S = \{a, b, d\}, c/S = \{a, c, d\}, d/S = \{b, c, d\}.$$

The lower and upper approximations defined by these tolerances are presented in Table 7.

X	X_R	X_S	X^R	X^S
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{a\}$	$\emptyset$	$\emptyset$	$\{a,b,d\}$	$\{a,b,c\}$
$\{b\}$	$\emptyset$	$\emptyset$	$\{a,b,c\}$	$\{a,b,d\}$
$\{c\}$	$\emptyset$	$\emptyset$	$\{b,c,d\}$	$\{a,c,d\}$
$\{d\}$	$\emptyset$	$\emptyset$	$\{a,c,d\}$	$\{b,c,d\}$
$\{a,b\}$	$\emptyset$	$\emptyset$	U	U
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\{c,d\}$	$\emptyset$	$\emptyset$	U	U
$\{a,b,c\}$	$\{b\}$	$\{a\}$	U	U
$\{a,b,d\}$	$\{a\}$	$\{b\}$	U	U
$\{a,c,d\}$	$\{d\}$	$\{c\}$	U	U
$\{b,c,d\}$	$\{c\}$	$\{d\}$	U	U
U	U	U	U	U

Table 7.

Now the relations $\approx_R$ and $\approx_S$ are equal, and they have the following six congruence classes $\{\emptyset,\ \{a\},\ \ldots,\ \{c,d\}\}$, $\{\{a,b,c\}\}$, $\{\{a,b,d\}\}$, $\{\{a,c,d\}\}$, $\{\{b,c,d\}\}$, and $\{U\}$.

Similarly, the relations $\approx^R$ and $\approx^S$ are identical and they have six congruence classes $\{\emptyset\}$, $\{\{a\}\}$, $\{\{b\}\}$, $\{\{c\}\}$, $\{\{d\}\}$, and $\{\{a,b\},\ \ldots,\ U\}$. It can be easily seen that also $\equiv_R$ and $\equiv_S$ are the same. They have 11 equivalence classes.

By Proposition 6.2.2,

$$\mathfrak{G}(\approx^R) = \{X_R \mid X \subseteq U\} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, U\}$$

and

$$\mathfrak{L}(\approx_R) = \{X^R \mid X \subseteq U\} = \{\emptyset, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}, U\}.$$

The isomorphic complete lattices $(\mathfrak{G}(\approx^R), \subseteq)$ and $(\mathfrak{L}(\approx_R), \subseteq)$ are presented in Figure 9. For simplicity, we denote subsets of U, which differ from U by sequences of letters. For instance, $\{a,b,c\}$ is written as abc.

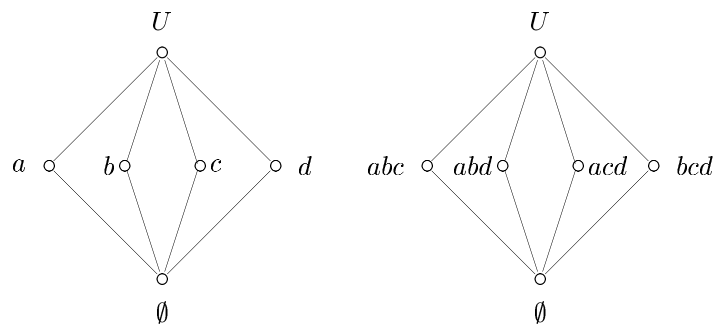


Figure 9.

We shall now turn our attention to the approximation operators of J. A. Pomykała [47, 48] mentioned above.

A family $\mathbb{E}$ of subsets of U is called a *cover* of U is $\bigcup \mathbb{E} = U$ and for all $B, C \in \mathbb{E}$, $B \subseteq C$ implies $B = C$ (see [53], for example). Let $R \in \mathrm{Tol}(U)$. A set $B \subseteq U$ is called R-*elementary* if B is a maximal set which satisfies $B \times B \subseteq R$. The set of all R-elementary sets is denoted by $\mathbb{E}(R)$.

Now the following lemma holds (cf. [47, 48, 53]).

Lemma 6.2.4. *If* $R \in \mathrm{Tol}(U)$*, then* $\mathbb{E}(R)$ *is a cover of* U *such that* $x/R = \bigcup\{B \in \mathbb{E}(R) \mid x \in B\}$ *for all* $x \in U$.

Proof. For any $x \in U$, let

$$\mathcal{H}_x = \{B \subseteq U \mid x \in B \text{ and } B \times B \subseteq R\}.$$

It is clear that $\mathcal{H}_x$ is nonempty since $\{x\} \times \{x\} \subseteq R$. Moreover, it is easy to see that the union $\bigcup \mathcal{C}$ of any nonempty chain $\mathcal{C} \subseteq \mathcal{H}_x$ is again in $\mathcal{H}_x$. This means by Zorn's Lemma that every $\mathcal{H}_x$ has a maximal element. Since it is clear that any set B maximal in $\mathcal{H}_x$ is also maximal with respect to the property $B \times B \subseteq R$, we may infer that $\bigcup \mathbb{E}(R) = U$. It is obvious that $B \subseteq C$ implies $B = C$ for all $B, C \in \mathbb{E}(R)$. Hence, $\mathbb{E}(R)$ is a cover of U.

If $y \in U$, then

$$\begin{aligned} y \in x/R &\iff \{x, y\} \times \{x, y\} \subseteq R \\ &\iff (\exists B \in \mathbb{E}(R))\ \{x, y\} \subseteq B \\ &\iff y \in \bigcup \{B \in \mathbb{E}(R) \mid x \in B\} \end{aligned}$$

□

Let $\mathbb{E}$ be a family of subsets of U such that $\bigcup \mathbb{E} = U$. Note that this property defines the notion of cover used by Pomykała [47, 48]. For all $x \in U$, we write

$$\mathbb{E}_x = \bigcup \{B \in \mathbb{E} \mid x \in B\}.$$

Let $X \subseteq U$. The operators $\overline{\mathbb{E}}_i \colon \wp(U) \to \wp(U)$ and $\underline{\mathbb{E}}_i \colon \wp(U) \to \wp(U)$, where $0 \leq i \leq 2$, are defined in [47, 48] as follows:

$$\begin{array}{rcl} \overline{\mathbb{E}}_2(X) & = & \{x \in U \mid (\forall y \in U)\ (x \in \mathbb{E}_y \Rightarrow \mathbb{E}_y \cap X \neq \emptyset)\}; \\ \underline{\mathbb{E}}_2(X) & = & \bigcup \{\mathbb{E}_x \mid \mathbb{E}_x \subseteq X\}; \\ \overline{\mathbb{E}}_1(X) & = & \bigcup \{B \in \mathbb{E} \mid B \cap X \neq \emptyset\}; \\ \underline{\mathbb{E}}_1(X) & = & \{x \in U \mid \mathbb{E}_x \subseteq X\}; \\ \overline{\mathbb{E}}_0(X) & = & \bigcup_{i \geq 0} (\overline{\mathbb{E}}_1)^i(X); \\ \underline{\mathbb{E}}_0(X) & = & \overline{\mathbb{E}}_0(X^{\complement})^{\complement}, \end{array}$$

where $(\overline{\mathbb{E}}_1)^i(X) = \overbrace{\overline{\mathbb{E}}_1 \overline{\mathbb{E}}_1 \cdots \overline{\mathbb{E}}_1}^{i-\text{times}}(X)$.

In [47, 48], $\underline{\mathbb{E}}_i(X^{\complement}) = \overline{\mathbb{E}}_i(X)^{\complement}$ was proved for $i = 1$ and $i = 2$. Also the following lemma holds.

