TUCS

Matteo Brunelli

Some Advances in Mathematical Models for Preference Relations

Turku Centre *for* Computer Science

TUCS Dissertations
No 136, June 2011

Some advances in mathematical models for preference relations

Matteo Brunelli

To be presented, with the permission of the Department of Information Technologies of Åbo Akademi University, for public criticism in Auditorium Gamma on June 10, 2011, at 12 noon.

Åbo Akademi University
Turku Centre for Computer Science
Institute for Advanced Management Systems Research
Joukahainengatan 3-5A

2011

Supervisor

Prof. Christer Carlsson
Institute for Advanced Management Systems Research
Department of Information Technologies
Åbo Akademi University
Finland

Reviewers

Prof. János Fodor
Institute of Intelligent Engineering Systems
Óbuda University
Hungary

Prof. Enrique Herrera-Viedma
Department of Computer Science and Artificial Intelligence
University of Granada
Spain

Opponent

Prof. Janusz Kacprzyk
Systems Research Institute
Polish Academy of Sciences
Poland

ISBN 978-952-12-2596-3
ISSN 1239-1883

Abstract

Preference relations, and their modeling, have played a crucial role in both social sciences and applied mathematics. A special category of preference relations is represented by cardinal preference relations, which are nothing other than relations which can also take into account the degree of relation. Preference relations play a pivotal role in most of multi criteria decision making methods and in the operational research.

This thesis aims at showing some recent advances in their methodology. Actually, there are a number of open issues in this field and the contributions presented in this thesis can be grouped accordingly.

The first issue regards the estimation of a weight vector given a preference relation. A new and efficient algorithm for estimating the priority vector of a reciprocal relation, i.e. a special type of preference relation, is going to be presented. The same section contains the proof that twenty methods already proposed in literature lead to unsatisfactory results as they employ a conflicting constraint in their optimization model.

The second area of interest concerns consistency evaluation and it is possibly the kernel of the thesis. This thesis contains the proofs that some indices are equivalent and that therefore, some seemingly different formulae, end up leading to the very same result. Moreover, some numerical simulations are presented. The section ends with some consideration of a new method for fairly evaluating consistency.

The third matter regards incomplete relations and how to estimate missing comparisons. This section reports a numerical study of the methods already proposed in literature and analyzes their behavior in different situations.

The fourth, and last, topic, proposes a way to deal with group decision making by means of connecting preference relations with social network analysis.

Acknowledgements

I have often been told about the difficulties of being a doctoral student. I will not lie: I have not found it as hard as I imagined and, to be honest, I enjoyed it a lot. Surely, if this has been the case, then the support of a number of people and institutions must be acknowledged.

My supervisor, **Christer Carlsson** has always believed in my capacities, sometimes even more than I believed in myself. Furthermore, I acknowledge his uncommon merit of having encouraged and challenged me to think with my own head, even when my conclusions were different from his.

Michele Fedrizzi — invisible co-author of this dissertation — has been my original mentor and has patiently helped me to move my first uncertain and goofy steps in the field of mathematical models.

While during my studies, he has never officially appeared as a supervisor, **Mario Fedrizzi** has been an invaluable educator of applied mathematics, and more.

I am not only grateful to Christer, Michele and Mario for their scientific support but also for having showed endless patience when dealing with my often unpredictable and bad temper.

I am thankful to **Enrique Herrera-Viedma** and **János Fodor** for having reviewed this manuscript and **Janusz Kacprzyk** for acting as opponent. Their time and resources have been appreciated.

My parents — the authors of the author of this thesis — have not only given me the freedom to choose the best for myself, but also their support, regardless on whether or not they agreed on my decisions.

If it is true that nothing of us is original and we are the combined effort of everyone we have ever known, then, perhaps, if **you** are reading this, you should also have been included in the acknowledgments of this thesis as you might have, without even know it, contributed to it.

My doctoral studies has been financially supported by: Turku Centre for Computer Science — Institute for Advanced Management Systems Research, Åbo Akademi University — Department of Computer and Management Sciences, University of Trento — Fondazione Trentino Università — Tekes — Mom and Dad!