Lemma 6.2.5. *If $\mathbb{E}$ is a family of subsets of U such that $\bigcup \mathbb{E} = U$, then for any $X \subseteq U$*

$$\underline{\mathbb{E}}_0(X) = \bigcap_{i \geq 0} (\underline{\mathbb{E}}_1)^i(X).$$

Proof. For any $X \subseteq U$,

$$\underline{\mathbb{E}}_0(X) = \overline{\mathbb{E}}_0(X^{\complement})^{\complement} = (\bigcup_{i \geq 0} (\overline{\mathbb{E}}_1)^i(X^{\complement}))^{\complement} = \bigcap_{i \geq 0} ((\overline{\mathbb{E}}_1)^i(X^{\complement}))^{\complement} = \bigcap_{i \geq 0} (\underline{\mathbb{E}}_1)^i(X).$$

□

Next we can write the following proposition which connects Pomykała's operators with our work.

Proposition 6.2.6. *Let $R \in \mathrm{Tol}(U)$. If $\mathbb{E} = \mathbb{E}(R)$ and $X \subseteq U$, then*

(a) $\overline{\mathbb{E}}_2(X) = (X^R)_R$ *and* $\underline{\mathbb{E}}_2(X) = (X_R)^R$,

(b) $\overline{\mathbb{E}}_1(X) = X^R$ *and* $\underline{\mathbb{E}}_1(X) = X_R$*,*

(c) $\overline{\mathbb{E}}_0(X) = \bigcup\{x/R^E \mid x/R^E \cap X \neq \emptyset\}$ *and* $\overline{\mathbb{E}}_0(X)$ *is the least* R*-definable set which includes* X*, and*

(d) $\underline{\mathbb{E}}_0(X) = \bigcup\{x/R^E \mid x/R^E \subseteq X\}$ *and* $\underline{\mathbb{E}}_0(X)$ *is the greatest* R*-definable set which is included in* X*.*

Proof. (a) By Proposition 6.1.1(g) and Lemma 6.2.4,

$$\underline{\mathbb{E}}_2(X) = \bigcup\{\mathbb{E}_x \mid \mathbb{E}_x \subseteq X\} = \bigcup\{x/R \mid x/R \subseteq X\} = (X_R)^R.$$

By Proposition 6.1.1(c),

$$\overline{\mathbb{E}}_2(X) = (\underline{\mathbb{E}}_2(X^{\complement}))^{\complement} = (((X^{\complement})_R)^R)^{\complement} = ((X^{\complement})_R)^{\complement})_R = (X^R)_R.$$

(b) Obviously, $\underline{\mathbb{E}}_1(X) = \{x \in U \mid \mathbb{E}_x \subseteq X\} = \{x \in U \mid x/R \subseteq X\} = X_R$ and $\overline{\mathbb{E}}_1(X) = (\underline{\mathbb{E}}_1(X^{\complement}))^{\complement} = ((X^{\complement})_R)^{\complement} = X^R$.

(c) If $y \in U$, then

$$\begin{aligned} y \in \overline{\mathbb{E}}_0(X) &\iff (\exists n \in \mathbb{N}_0)(y \in (\overline{\mathbb{E}}_1)^n(X)) \\ &\iff (\exists x \in X)(\exists n \in \mathbb{N}_0)(\exists c_0, \ldots, c_n \in U) \\ &\qquad (c_0 = x, c_n = y \text{ and } c_i R c_{i+1} \text{ for all } 0 \leq i \leq n-1) \\ &\iff (\exists x \in X)(x, y) \in R^E \\ &\iff y \in \bigcup\{x/R^E \mid x/R^E \cap X \neq \emptyset\}. \end{aligned}$$

Because $\overline{\mathbb{E}}_0(X)$ is a union of x/R^E-classes, it is R-definable by Proposition 6.1.8. Suppose that $X \subseteq Y$ for some R-definable set Y. Then $\overline{\mathbb{E}}_0(X) \subseteq \overline{\mathbb{E}}_0(Y) = Y \cup Y^R \cup (Y^R)^R \cup \cdots = Y$ because $\overline{\mathbb{E}}_0$ is obviously order-preserving.

(d) If $y \in U$, then

$$\begin{aligned} y \in \underline{\mathbb{E}}_0(X) &\iff y \notin \overline{\mathbb{E}}_0(X^{\complement}) \\ &\iff y \notin \bigcup\{x/R^E \mid x/R^E \cap X^{\complement} \neq \emptyset\} \\ &\iff y/R^E \cap X^{\complement} = \emptyset \\ &\iff y/R^E \subseteq X \\ &\iff y \in \bigcup\{x/R^E \mid x/R^E \subseteq X\}. \end{aligned}$$

The set $\underline{\mathbb{E}}_0(X)$ is R-definable because it is a union of x/R^E-classes. If $Y \subseteq X$ and Y is R-definable, then $\underline{\mathbb{E}}_0(X) \supseteq \underline{\mathbb{E}}_0(Y) = Y \cap Y_R \cap (Y_R)_R \cap \cdots = Y$ because $\underline{\mathbb{E}}_0$ is clearly order-preserving. □

By the previous proposition, $\overline{\mathbb{E}}_1(X)$ is the upper R-approximation of X, $\overline{\mathbb{E}}_2(X)$ is the greatest element in the $\approx^R$-class of X, and $\overline{\mathbb{E}}_0(X)$ is the least R-definable set including X. Similarly, $\underline{\mathbb{E}}_1(X)$ is the lower R-approximation of X, $\underline{\mathbb{E}}_2(X)$ is the least element in the $\approx_R$-class of X, and $\underline{\mathbb{E}}_0(X)$ is the greatest R-definable set included in X.

We call a binary relation Θ on $\wp(U)$ a *rough bottom equality* if there exists a tolerance $R \in \mathrm{Tol}(U)$ such that $\Theta = \approx_R$. As we already mentioned, Novotný and Pawlak [28, 29] have characterized all three types of rough equalities defined by equivalences, and Steinby [54] generalized these characterizations by omitting the assumption of finiteness.

Next we will present our proposition, which characterizes rough bottom equalities defined by tolerances.

Proposition 6.2.7. *A relation Θ on $\wp(U)$ is a rough bottom equality if and only if Θ is a complete congruence on $(\wp(U), \cap)$ and there exists a Θ-dense family $\{R_x \mid x \in U\}$ such that for all $x, y \in U$,*

(a) $x \in R_x$ *and*

(b) $y \in R_x$ *implies* $x \in R_y$.

Proof. Suppose $\Theta = \approx_R$ for some $R \in \mathrm{Tol}(U)$. Then Θ is a complete congruence on $(\wp(U), \cap)$ by Proposition 6.2.2(b). Let $R_x = x/R$ for all $x \in U$. By Proposition 6.2.2(g), $\{R_x \mid x \in U\}$ is $\approx_R$-dense. Conditions (a) and (b) hold because R is a tolerance.

Conversely, let Θ be a complete congruence on $(\wp(U), \cap)$ and assume that there exists a Θ-dense family $\{R_x \mid x \in U\}$ which satisfies (a) and (b). Let us define a binary relation R on U so that xRy if and only if $x \in R_y$. By (a), xRx for all $x \in X$. If xRy, then $x \in R_y$ which implies $y \in R_x$ and yRx by (b). Hence, R is a tolerance.

Next we show that $\mathfrak{L}(\Theta) = \mathfrak{L}(\approx_R)$. Let $X \in \mathfrak{L}(\Theta)$. Because $\{R_x \mid x \in U\}$ is Θ-dense, it is also join-dense in $\mathfrak{L}(\Theta)$ by the dual of Proposition 3.4.9. Hence, there exists an $\mathcal{H} \subseteq \{R_x \mid x \in U\}$ such that $X = \bigcup \mathcal{H}$. Since for all $x \in U$, $R_x = x/R = \{x\}^R$, we get by Proposition 6.2.2(d) that $\mathcal{H} \subseteq \mathfrak{L}(\approx_R)$. Because $\mathfrak{L}(\approx_R)$ is an interior system, $\bigcup \mathcal{H} = X$ is in $\mathfrak{L}(\approx_R)$. On the other hand, assume that $X \in \mathfrak{L}(\approx_R)$. The set $\{x/R \mid x \in U\}$ is join-dense in $\mathfrak{L}(\approx_R)$, because by Proposition 6.2.2(g) it is $\approx_R$-dense. Hence, there exists an $\mathcal{H} \subseteq \{x/R \mid x \in U\}$ such that $X = \bigcup \mathcal{H}$. For all $x \in X$, $x/R = R_x \in \mathfrak{L}(\Theta)$ by the dual of Lemma 3.4.8(a), since $\{R_x \mid x \in U\}$ is Θ-dense. The fact that $\mathfrak{L}(\Theta)$ is an interior system implies $X \in \mathfrak{L}(\Theta)$.

Now we have shown $\mathfrak{L}(\Theta) = \mathfrak{L}(\approx_R)$. This implies by the dual of Lemma 3.2.6(a) that $\Theta = \approx_R$. □

We say that a binary relation Θ on $\wp(U)$ is a *rough top equality* if there exists a tolerance $R \in \mathrm{Tol}(U)$ such that $\Theta = \approx^R$. In our following proposition we characterize rough top equalities.

Proposition 6.2.8. *A relation Θ on $\wp(U)$ is a rough top equality if and only if Θ is a complete congruence on $(\wp(U), \cup)$ and there exists a Θ-dense family $\{D_x \mid x \in U\}$ such that for all $x, y \in U$,*

(a) $x \notin D_x$ *and*

(b) $y \in D_x$ *implies* $x \in D_y$.

Proof. If $\Theta = \approx^R$ for some $R \in \mathrm{Tol}(U)$, then Θ is a complete congruence on $(\wp(U), \cup)$ by Proposition 6.2.2(a). Let us set $D_x = (x/R)^{\complement}$ for all $x \in U$. By Proposition 6.2.2(f), $\{D_x \mid x \in U\}$ is $\approx^R$-dense. Because R is a tolerance (a) and (b) hold.

On the other hand, let Θ be a complete congruence on $(\wp(U), \cup)$ and suppose that there exists a Θ-dense family $\{D_x \mid x \in U\}$ which satisfies (a) and (b). We define a binary relation R on U so that xRy if and only if $x \notin D_y$. By (a), xRx for all $x \in U$. If xRy, then $x \notin D_y$ and this implies $y \notin D_x$ and yRx by (b). Thus, R is a tolerance.

Next we show that $\mathfrak{G}(\Theta) = \mathfrak{G}(\approx^R)$. Let $X \in \mathfrak{G}(\Theta)$. Because $\{D_x \mid x \in U\}$ is Θ-dense, it is meet-dense in $\mathfrak{G}(\Theta)$ by Proposition 3.4.9. Hence, there exists an $\mathcal{H} \subseteq \{D_x \mid x \in U\}$ such that $X = \bigcap \mathcal{H}$. Since for all $x \in U$, $D_x = (x/R)^{\complement} = (\{x\}^{\complement})_R$, this implies by Proposition 6.2.2(c) that $\mathcal{H} \subseteq \mathfrak{G}(\approx^R)$. The fact that $\mathfrak{G}(\approx^R)$ is a closure system implies $X \in \mathfrak{G}(\approx^R)$. Conversely, assume that $X \in \mathfrak{G}(\approx^R)$. The set $\{(x/R)^{\complement} \mid x \in U\}$ is meet-dense in $\mathfrak{G}(\approx^R)$ since by Proposition 6.2.2(f) it is $\approx^R$-dense. Hence, there is an $\mathcal{H} \subseteq \{(x/R)^{\complement} \mid x \in U\}$ such that $X = \bigcap \mathcal{H}$. For any $x \in U$, $(x/R)^{\complement} = D_x \in \mathfrak{G}(\Theta)$ by Lemma 3.4.8 because $\{D_x \mid x \in U\}$ is Θ-dense. Since $\mathfrak{G}(\Theta)$ is a closure system, $\bigcap \mathcal{H} = X$ belongs to $\mathfrak{G}(\Theta)$. Thus, $\mathfrak{G}(\Theta) = \mathfrak{G}(\approx^R)$ which implies by Lemma 3.2.6(a) that $\Theta = \approx^R$. □

We say that a binary relation Θ on $\wp(U)$ is a *rough equality* if there exists a tolerance $R \in \mathrm{Tol}(U)$ such that $\Theta = \equiv_R$. Rough equality relations $\equiv_R$ are equivalences on $\wp(U)$, but they are not usually congruences on $(\wp(U), \cup)$ or on $(\wp(U), \cap)$. Before we characterize the rough equality relations, we introduce a notion which we shall need.

Let Θ be an equivalence on $\wp(U)$ and $\mathcal{H}, \mathcal{K} \subseteq \wp(U)$. We say that the pair $(\mathcal{H}, \mathcal{K})$ *induces* Θ if $\Theta = \Theta_{\mathcal{H}} \cap \Omega_{\mathcal{K}}$. It is possible that not all Θ-classes have smallest elements. Let us denote by $\Lambda(\Theta)$ the set of the least elements of those Θ-classes which have a least element. Similarly, we denote by $\Gamma(\Theta)$ the set of the greatest elements of those Θ-classes which have a greatest element.

Lemma 6.2.9. *Let U be a set and let $R \in \mathrm{Tol}(U)$.*
(a) $\mathfrak{G}(\approx^R) \subseteq \Gamma(\equiv_R)$ *and* $\mathfrak{L}(\approx_R) \subseteq \Lambda(\equiv_R)$.
(b) *The pair* $(\mathfrak{G}(\approx^R), \mathfrak{L}(\approx_R))$ *induces* $\equiv_R$.

Proof. (a) Let $X \in \mathfrak{G}(\approx^R)$, which is equivalent to $(X^R)_R = X$. If $X \equiv_R Y$, then $X^R = Y^R$ and $Y \subseteq (Y^R)_R = (X^R)_R = X$, which means that X is the greatest element in its $\equiv_R$-class. The other part is similar.

(b) Because $\Theta_{\mathfrak{G}(\approx^R)} = \approx^R$ and $\Omega_{\mathfrak{L}(\approx_R)} = \approx_R$, and $\equiv_R$ is the intersection of $\approx^R$ and $\approx_R$, the claim is obvious. □

For any $\mathcal{H} \subseteq \wp(U)$, we write $\mathcal{H}' = \{X^{\complement} \mid X \in \mathcal{H}\}$. Next we give a proposition characterizing the rough equalities.

Proposition 6.2.10. *An equivalence Θ on $\wp(U)$ is a rough equality if and only if there exists an interior system $\mathcal{H} \subseteq \Lambda(\Theta)$ such that the pair $(\mathcal{H}', \mathcal{H})$ induces Θ and there exists a join-dense family $\{R_x \mid x \in U\}$ in $\mathcal{H}$, such that for all $x, y \in U$,*
(a) $x \in R_x$ *and*
(b) $y \in R_x$ *implies* $x \in R_y$.

Proof. Suppose $\Theta = \equiv_R$ for some $R \in \mathrm{Tol}(U)$. Let us denote $\mathcal{H} = \mathfrak{L}(\approx_R)$. Then $\mathcal{H}$ is obviously an interior system and $\mathcal{H} \subseteq \Lambda(\Theta)$ by Lemma 6.2.9(a). By Proposition 6.2.2(c), $\mathcal{H}' = \mathfrak{G}(\approx^R)$ and this implies by Lemma 6.2.9(b) that the pair $(\mathcal{H}', \mathcal{H})$ induces Θ. If we set $R_x = x/R$ for all $x \in U$, then by Propositions 3.4.9 and 6.2.2(g) the family $\{x/R \mid x \in U\}$ is join-dense in $\mathcal{H}$. Because R is a tolerance, (a) and (b) hold.