List of original publications

1. Brunelli M., Fedrizzi M. and Giove S. (2007). Reconstruction methods for incomplete fuzzy preference relations: a numerical comparison. Masulli F. et al. (Eds.): *WILF 2007, LNAI 4578* (pp. 86–93). Springer-Verlag, Berlin Heidelberg
2. Brunelli M. and Fedrizzi M (2007). Fair consistency evaluation in fuzzy preference relations and in AHP. Apolloni B. et al. (Eds.): *KES 2007/ WIRN 2007, Part II, LNAI 4693* (pp. 612–618). Springer-Verlag, Berlin Heidelberg
3. Fedrizzi M. and Brunelli M. (2009). Fair consistency evaluation for reciprocal relations and in group decision making, *New Mathematics and Natural Computation*, 5(**2**), 407–420
4. Fedrizzi M. and Brunelli M. (2009). On the normalisation of a priority vector associated with a reciprocal relation, *International Journal of General Systems*, 38(**5**), 579–586
5. Brunelli M. and Fedrizzi M. (2009): A fuzzy logic approach to social network analysis. *ASONAM 2009*, (pp. 225–230), IEEE Computer Society
6. Fedrizzi M. and Brunelli M. (2010): On the priority vector associated with a reciprocal relation and with a pairwise comparison matrix, *Soft Computing*, 14(**6**), 639–645
7. Brunelli M., Fedrizzi M and Fedrizzi M. (2011): OWA-based fuzzy m-ary adjacency relations in Social Network Analysis. Yager R.R., Kacprzyk J. and Beliakov G. (Eds.): *Recent Developments in the Ordered Weighted Averaging Operators: Theory and Practice*, in: Studies in Fuzziness and Soft Computing, Vol. 265, (pp. 255–267) Springer-Verlag, Berlin Heidelberg
8. Brunelli M., Critch A. and Fedrizzi M. (2011): A note on the proportionality between some consistency indices in the AHP, submitted to *Applied Mathematics and Computation*

Contents

Part I

Research summary

Chapter 1

Introduction

Decision making is one of the most natural and omnipresent activities that everybody experiences, and of which nobody can do without. Decisions can be more or less complex but they generally share some common factors. There exists a large literature on the architecture of a decision and, as it usually is the case, a scarce meeting of minds on the main definitions.

In the widest framework for multi-criteria decision making, Zelený [142] drew elegant distinctions between the concepts of objective, attribute and criterion. His conceptual framework is particularly appealing and of broad interests and so it results in being very general. However, as this thesis deals only with a part of decision theory, some concepts of the most general framework, albeit very interesting, can be simplified. This thesis is then going to refer to the simplier scheme proposed by Keeney and Raiffa [63] and Saaty [98]. That is, a decision involves a set of alternatives and a set of criteria and attributes, according to which an alternative can be judged better, worse, indifferent or sometimes also be incomparable to another one. A number of notions are generally associated with the idea of a decision. In this thesis

- The *decision maker* represents the subject in charge of the decision. A subject does not need to be a person but can be an organization. Moreover, later on, the case of decisions where a group of decision makers are present will also be treated.

- A set of *alternatives* represent the domain of the decision making activity and therefore an alternative means a possible course of action that the decision maker can take

- *Criteria* are characteristics, objectives, goals and attributes which have been judged relevant in a given situation by a particular decision maker. Criteria can then make one alternative preferable to another.

Finally, a decision making model is an algorithm which leads to a result, possibly an element from the set of alternatives.

Example 1 (Purchase decision)**.** Whenever we purchase some goods, we face a decision. Imagine that a decision maker is going to buy a car. Then, the set of alternatives is represented by the set of cars available on the market and the set of criteria is represented by the characteristics of a car, e.g. power, level of safety, maximum speed. A similar problem, which is that of buying a house, is a classic toy-example [100], used to introduce novices to decision making with hierarchies. ⧫

In multiobjective optimization theory [36], the presence of criteria is fundamental to induce a mapping from the set of alternatives to a set of tuples where each element of the tuple represents the degree to which that criterion is satisfied. The set of alternatives is generally called *decision* space and the set of tuples is called *objective* space.

It is just the case to recall an activity which often goes arm-in-arm with the process of *selection*: the so-called *screening*. Screening alternatives means constructing a set of 'most' feasible alternatives as a subset of the initial set of alternatives. This is often considered a fundamental step in order to reduce the complexity of the problem.

Going back to the discussion, it is even easier to picture the relevance of decision making activities in our daily life if we reckon the cascade effect of decisions, as often the decision at stake is the fruit of another process of decision. For instance, the decision of what car to buy surely follows the decision on whether of not to buy a car. By the same token, the purchase of the car induces more decisions. Choosing the insurance company could be a proper example in this direction.
The importance of making good decisions is crucial in some other ways which could not be understood at first sight. For instance, given a set of alternatives and a proper set of criteria, we could be able to perform predictions; thus, mathematical methods for decision making can act as alternative tools to statistics and artificial intelligence methods for forecasting.

Example 2 (Football world cup)**.** The football world cup is forthcoming and we want to guess the winner. The set of alternatives is then the set of competing teams and the set of criteria is the set of characteristics which could determine if a given team is better than another, e.g. stamina, moral, experience. At this point, it is clear that if we had a sound mathematical model and full information about the teams, then we could be able to forecast what team is most likely to win the world cup. A real application was made to predict the winner of the chess match between Spassky and Fisher [101]. ⧫

Hereafter, until the end of this introductory chapter, the general framework for this thesis is going to be presented. Two more sections will introduce decision theory and preference relations.

1.1 Decision theory

The inception of decision theory could be reasonably dated back to the seminal papers of de Condorcet [32] and de Borda [31] on voting theory. Further development have occurred over time, as, e.g. the mathematical recognition of the concept of utility by Daniel Bernoulli [11]. Nevertheless, decision theory emerged as a discipline only later on and its organic systematization started at the beginning of the twentieth century. Perhaps, the most critical contribution was the axiomatic work by von Neumann and Morgenstern [117]. However sometimes criticized, it has remained the basis for most of the further development in decision theory and the grand merit of their treatise is that, since then, decisions have far and wide been examined from a rigorous point of view. Although the work achieved so far generally goes under the name of decision theory, this does not appear as a unified subject and there exists different research traditions. These different approaches and schools have often generated dichotomies within the theory.

Examining some of them is not just a pleasant exercise but it can help in positioning this dissertation. Namely, it would answer the question where this thesis stands in the framework of decision theory.

One of the most natural dichotomies in decision theory is that between *stated* and *revealed* preferences. The difference between these two theories is that the first describes preferences which are directly expressed by a decision maker, whereas the second is interested in preferences as deduced from the behavior of the decision maker. This thesis deals with stated preferences because, with very few exceptions, it is assumed that the decision maker is capable of expressing his/her preferences on pairs of alternatives.

The need of generalization has led several scholars to pursue two very interesting directions. These directions are those of decisions under uncertainty, also called robust decisions in some specific contexts, and dynamic decisions, which are intertemporal decisions. Neither of these directions are explicitly considered in this thesis but it is definitely worth spending some few more lines on them.

The idea of decisions under *uncertainty* [63] has been studied by a large number of scholars, perhaps also in light of the plurality of measures and techniques which have been employed to capture uncertainty [65]. However, in this thesis, preference relations are tackled in such a way that, from first view, there is no uncertainty. Uncertainty will be implicitly disclosed when we shall talk about incomplete preference relations and reciprocal relations as, for instance, some authors [28] have rightly pointed out that uncertainty can be embedded in a reciprocal relation, i.e a family of preference relations which can be interpreted as probabilistic relations. Despite this, different interpretations are not going to invalidate the results exposed in this thesis and for sake of simplicity we are going to speak of imprecision only.

The concept of *intertemporal* decisions is a generalization which was essentially introduced to take into account the fact that a choice made now can virtually affect, e.g. constrain, any choice which can be made in the future. Nevertheless, this field of investigation is generally treated within the framework of microeconomics and it does not interest this dissertation.