Conversely, suppose Θ is an equivalence on $\wp(U)$ and assume that there exists an interior system $\mathcal{H} \subseteq \Lambda(\Theta)$ such that the pair $(\mathcal{H}', \mathcal{H})$ induces Θ and for every $x \in U$, there exists an $R_x \in \mathcal{H}$ which satisfies (a) and (b) and the family $\{R_x \mid x \in U\}$ is join-dense in $\mathcal{H}$. Let us define a binary relation R on U by xRy if and only if $x \in R_y$. We have shown in the proof of Proposition 6.2.7 that R is a tolerance. It suffices to show that $\mathcal{H} = \mathfrak{L}(\approx_R)$, since this implies $\mathcal{H}' = \mathfrak{G}(\approx^R)$ by Proposition 6.2.2(d), and furthermore

$$\Theta = \Theta_{\mathcal{H}'} \cap \Omega_{\mathcal{H}} = \Theta_{\mathfrak{G}(\approx^R)} \cap \Omega_{\mathfrak{L}(\approx_R)} = \approx^R \cap \approx_R = \equiv_R .$$

Let $X \in \mathcal{H}$. Because $\{R_x \mid x \in U\}$ is join-dense in $\mathcal{H}$, there exists a $\mathcal{K} \subseteq \{R_x \mid x \in U\}$ such that $X = \bigcup \mathcal{K}$. For all $x \in U$, $R_x = x/R = \{x\}^R$, which implies by Proposition 6.2.2(d) that $\mathcal{K} \subseteq \mathfrak{L}(\approx_R)$. The fact that $\mathfrak{L}(\approx_R)$ is an interior system implies that $\bigcup \mathcal{K} = X$ is in $\mathfrak{L}(\approx_R)$. On the other hand, assume $X \in \mathfrak{L}(\approx_R)$. The set $\{x/R \mid x \in U\}$ is join-dense in $\mathfrak{L}(\approx_R)$, because by Proposition 6.2.2(g) it is $\approx_R$-dense. Hence, there exists a $\mathcal{K} \subseteq \{x/R \mid x \in U\}$ such that $X = \bigcup \mathcal{K}$. For all $x \in X$, $x/R = R_x \in \mathcal{H}$ by the dual of Lemma 3.4.8(a) since $\{R_x \mid x \in U\}$ is join-dense in $\mathcal{H}$, and thus $\mathcal{K} \subseteq \mathcal{H}$. Because $\mathcal{H}$ is an interior system, $X = \bigcup \mathcal{K}$ is in $\mathcal{H}$. Hence, $\mathcal{H} = \mathfrak{L}(\approx_R)$ which completes the proof. □

6.3 Structure of Rough Sets

Here we generalize Pawlak's notion by defining rough sets in terms of tolerances. Let $R \in \mathrm{Tol}(U)$. We call the equivalence classes of $\equiv_R$ *R-rough sets*. The set of all R-rough sets is denoted by $\mathcal{R}(R)$. We usually talk simply about rough sets, if R is understood. Now we can define an order $\leq$ on $\mathcal{R}(R)$ by setting for all $\mathcal{B}, \mathcal{C} \in \mathcal{R}(R)$,

$$\mathcal{B} \leq \mathcal{C} \iff X_R \subseteq Y_R \text{ and } X^R \subseteq Y^R,$$

where $X \in \mathcal{B}$ and $Y \in \mathcal{C}$.

First we consider rough sets defined by an equivalence relation $E \in \mathrm{Eq}(U)$. J. Pomykała and J. A. Pomykała [46] have shown that there exists a uniform set of representatives of $\mathcal{R}(E)$, which forms a complete sublattice of $(\wp(U), \subseteq)$. Getting this uniform set of representatives does require the Axiom of Choice. Let $f\colon U/E \to U$ be a choice function which picks an element from each U/E-class. We denote by $Rg(f)$ the range of f.

Let us denote (cf. [13]) for any $X \subseteq U$,

$$X^f = X_E \cup (X^E \cap Rg(f)).$$

It is clear that $(X_E)^f = X_E$ and $(X^E)^f = X^E$ for all $X \subseteq U$, and also $X \equiv_E X^f$ holds. For every rough set $\mathcal{C} \in \mathcal{R}(E)$, there exists a representative $\mathcal{C}^f$ which is defined by $\mathcal{C}^f = X^f$, where X is any member of $\mathcal{C}$. Note that $\mathcal{C}^f$ does not depend on the particular $X \in \mathcal{C}$ chosen. Now we can write the following lemma.

Lemma 6.3.1. *If $E \in \mathrm{Eq}(U)$ and $f\colon U/E \to U$ is a choice function, then*

$$(\mathcal{R}(E), \leq) \cong (\{\mathcal{C}^f \mid \mathcal{C} \in \mathcal{R}(E)\}, \subseteq).$$

Proof. It is obvious that the map $\mathcal{C} \mapsto \mathcal{C}^f$ is onto $\{\mathcal{C}^f \mid \mathcal{C} \in \mathcal{R}(E)\}$. Suppose $\mathcal{B} \leq \mathcal{C}$ holds in $(\mathcal{R}(E), \leq)$, $X \in \mathcal{B}$, and $Y \in \mathcal{C}$. Then $X_E \subseteq Y_E$ and $X^E \subseteq Y^E$. This implies that

$$\mathcal{B}^f = X^f = X_E \cup (X^E \cap Rg(f)) \subseteq Y_E \cup (Y^E \cap Rg(f)) = Y^f = \mathcal{C}^f.$$

On the other hand, if $\mathcal{B}^f \subseteq \mathcal{C}^f$, then for all $X \in \mathcal{B}$ and $Y \in \mathcal{C}$,

$$X_E = (X^f)_E = (\mathcal{B}^f)_E \subseteq (\mathcal{C}^f)_E = (Y^f)_E = Y_E$$

and

$$X^E = (X^f)^E = (\mathcal{B}^f)^E \subseteq (\mathcal{C}^f)^E = (Y^f)^E = Y^E.$$

Hence, $\mathcal{B} \leq \mathcal{C}$. □

Let us consider a family $\{(X_E, X^E) \mid X \in \mathcal{H}\}$ for some $\mathcal{H} \subseteq \wp(U)$. It is not clear that $(\bigcup_{X\in\mathcal{H}} X_E, \bigcup_{X\in\mathcal{H}} X^E)$ and $(\bigcap_{X\in\mathcal{H}} X_E, \bigcap_{X\in\mathcal{H}} X^E)$ are of the form (Y_E, Y^E), where $Y \subseteq U$. In particular, it is not generally true that $\bigcup_{X\in\mathcal{H}} X_E = (\bigcup_{X\in\mathcal{H}} X)_E$ and $\bigcap_{X\in\mathcal{H}} X^E = (\bigcap_{X\in\mathcal{H}} X)^E$. Our next lemma, which appears also in [46] in a different form describes $\bigcup_{X\in\mathcal{H}} X_E$ and $\bigcap_{X\in\mathcal{H}} X^E$ in the terms of representatives.

Lemma 6.3.2. *Let $E \in \mathrm{Eq}(U)$ be an equivalence and $f\colon U/E \to U$ a choice function. For any $\mathcal{H} \subseteq \wp(U)$,*

(a) $(\bigcup\{X^f \mid X \in \mathcal{H}\})_E = \bigcup\{X_E \mid X \in \mathcal{H}\}$, *and*
(b) $(\bigcap\{X^f \mid X \in \mathcal{H}\})^E = \bigcap\{X^E \mid X \in \mathcal{H}\}$.