The part of decision theory which deals with mathematical methods for subjective decisions, is commonly called *decision analysis*. Within this field of study, it is possible to see that two main schools have emerged, as highlighted in a recent survey on future directions for multi-criteria decision making research [118]. The *American school* is well represented by the AHP [95] and MAUT [63] whereas the *European school*, sometimes improperly called the French school, has in PROMETHEE [13], ELECTRE [92] and the many outranking methods, its foremost models. Preference relations tackled in this thesis are particularly useful for the AHP and therefore this thesis is connected and may be interesting for the American school rather than for the European.

Another dichotomy in research in decision theory is made between the normative and the descriptive approach. A *normative* decision theory is a theory about how decisions should be made. Conversely, a *descriptive* theory is more concerned about how decisions are actually made. The normative approach tends to be more elegant than the descriptive one, but this greater elegance is often achieved at the price of the further assumptions which can make the model quite unsuitable for practical approaches. Hence, there is an important and fuzzy trade-off between elegance and pragmatism. As it will be clear at the end of chapter 3, this thesis introduces concepts of normative decision theory in the sense that it describes how decisions with preference relations would be made in a world where there is perfect information and everybody is rational. On the other hand, still in the same chapter, some relaxations will be applied to the normative framework as the whole thesis takes into account more realistic cases as, e.g., incomplete preferences and irrationality.

1.2 Preference relations

This thesis is going to explore some advances in the methodology of preference relations. In classic literature on social choice and voting systems, preference relations are identified as ordering relations on some reference set [81]. Ordering relations are good at defining a lattice on the set of alternatives but they do not provide information regarding the degree of preference. That is, they provide a ranking of alternatives on an ordinal scale [111]. More sophisticated and information demanding preference relations have then been introduced to take into account degrees of preference. Their importance can be explained by means of an example.
Suppose that there are three decision makers and they have to choose one alternative out of two. Suppose that two out of three decision makers are almost indifferent between the two alternatives but, at the end of the day, they slightly prefer the first alternative over the second. Conversely, the remaining decision maker strongly prefers the second alternative over the first. Certainly, if it was a matter of votes, the first alternative would be chosen but, for sake of fairness, the best decision should take into account how strongly decision makers prefer one alternative over another [30]. As it will be presented, starting from chapter 3, preference relations expressing these degrees of preference are usually called *cardinal*.

Cardinal preference relations are used in several methods for decision making, as, for instance, the Analytic Hierarchy Process (AHP) [94, 95, 96, 98] and the Measuring Attractiveness by a Categorical Based Evaluation Technique (MACBETH) [8].

Preference relations are often used to compare intangible alternatives and criteria. Especially for the criteria, as they are very often intangibles, one can see that preference relations can be used for a very wide range of purposes as, for instance, the determination of the weights of criteria in potentially every multi-criteria optimization model [142].

So, to sum up, among the family of multi criteria decision making models, those involving preference relations are particularly appealing to be used when the number of alternatives and criteria is finite and when alternatives and/or criteria are intangibles.

Preference relations are mathematical objects collecting opinions of an expert expressed as pairwise comparisons between alternatives [48]. Their scope is that of representing these pairwise preferences in such a way that they can be analyzed and a rating of the alternatives can be derived. This rating of alternatives will, hereafter, be exchangeably called *priority vector* or *weight vector*.

Example 3 (Naïve example)**.** Imagine that Mr. Smith got three alternative months, June, July, and August for going on vacation and that he is asked to select one. For sake or simplicity and coherence with standard notation, let us assume that the symbol $\succ$ means 'is preferred to'. Then, the following statements

$$\text{June} \succ \text{July}, \quad \text{June} \succ \text{August}, \quad \text{August} \succ \text{July} \tag{1.1}$$

can be representative of his preference relation. With some imagination, we could also guess a way to construct a weight vector, which is nothing else but a coherent assignment of some values to alternatives. For instance we can assume that the higher the assigned number, the more preferable the alternative. Equivalently, each alternative can be associated with the number of times that it beats the other competing alternatives. This said, the assignment June $= 2$, July $= 0$, August $= 1$ seems to be coherent with the preferences of Mr. Smith (1.1). ♦

In hierarchically structured problems, weight vectors would be then algebraically combined in order to scale the hierarchy, but this is another story, which does not concern this thesis. Instead, it is definitely worth dwelling on the *raison d'être* of preference relations and justifying their relevance. In other words, could not we just make it without preference relations, perhaps estimating the weight vector directly, by assigning scores to alternatives? This can clearly be done but it loses significance when the number of alternatives is large enough. This was first conceived by Thurstone [114] in his pioneering work on pairwise comparisons and then widely recognized by Miller [77] when claiming that an individual cannot simultaneously compare more than 7 ± 2 objects without being confused. Comparing alternatives two at a time, which is what preference relations do, overcomes this limitation and helps to make better decisions. There would actually be another, and more hidden, reason for using the methodology of pairwise comparisons and it is connected with consistency evaluation. Namely, it is impossible to estimate the inconsistency of a decision maker on the basis of the priority vector that he/she associates to the set of alternatives. Conversely, on the basis of his/her preference relations, estimating consistency is possible, thanks to some conditions of transitivity, which are going to be explored and analyzed later on.

1.3 Research problems

To my eyes, research in the topic of cardinal preference relations has reached a very high level of sophistication and it is safe to say that many among the developed mathematical models are too complex to be implemented in practice. However, it is also true that a number of problems are still open. Seemingly, most of the research has been oriented at building new models *tout court*, but, in this maze, not enough has been done in order to validate and compare these very same models and, if possible, build new ones showing that they outperform their ancestors. This thesis is concerned with three major points in the theory of preference relations

- How can one derive a priority vector from a preference relation? What properties shall this vector satisfy?
- How can we verify if the decision maker is not contradictory when he expresses his opinions? If he is contradictory, how can we measure the intensity of this phenomena.
- Supposing that the decision maker is not able to state his preferences over all the pairs of alternatives, how can we deal with this lack of information?

Research in the computational methodology of preference relations is foremost and research in the above mentioned research themes is wide. However, sometimes it can happen that some big pieces of the puzzle were left, undiscovered, along the road. If this thesis were taken as a contribution in going back to look to these, hopefully precious, missing pieces, then I would be flattered and my goal be achieved.

1.4 Structure of this dissertation

This thesis is organized as a structured exposition of some co-authored contributions to the computational methodology of preference relations. As explained, the research has concerned a number of topics which can be better presented disjointly. Consequently, sometimes the understanding of a given section does not require the understanding of the previous ones. Dependencies, in terms of sections, are graphically exposed in figure 1.1, where chapters are naturally indicated by numbers from one to eight.

This introduction is represented by chapter 1, followed by chapter 2, devoted to the exposition of the methodology. Chapter 3, which really is the 'base camp' of this dissertation, presents some fundamentals on preference relations, their consistency and priority vectors. Besides its explanatory role, chapter 3 ends up with a section which outlines this thesis more extensively, with the aid of more technical notions and referring to some co-authored papers. Chapter 4 deals with the issues of priority vectors for a special class of preference relations. Chapter 5, which possibly is the most central and rich in this thesis, presents some theoretical and numerical advances in the

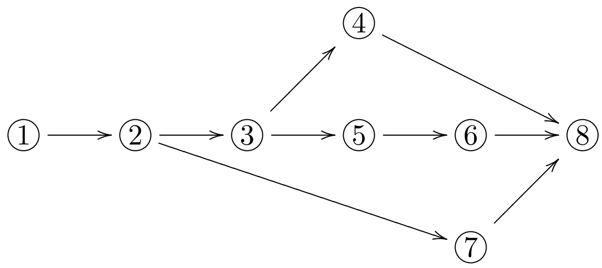


Figure 1.1: Structure of the thesis

estimation of inconsistency. Chapter 6 is, instead, comparatively short and briefly describes and compares some methods to fill incomplete preference relations. Unlike chapters 3–5, in order to be more accessible, chapter 7 does not assume any knowledge of preference relations as it treats the natural connection between social network analysis and group decision making in an independent framework. Finally, chapter 8 contains some conclusions and suggestions for future works. Two appendices, which are not part of the diagram, integrate the thesis by providing some useful material.