Proof. (a)

$$\begin{aligned}
(\bigcup_{X\in\mathcal{H}} X^f)_E &= (\bigcup\{X_E \cup (X^E \cap Rg(f)) \mid X \in \mathcal{H}\})_E \\
&= (\bigcup\{X_E \mid X \in \mathcal{H}\} \cup \bigcup\{X^E \cap Rg(f) \mid X \in \mathcal{H}\})_E \\
&= (\bigcup\{X_E \mid X \in \mathcal{H}\})_E \cup (\bigcup\{X^E \cap Rg(f) \mid X \in \mathcal{H}\})_E \\
&= \bigcup\{X_E \mid X \in \mathcal{H}\} \cup (\bigcup\{X^E \mid X \in \mathcal{H}\} \cap Rg(f))_E.
\end{aligned}$$

If $x \in (\bigcup\{X^E \mid X \in \mathcal{H}\} \cap Rg(f))_E$, then $x/E \subseteq Rg(f)$, which implies $x/E = \{x\}$ and $x \in \bigcup\{X^E \mid X \in \mathcal{H}\}$. Thus, there exists an $X \in \mathcal{H}$ such that $x \in X^E$. Because $x/E = \{x\}$, this implies $x \in X$ and $x \in X_E$. So, $x \in \bigcup\{X_E \mid X \in \mathcal{H}\}$

and hence $(\bigcup\{X^E \mid X \in \mathcal{H}\} \cap Rg(f))_E \subseteq \bigcup\{X_E \mid X \in \mathcal{H}\}$. This implies $(\bigcup\{X^f \mid X \in \mathcal{H})_E = \bigcup\{X_E \mid X \in \mathcal{H}\}$.

(b)

$$\begin{aligned}
(\bigcap_{X\in\mathcal{H}} X^f)^E &= (\bigcap\{X_E \cup (X^E \cap Rg(f)) \mid X \in \mathcal{H}\})^E \\
&= (\bigcap\{(X_E \cup X^E) \cap (X_E \cup Rg(f)) \mid X \in \mathcal{H}\})^E \\
&= (\bigcap\{X^E \mid X \in \mathcal{H}\} \cap \bigcap\{X_E \cup Rg(f) \mid X \in \mathcal{H}\})^E \\
&= (\bigcap\{X^E \mid X \in \mathcal{H}\})^E \cap (\bigcap\{X_E \cup Rg(f) \mid X \in \mathcal{H}\})^E \\
&= \bigcap\{X^E \mid X \in \mathcal{H}\} \cap (\bigcap\{X_E \mid X \in \mathcal{H}\} \cup Rg(f))^E \\
&= \bigcap\{X^E \mid X \in \mathcal{H}\} \cap U \\
&= \bigcap\{X^E \mid X \in \mathcal{H}\}.
\end{aligned}$$

□

It is mentioned in [13] without proof that $(\{\mathcal{C}^f \mid \mathcal{C} \in \mathcal{R}(E)\}, \subseteq)$ is a sublattice of $(\wp(U), \subseteq)$. Our next lemma extends this result.

Proposition 6.3.3. *If* $E \in \mathrm{Eq}(U)$ *and* $f\colon U/E \to U$ *is a choice function, then* $(\{\mathcal{C}^f \mid \mathcal{C} \in \mathcal{R}(E)\}, \subseteq)$ *is a complete sublattice of* $(\wp(U), \subseteq)$.

Proof. It is obvious that $\emptyset^f = \emptyset$ and $U^f = U$, and hence $\bigcup\emptyset$ and $\bigcap\emptyset$ are in $\{\mathcal{C}^f \mid \mathcal{C} \in \mathcal{R}(E)\}$. Let $\{X^f \mid X \in \mathcal{H}\}$ be a nonempty subset of $\{\mathcal{C}^f \mid \mathcal{C} \in \mathcal{R}(E)\}$. Then by Lemmas 6.1.2 and 6.3.2,

$$\begin{aligned}
\bigcup_{X\in\mathcal{H}} X^f &= \bigcup_{X\in\mathcal{H}} (X_E \cup (X^E \cap Rg(f)) \\
&= \bigcup_{X\in\mathcal{H}} X_E \cup \bigcup_{X\in\mathcal{H}} (X^E \cap Rg(f)) \\
&= (\bigcup_{X\in\mathcal{H}} X^f)_E \cup (\bigcup_{X\in\mathcal{H}} X^E \cap Rg(f)) \\
&= (\bigcup_{X\in\mathcal{H}} X^f)_E \cup (\bigcup_{X\in\mathcal{H}} (X^f)^E \cap Rg(f)) \\
&= (\bigcup_{X\in\mathcal{H}} X^f)_E \cup ((\bigcup_{X\in\mathcal{H}} X^f)^E \cap Rg(f)) \\
&= (\bigcup_{X\in\mathcal{H}} X^f)^f.
\end{aligned}$$

Hence, $\bigcup_{X \in \mathcal{H}} X^f \in \{\mathcal{C}^f \mid \mathcal{C} \in \mathcal{R}(E)\}$. The other part can be proved in a similar way:

$$\begin{aligned}
\bigcap_{X \in \mathcal{H}} X^f &= \bigcap_{X \in \mathcal{H}} (X_E \cup (X_E \cap Rg(f)) \\
&= \bigcap_{X \in \mathcal{H}} ((X_E \cup X^E) \cap (X_E \cup Rg(f))) \\
&= \bigcap_{X \in \mathcal{H}} X^E \cap \bigcap_{X \in \mathcal{H}} (X_E \cup Rg(f)) \\
&= \bigcap_{X \in \mathcal{H}} X^E \cap (\bigcap_{X \in \mathcal{H}} X_E \cup Rg(f)) \\
&= (\bigcap_{X \in \mathcal{H}} X^f)^E \cap (\bigcap_{X \in \mathcal{H}} (X^f)_E \cup Rg(f)) \\
&= (\bigcap_{X \in \mathcal{H}} X^f)^E \cap ((\bigcap_{X \in \mathcal{H}} X^f)_E \cup Rg(f)) \\
&= ((\bigcap_{X \in \mathcal{H}} X^f)^E \cap (\bigcap_{X \in \mathcal{H}} X^f)_E) \cup ((\bigcap_{X \in \mathcal{H}} X^f)^E \cap Rg(f)) \\
&= (\bigcap_{X \in \mathcal{H}} X^f)_E \cup ((\bigcap_{X \in \mathcal{H}} X^f)^E \cap Rg(f)) \\
&= (\bigcap_{X \in \mathcal{H}} X^f)^f.
\end{aligned}$$

□

By Lemma 6.3.1 the previous proposition has the following corollary.

Corollary 6.3.4. *If $E \in \mathrm{Eq}(U)$, then $(\mathcal{R}(E), \leq)$ can be completely embedded in $(\wp(U), \subseteq)$.*

Example 6.3.5. Let $U = \{a, b, c\}$ and let E be an equivalence on U such that $a/E = b/E = \{a, b\}$ and $c/E = \{c\}$. Let f be the choice function $U/E \to U$ which picks from each E-class its first element. Then $Rg(f) = \{a, c\}$. The sets X_E, X^E, and X^f are presented in Table 8 for all $X \subseteq U$.

X	X_E	X^E	X^f
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{a\}$	$\emptyset$	$\{a,b\}$	$\{a\}$
$\{b\}$	$\emptyset$	$\{a,b\}$	$\{a\}$
$\{c\}$	$\{c\}$	$\{c\}$	$\{c\}$
$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
$\{a,c\}$	$\{c\}$	U	$\{a,c\}$
$\{b,c\}$	$\{c\}$	U	$\{a,c\}$
U	U	U	U

Table 8.