Chapter 2

Methodology

During my research I have always clung to the idea that a mathematical model is a formal description of a system. Decision making is a system and, as such, it can be described in mathematical terms. In doing so, not only we formalize the model but we generalize it, and we also tap the full power or mathematics.

The principles guiding this thesis, and the related contributions, are strictly connected by the present needs of the research field which, in this case, I consider being *Operational Research* in its broadest sense. The aim of Operational Research is often that of helping to make the best decision or take the best course of action. Thus, although based on subjective information, the study of preference relations should be driven by the principles of Operational Research, as much as possible. As Operational Research is considered a formal science, the methodology of this thesis is inspired by the research techniques applied for formal sciences.

This thesis deals with the mathematics of a very small part of Operational Research. Nevertheless, this does not mean that the mathematical content of this thesis is supposed to be complicated but just that non-mathematical aspects of the topic will tendentially be ignored. Such a strategy is necessary to keep the thesis focused and not to dissipate energy; however, there is absolutely no intention to categorize non-mathematical aspects of Operational Research as secondary, in terms of importance [37].

2.1 Operational Research

Operational Research is an interdisciplinary branch of mathematics, engineering and economics which aims at solving problems by means of analytic methods, whenever possible. Several definitions of Operational Research has been given; a very broad one is 'the science concerned with the improvement of systems, organizations and institutions which uses whatever art, science and humanities are necessary to do so' [64]. Some other definitions were collected by Saaty [93] and some curious and thought-provoking ones can be found: Operational Research was defined as 'quantitative common sense' or, perhaps in the intent of underlining its limitations, as 'the art of giving bad answers to problems to which otherwise worse answers are given'.

Historically, Operational Research has always privileged the study and the application of classical methods, leading to the optimal solution without compromises. However, lately, perhaps discouraged by the complexity of emerging problem, some so-called soft computing methods have gained wide acceptance. By soft computing methods we mean all the methods, e.g. heuristics and meta heuristics, which lead to a suboptimal solution in exchange of a smaller computational effort. Another direction which has emerged in Operational Research, which should not be confused with the soft computing methods, is that of 'soft' Operational Research which includes all those techniques which can be used to solve problems which cannot be structured mathematically [78].

In the most recent years, the range of Operational Research, and the influence of its methodology has reached and contaminated a number of other disciplines. The reason for this expansion is probably caused by a generalized trend of making things scientifically, whenever possible. In fact, one of the most successful disciplines is probably *management science*, which can be roughly defined as the effort of applying scientific reasoning and mathematical models in the field of management [1].

Clearly, as it often happens in modern sciences, boundaries of the previously mentioned disciplines are vague and nowadays, it is quite difficult to state whether a given problem is in the domain of Operational Research. Indeed, it would be hard to draw a diagram to show relations between subjects related with Operational Research.

In spite of these difficulties in sharply defining modern Operational Research, its activity is conventionally divided into several phases [60]:

1. Examination of the real world situation and collection of information;
2. Formulation of the problem, choice of the variables and of the function to be optimized;
3. Construction of the mathematical model, which is a good representation of the problem; it ought to be easy to be used, representative of the problem, using, and returning, all the necessary information to solve the problem;

[1] For the sake of truth, sometimes, Operational Research and management science are treated as a unique subject, without distinctions.

4. Solution of the model keeping in mind that there may be several different ways to obtain it;

5. Analysis and validation of the obtained solutions and of the theoretical function and representativity of the model;

6. Implementation of the solution.

Within this framework, the research reported in this thesis aims to find new models and to analyze and possibly enhance some already proposed ones. This means that the points of the activities of Operational Research involved in this thesis are 3–5. Moreover, if there is a method which uses preference relations as they are described in this dissertation more than any other, then this is surely the AHP. Describing such a method would improve the understanding of any thesis so closely related with it, but this would be far beyond the scope of this work. As a compromise, appendix B offers a very short, yet possibly self-contained, description of the AHP as a method for hierarchically combining preferences.