The Hasse diagram of $(\{\mathcal{C}^f \mid \mathcal{C} \in \mathcal{R}(E)\}, \subseteq)$ is presented in Figure 10.

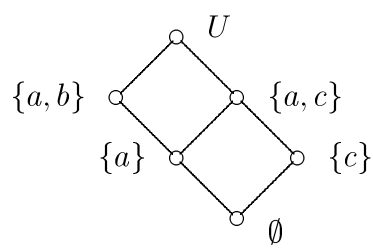


Figure 10.

By Lemma 6.3.1 $\mathcal{C}^f \mapsto \mathcal{C}^f/\equiv_E$ is an order-isomorphism between $(\{\mathcal{C}^f \mid \mathcal{C} \in \mathcal{R}(E)\}, \subseteq)$ and $(\mathcal{R}(E), \leq)$. This observation implies by Lemma 6.3.3 the following proposition, which originally appeared in [46].

Proposition 6.3.6. *If $E \in \mathrm{Eq}(U)$, then $(\mathcal{R}(E), \leq)$ is a complete lattice such that for all $\mathcal{H} \subseteq \mathcal{R}(E)$,*

$$\begin{aligned} \bigvee \mathcal{H} &= \bigcup \{\mathcal{C}^f \mid \mathcal{C} \in \mathcal{H}\}/\equiv_E; \\ \bigwedge \mathcal{H} &= \bigcap \{\mathcal{C}^f \mid \mathcal{C} \in \mathcal{H}\}/\equiv_E, \end{aligned}$$

where $f\colon U/E \to U$ is an arbitrary choice function. □

Next we consider rough sets defined by tolerances.

Example 6.3.7. Let $U = \{a, b, c\}$ and let R be a tolerance on U such that

$$a/R = \{a, b\}, b/R = \{a, b, c\}, c/R = \{b, c\}.$$

The lower and upper approximations defined by R are presented in Table 9.

X	X_R	X^R
$\emptyset$	$\emptyset$	$\emptyset$
$\{a\}$	$\emptyset$	$\{a, b\}$
$\{b\}$	$\emptyset$	U
$\{c\}$	$\emptyset$	$\{b, c\}$
$\{a, b\}$	$\{a\}$	U
$\{a, c\}$	$\emptyset$	U
$\{b, c\}$	$\{c\}$	U
U	U	U

Table 9.

The Hasse diagram of $(\mathcal{R}(R), \leq)$ is presented in Figure 11.

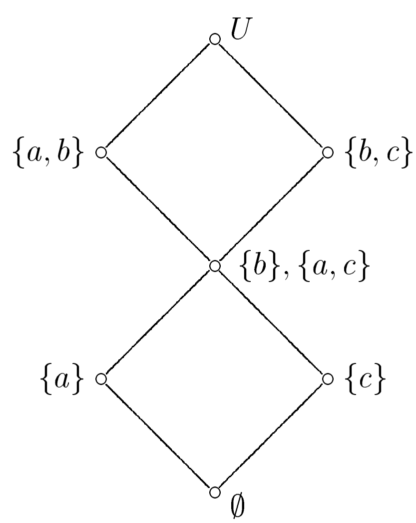


Figure 11.

The rough sets defined by tolerances differ essentially from the ones defined by equivalences. For example, it is not possible to pick a representative from the class $\{\{b\}, \{a, c\}\}$ so that the set of representatives of $\equiv_R$-classes forms a sublattice of $(\wp(U), \subseteq)$. It is also clear that $(\mathcal{R}(R), \leq)$ cannot be embedded into $(\wp(U), \cup)$.

The previous example shows that if $R(\in \mathrm{Tol}(U))$ is a tolerance, it is not necessarily possible to pick a representative for each element in $\mathcal{R}(R)$ such that this set of representatives is a sublattice of $(\wp(U), \subseteq)$. Furthermore, we do not yet know whether $(\mathcal{R}(R), \leq)$ is necessarily a semilattice.

Next we present an another approach to the structure of rough sets introduced by Iwiński [17]. It is based on a fact that if E is an equivalence, then E-rough sets can be equivalently viewed as pairs (X_E, X^E), where $X \subseteq U$, since each $\mathcal{C} \in \mathcal{R}(E)$ is uniquely determined by the pair (X_E, X^E), where X is any member of $\mathcal{C}$.

Let $R \in \mathrm{Tol}(U)$. For any $X \subseteq U$, the pair $R\langle X\rangle = (X_R, X^R)$ is called the *R-approximation* of X. The set of all R-approximations of the subsets of U is $\mathcal{A}(R) = \{R\langle X\rangle \mid X \subseteq U\}$.

There is a canonical order-relation $\leq$ on $\wp(U) \times \wp(U)$ defined by

$$(X_1, X_2) \leq (Y_1, Y_2) \text{ iff } X_1 \subseteq Y_1 \text{ and } X_2 \subseteq Y_2.$$

Because $\mathcal{A}(R) \subseteq \wp(U) \times \wp(U)$ for all $R \in \mathrm{Tol}(U)$, the set $\mathcal{A}(R)$ may be ordered by $\leq$. The next lemma is a generalization of a result presented in [46] for equivalences.

Lemma 6.3.8. *If $R \in \mathrm{Tol}(U)$, then*

$$(\mathcal{R}(R), \leq) \cong (\mathcal{A}(R), \leq)$$

Proof. Let us denote the map $X/\equiv_R \mapsto R\langle X\rangle$ by f. If $(B, C) \in \mathcal{A}(R)$, then there is an $X \subseteq U$ such that $X_R = B$ and $X^R = C$. Obviously, $f(X/\equiv_R) = (B, C)$. Thus, f is onto.

If $\mathcal{B}, \mathcal{C} \in \mathcal{R}(R)$, $X \in \mathcal{B}$, and $Y \in \mathcal{C}$, then

$$\mathcal{B} \leq \mathcal{C} \text{ in } \mathcal{R}(R) \iff X_R \subseteq Y_R \text{ and } X^R \subseteq Y^R \iff f(\mathcal{B}) \leq f(\mathcal{C}) \text{ in } \mathcal{A}(R).$$

□

It is clear that the ordered set $(\mathcal{A}(R), \leq)$ is bounded; the bottom element is $R\langle\emptyset\rangle$ and the top element is $R\langle U\rangle$.

Next we present some properties of $(\mathcal{A}(E), \leq)$, where $E \in \mathrm{Eq}(U)$, which can be found in the literature. Because $(\wp(U), \subseteq)$ is a complete lattice, $(\wp(U) \times \wp(U), \leq)$ is a complete lattice (see e.g. [5]) such that

$$\begin{aligned} \bigvee\{(X_i, Y_i) \mid i \in I\} &= (\bigcup\{X_i \mid i \in I\}, \bigcup\{Y_i \mid i \in I\}); \\ \bigwedge\{(X_i, Y_i) \mid i \in I\} &= (\bigcap\{X_i \mid i \in I\}, \bigcap\{Y_i \mid i \in I\}) \end{aligned}$$

for all $\{(X_i, Y_i) \mid i \in I\} \subseteq \wp(U) \times \wp(U)$. The following proposition, which can be found in [46], shows that $(\mathcal{A}(E), \leq)$ is a complete sublattice of $(\wp(U) \times \wp(U), \leq)$.