2.2 Positivist research

Operational research has been far and wide influenced by the positivist methodology. Contrary to what I originally and ingenuously believed, the term positivism does not stem from the word positive but from the Latin word *positum*, roughly translatable with the English words posed and placed. Positivist research tries, as much as possible to distance the research from the researcher and the values of that particular time, thus creating an ideal of objectivity and independence around the results. In this sense positivist research should be based on concrete and unarguable facts only. This goal can be obtained by attaining a number of prescriptions:

- Scientific research derives knowledge only from the tools of experimental sciences and not from intuition. The implication is that only scientific statements are valid whereas all other affirmations are simply legitimate, however not scientific. A statement is scientific if it refuses all the hypotheses which cannot be verified

- Positivist researchers should trust reason and science

- The positivist approach tends to extend the scientific method to fields which formerly used to be the exclusive domain of moral and metaphysics

- The results of research should be, as far as possible, based on logical or empirical observations. Therefore, research should be mostly deductive rather than inductive.
- Progress is supported by the expansion of the scientific knowledge

It follows that positive research is a very robust methodology and results are seldom contestable. Alas, this desirable feature is obtained at the price of the assumptions which are made in order to derive the results. Such often very elegant assumptions are simplifications of the real-world and for this reason they are not as incontestable as the results built on them.

2.3 Working methodology

The findings of this thesis can very often be classified as based, as much as possible, on mathematical evidence. Consequently, also the working methodology behind this thesis has been inspired by positive methodology. In this direction, chapter 4 and section 5.2 can be taken as a representative examples, since they contain propositions which are validated by mathematical proofs. Unfortunately, it is not always the case that everything can be proved and therefore, whenever this supremely elegant way cannot be taken, statements can be rooted on observation of very large samples and thus, we say that they are statistically based. This procedure can be observed in section 5.3 and the whole of chapter 6.

Chapter 3

Preference Relations

> Beauty started when people began to choose.
>
> Roberto Benigni

As this thesis deals quite extensively with the concept of preference relation, this chapter introduces the main definitions for a number of preference relations and related concepts such as consistency and priority vector. Nevertheless, these latter fundamental concepts are only introduced here, as their importance is going to be exposed and dwelt on in the next chapters. At the very end of this chapter, the rest of the thesis will be analytically summarized and the relevance of the contributions synthesized.

3.1 Preference relations

The kernel of this thesis is the idea of preference relation. Although only two types of preference relations are going to be examined in the rest of the thesis, it is always good to have a view on the entire forest in order to better understand single trees. Here is an operative definition of 'forest'.

Definition 1 (Preference relation [4])**.** A preference relation P on the set $X = \{x_1, \dots, x_n\}$ is characterized by a function $\mu_P : X \times X \to S$, where S is the domain of representation of preference degrees provided by the decision maker for each pair of alternatives.

It is the case to note that, to my best knowledge, in literature it is generally assumed that S is a lattice.

Apparently this definition is sufficiently ambiguous but, as suggested before, it comprehends a lot of specific kinds of preference relation.

It is the case to notice that, equivalently, a preference relation can be seen as a set of pairs

$$P = \{((x_i, x_j), \mu_P(x_i, x_j)) \,|(x_i, x_j) \in X \times X, \mu_P(x_i, x_j) \in S \; \forall i, j\}$$

As the cardinality of set X is likely to be a reasonably small natural number, then a preference relation can conveniently be represented by means of a matrix [1] whose entry on the i-th row and j-th column is an element of S which represents the degree of preference of alternative x_i over x_j, i.e.. $\mu_P(x_i, x_j)$. This representation is especially convenient as it allows an intuitive and compact representation and ease of algebraic manipulation. Indeed, once we have a priori defined X and S, then a preference relation can be univocally associated with its representing matrix. Hence, thanks to this fact, in the following, no difference will be made between the relational and the matrix form of a preference relation.