Proposition 6.3.9. *If $E \in \mathrm{Eq}(U)$, then $(\mathcal{A}(E), \leq)$ is a complete sublattice of $(\wp(U) \times \wp(U), \leq)$.*

Proof. Let $\{(X_E, X^E) \mid X \in \mathcal{H}\}$ be a subset of $\mathcal{A}(E)$ and let $f\colon U/E \to U$ be an arbitrary choice function. Then

$$\begin{aligned}\bigvee\{(X_E, X^E) \mid X \in \mathcal{H}\} &= (\bigcup\{X_E \mid X \in \mathcal{H}\}, \bigcup\{X^E \mid X \in \mathcal{H}\}) \\ &= (\bigcup\{X_E \mid X \in \mathcal{H}\}, \bigcup\{(X^f)^E \mid X \in \mathcal{H}\}) \\ &= ((\bigcup\{X^f \mid X \in \mathcal{H}\})_E, (\bigcup X^f \mid X \in \mathcal{H}\})^E).\end{aligned}$$

Hence, $\bigvee\{(X_E, X^E) \mid X \in \mathcal{H}\} \in \mathcal{A}(E)$. Similarly, we can show that $\bigwedge\{(X_E, X^E) \mid X \in \mathcal{H}\} \in \mathcal{A}(E)$. □

If $E \in \mathrm{Eq}(U)$ and $f\colon U/E \to U$ is a choice function, then by Lemmas 6.3.1 and 6.3.8,

$$(\{\mathcal{C}^f \mid \mathcal{C} \in \mathcal{R}(E)\}, \subseteq) \cong (\mathcal{R}(E), \leq) \cong (\mathcal{A}(E), \leq).$$

It follows from Proposition 6.3.3 that $(\mathcal{A}(E), \leq)$ can be embedded into $(\wp(U), \subseteq)$.

A lattice $\mathcal{L} = (L, \leq)$ is *distributive* if it satisfies

(D1) $\quad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

or

(D2) $\quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

for all $x, y, z \in L$; it is well-known that if a lattice satisfies one of the identities (D1) and (D2), it satisfies both of them (cf. [15], for example).

Suppose that $\mathcal{L}$ has a zero 0. An element x^* is a *pseudocomplement* of $x \in L$, if $x \wedge x^* = 0$ and for all $a \in L$, $x \wedge a = 0$ implies $a \leq x^*$. An element can have at most one pseudocomplement. A lattice is *pseudocomplemented* if each element has a pseudocomplement.

A bounded pseudocomplemented lattice $\mathcal{L}$ which satisfies the identity $x^* \vee x^{**} = 1$ is called a *Stone lattice*. It is known [46] that for any $E \in \mathrm{Eq}(U)$, $(\mathcal{A}(E), \leq)$ is a Stone lattice such that for any $X \subseteq U$, the pseudocomplement of (X_E, X^E) is $((X^E)^{\complement}, (X^E)^{\complement})$. Moreover, the lattice $(\mathcal{A}(E), \leq)$ is isomorphic to the lattice $(\mathbf{2}^I \times \mathbf{3}^J, \leq)$, where $I = \{a/E \mid |a/E| = 1\}$ and $J = \{a/E \mid |a/E| > 1\}$ (see [13]).

It is now clear that if $E \in \mathrm{Eq}(U)$, then also $(\mathcal{R}(E), \leq)$ is a Stone lattice isomorphic to $(\mathbf{2}^I \times \mathbf{3}^J, \leq)$ such that for any $X \subseteq U$, the pseudocomplement of $X/{\equiv_E}$ is $(X^E)^{\complement}/{\equiv_E}$. Similarly, if $f\colon U/E \to U$ is a choice function, then $(\{\mathcal{C}^f \mid \mathcal{C} \in \mathcal{R}(E)\}, \leq)$ is a Stone lattice isomorphic to $(\mathbf{2}^I \times \mathbf{3}^J, \leq)$, in which $\mathcal{C}^f$ has a pseudocomplement $((\mathcal{C}^f)^E)^{\complement}$.

Next we shall consider the ordered set $(\mathcal{A}(R), \leq)$, where $R \in \mathrm{Tol}(U)$.

Proposition 6.3.10. *If $R \in \mathrm{Tol}(U)$, then $(\mathcal{A}(R), \leq)$ can be embedded into $(\mathbf{2}^I \times \mathbf{3}^J, \leq)$, where $I = \{a/R \mid |a/R| = 1\}$ and $J = \{a/R \mid |a/R| > 1\}$.*

Proof. Let us define the map $\varphi\colon \mathcal{A}(R) \to \mathbf{2}^I \times \mathbf{3}^J$ by setting $\varphi((X_R, X^R)) = (f, g)$, where the maps $f\colon I \to \mathbf{2}$ and $g\colon J \to \mathbf{3}$ are defined by

$$f(x/R) = \begin{cases} 1 & \text{if } x \in X, \\ 0 & \text{if } x \notin X; \end{cases}$$

and

$$g(x/R) = \begin{cases} 2 & \text{if } x/R \subseteq X^R \text{ and } x/R \cap X_R \neq \emptyset, \\ 1 & \text{if } x/R \subseteq X^R \text{ and } x/R \cap X_R = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Let us denote $\varphi((X_R, X^R)) = (f_1, g_1)$ and $\varphi((Y_R, Y^R)) = (f_2, g_2)$. Assume that $(X_R, X^R) \leq (Y_R, Y^R)$. We show that $(f_1, g_1) \leq (f_2, g_2)$, which means that $f_1(x/R) \leq f_2(x/R)$ for all $x/R \in I$ and $g_1(y/R) \leq g_2(y/R)$ for all $y/R \in J$.

If $f_1(x/R) = 1$ for some $x/R \in I$, then $x \in X$, and since $x/R = \{x\}$, we get $x \in X_R \subseteq Y_R$. This implies $x \in Y$ and hence $f_2(x/R) = 1$. Thus, $f_1 \leq f_2$. If $g_1(y/R) = 2$ for some $y/R \in J$, then $y/R \subseteq X^R$ and $y/R \cap X_R \neq \emptyset$. This implies $y/R \subseteq Y^R$ and $y/R \cap Y_R \neq \emptyset$ and thus $g_2(y/R) = 2$. If $g_1(y/R) = 1$ for some $y/R \in J$, then $y/R \subseteq X^R$ and this implies $y/R \subseteq Y^R$. Hence, $g_2(y/R) \geq 1$ and thus $g_1 \leq g_2$. We have now shown that $(f_1, g_1) \leq (f_2, g_2)$.

Conversely, assume that $(f_1, g_1) \leq (f_2, g_2)$. We will show that $(X_R, X^R) \leq (Y_R, Y^R)$. Let us recall that by the dual of Proposition 3.4.9,

$$B = \bigcup \{x/R \mid x/R \subseteq B\} \tag{6.1}$$

for all $B \in \mathfrak{L}(\approx_R)$, and

$$C^{\complement} = \bigcup \{x/R \mid x/R \subseteq C^{\complement}\} = \bigcup \{x/R \mid x/R \cap C = \emptyset\} \tag{6.2}$$

for all $C \in \mathfrak{G}(\approx^R)$. Let $y \in X^R$. If $y/R \in I$, then $y \in X$ and $f_1(y/R) = 1$. This implies $f_2(y/R) = 1$ and thus $y \in Y \subseteq Y^R$. If $y/R \in J$, then $y \in X^R$ and $X^R \in \mathfrak{L}(\approx_R)$ imply by (6.1) that there exists a set x/R such that $y \in x/R$ and $x/R \subseteq X^R$. Hence, $g_1(x/R) \geq 1$ and $g_2(x/R) \geq 1$. This implies $y \in x/R \subseteq Y^R$. Now we have shown that $X^R \subseteq Y^R$.

Let $y \notin Y_R$. If $y/R \in I$, then $y \notin Y$ and $f_2(Y/R) = 0$. Hence, $f_1(y/R) = 0$ and $y \notin X$, which implies $y \notin X_R$. If $y/R \in J$, then $y \notin Y_R$ and $Y_R \in \mathfrak{G}(\approx^R)$ imply by (6.2) that there exists an $x \in U$ such that $y \in x/R$ and $x/R \cap Y_R = \emptyset$. This means that $g_2(x/R) \leq 1$. If $g_2(x/R) = 0$, then necessarily $g_1(x/R) = 0$ and hence $x/R \not\subseteq X^R$. This implies $x \notin X$ and thus for all $z \in X_R$, $(x, z) \notin R$. We get that $x/R \cap X_R = \emptyset$. This means that

$$y \in (X_R)^{\complement} = \bigcup\{x/R \mid x/R \cap X_R = \emptyset\}.$$

If $g_2(x/R) = 1$, then it suffices to consider the case $g_1(x/R) = 1$. This implies directly $x/R \cap X_R = \emptyset$ and hence $y \notin X_R$. Thus, also $X_R \subseteq Y_R$. $\square$

Example 6.3.11. Let us consider the tolerance R of Example 6.2.3. The Hasse diagram of $(\mathcal{A}(R), \leq)$ is given in Figure 12. For simplicity we denote the subsets of U, which differ from $\emptyset$ and U by sequences of letters. For example, $\{a, b, c\}$ is written as abc.

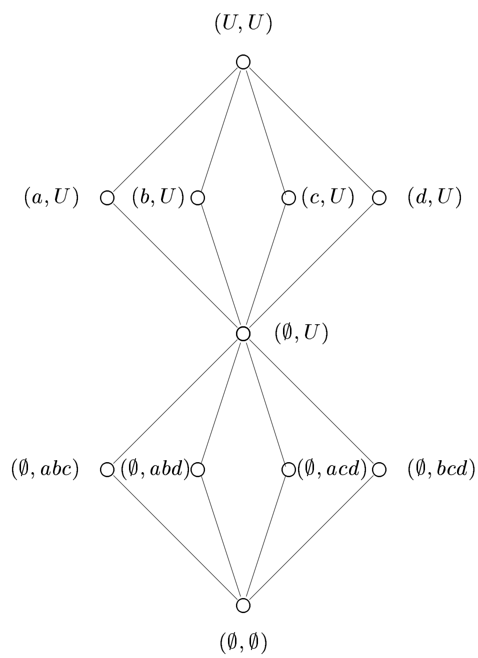


Figure 12.

Even though $(\mathcal{A}(R), \leq)$ is a lattice, it is not distributive because, for example,

$$(\emptyset, abc) \wedge ((\emptyset, abd) \vee (\emptyset, acd)) = (\emptyset, abc) \wedge (\emptyset, U) = (\emptyset, abc),$$

but

$$((\emptyset, abc) \wedge (\emptyset, abd)) \vee ((\emptyset, abc) \wedge (\emptyset, acd)) = (\emptyset, \emptyset) \vee (\emptyset, \emptyset) = (\emptyset, \emptyset).$$

In addition to this, the lattice $(\mathcal{A}(R), \leq)$ is not pseudocomplemented, since, for instance, the element $(\emptyset, abc)$ does not have a pseudocomplement.

Our next example shows that $(\mathcal{A}(R), \leq)$ is not necessarily even a semilattice.

Example 6.3.12. Let $U = \{1, 2, 3, 4, 5\}$ and let R be a tolerance on U such that

$$1/R = \{1, 2\}, 2/R = \{1, 2, 3\}, 3/R = \{2, 3, 4\}, 4/R = \{3, 4, 5\}, 5/R = \{4, 5\}.$$

The lower and upper approximations defined by R are presented in Table 10.

X	$R\langle X\rangle$	X	$R\langle X\rangle$
$\emptyset$	$(\emptyset, \emptyset)$	$\{1, 2, 3\}$	$(12, 1234)$
$\{1\}$	$(\emptyset, 12)$	$\{1, 2, 4\}$	$(1, U)$
$\{2\}$	$(\emptyset, 123)$	$\{1, 2, 5\}$	$(1, U)$
$\{3\}$	$(\emptyset, 234)$	$\{1, 3, 4\}$	$(\emptyset, U)$
$\{4\}$	$(\emptyset, 345)$	$\{1, 3, 5\}$	$(\emptyset, U)$
$\{5\}$	$(\emptyset, 45)$	$\{1, 4, 5\}$	$(5, U)$
$\{1, 2\}$	$(1, 123)$	$\{2, 3, 4\}$	$(3, U)$
$\{1, 3\}$	$(\emptyset, 1234)$	$\{2, 3, 5\}$	$(\emptyset, U)$
$\{1, 4\}$	$(\emptyset, U)$	$\{2, 4, 5\}$	$(5, U)$
$\{1, 5\}$	$(\emptyset, 1245)$	$\{3, 4, 5\}$	$(45, 2345)$
$\{2, 3\}$	$(\emptyset, 1234)$	$\{1, 2, 3, 4\}$	$(123, U)$
$\{2, 4\}$	$(\emptyset, U)$	$\{1, 2, 3, 5\}$	$(12, U)$
$\{2, 5\}$	$(\emptyset, U)$	$\{1, 2, 4, 5\}$	$(15, U)$
$\{3, 4\}$	$(\emptyset, 2345)$	$\{1, 3, 4, 5\}$	$(45, U)$
$\{3, 5\}$	$(\emptyset, 2345)$	$\{2, 3, 4, 5\}$	$(345, U)$
$\{4, 5\}$	$(5, 345)$	U	(U, U)

Table 10.

The Hasse diagram of $(\mathcal{A}(R), \leq)$ is given in Figure 13.

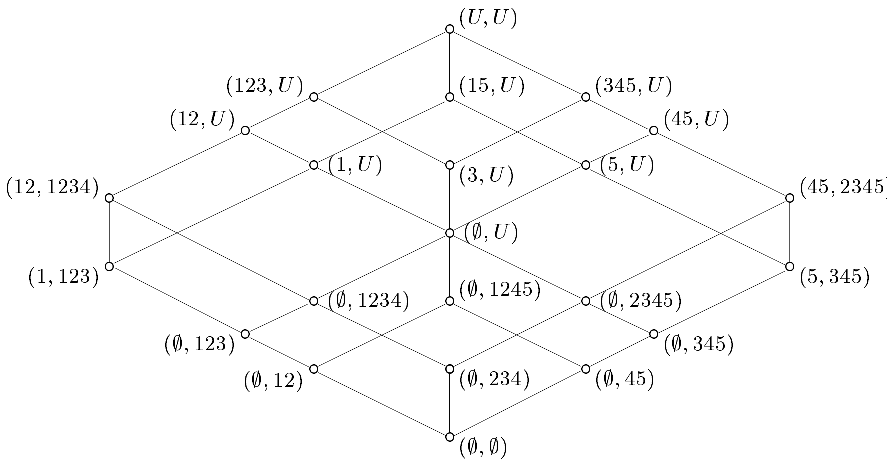


Figure 13.

Note that $(\mathcal{A}(R),\leq)$ is not a join-semilattice because, for instance, the elements $(1,123)$ and $(\emptyset,1234)$ do not have a least upper bound. Similarly, $(\mathcal{A}(R),\leq)$ is not a meet-semilattice since the elements $(12,1234)$ and $(1,U)$ do not have a greatest lower bound.

Because $(\mathcal{R}(R),\leq) \cong (\mathcal{A}(R),\leq)$ for any $R \in \mathrm{Tol}(U)$, $(\mathcal{R}(R),\leq)$ is not always a semilattice. By Example 6.3.11 it is clear that even if $(\mathcal{R}(R),\leq)$ is a lattice, it is not necessarily distributive or pseudocomplemented.

We end this thesis by noting that $(\mathcal{A}(R),\leq)$ is always a lattice, if $R \in \mathrm{Tol}(U)$ and $|U| \leq 4$.

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