Example 4 (Linguistic preference relation)**.** A *linguistic* preference relation L is a relation associated with $\mu_L : X \times X \to S$ with S being a set of linguistic labels, for instance {far worse, worse, equivalent, better, much better}. Then, given a set of three alternatives, L can be represented by means of the following matrix

$$L = \begin{pmatrix} \text{equivalent} & \text{better} & \text{much better} \\ \text{worse} & \text{equivalent} & \text{much worse} \\ \text{worse} & \text{worse} & \text{equivalent} \end{pmatrix} . \;\blacklozenge$$

Example 5 (Partial order)**.** A partial order (poset) is defined as a reflexive, antisymmetric and transitive relation $\succeq$, associated with $\mu_{\succeq} : X \times X \to \{0, 1\}$. A partial order can be represented by means of a matrix $\succeq \in \{0, 1\}^{n \times n}$. An example could be

$$\succeq = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

from which we can easily derive that $x_3 \succeq x_1 \succeq x_2$. ♦

Hereafter we are going to focus on two other types of preference relations called reciprocal relations and pairwise comparison matrices. These two types of preference relations have proved to be the most popular cardinal preference relations. By *cardinal preference relations* we indicate the family of preference relations where the domain or representation of the preferences,

[1] At this preliminary stage of the thesis, a matrix is just an array and it does not need to be defined on a field.

i.e. set S, allows the decision maker to express the intensity to which he/she prefers an alternative over another. Taking into account examples 4 and 5 it can be seen that in the first case, linguistic terms represent degrees of preference, whereas in the second case, the partial order does not allow any such distinction.

3.2 Reciprocal Relations

One of the most used types of preference relations is based on the idea of *reciprocal relation*, often called fuzzy preference relation in literature. Nevertheless, in order to disambiguate between these two terms we prefer to adopt the terminology reciprocal relations following the approach proposed by De Baets et al. [29]. Thus, we consider fuzzy preference relations [49, 83, 141] as a different kind of preference relations whose particularity is that of using an unipolar scale [34]. Seemingly, and in general, bipolar representations of preferences have gained wider acceptance than unipolar, perhaps because preferences, unlike some other relations, e.g. similarity, seem to be naturally bipolar.

Definition 2 (Reciprocal relation)**.** A reciprocal relation R on the set X is characterized by a function $\mu_R : X \times X \to [0,1]$, such that $\mu_R(x_i, x_j) = 0.5\ \forall i = j$ and $\mu_R(x_i, x_j) + \mu_R(x_j, x_i) = 1\ \forall i, j$.

Furthermore, the unit interval scale has the following associated semantic

$$\mu_R(x_i, x_j) = \begin{cases} 1, & \text{if } x_i \text{ is definitely preferred over } x_j \\ \alpha \in]0.5, 1[, & \text{if } x_i \text{ is preferred over } x_j \\ 0.5, & \text{if there is indifference between } x_i \text{ and } x_j \\ \beta \in]0, 0.5[, & \text{if } x_j \text{ is preferred over } x_i \\ 0, & \text{if } x_j \text{ is definitely preferred over } x_i. \end{cases}$$

From now on, for sake of convenience, by means of $r_{ij} := \mu_R(x_i, x_j)$, in the following we will not distinguish between a reciprocal relation R and the matrix $\mathbf{R} = (r_{ij}) \in [0,1]^{n \times n}$.
Some consistency conditions are presented in the following, mostly under the form of transitivity conditions. The importance of consistency, a condition which generally entails the rationality of the decision maker, will be examined later on and for the moment it is enough to go through the key concepts.

Definition 3 (Additively consistent reciprocal relations [112])**.** A reciprocal relation $\mathbf{R} = (r_{ij})_{n \times n}$ is *additively* consistent if the following condition holds

$$(r_{ik} - 0.5) = (r_{ij} - 0.5) + (r_{jk} - 0.5) \quad \forall i, j, k \tag{3.1}$$

We call $\mathcal{R}^+$ the set of all the additively consistent reciprocal relations.

Furthermore, consistency condition (3.1) guarantees the existence of a priority vector which represents the rating of the alternatives.

Proposition 1 ([112])**.** *If and only if* $\mathbf{R}$ *is additively consistent, i.e.* $\mathbf{R} \in \mathcal{R}^+$*, then there exists a non-negative vector* $\mathbf{u} = (u_1, \ldots, u_n)$ *such that*

$$r_{ij} = 0.5 + 0.5(u_i - u_j) \quad \forall i, j \tag{3.2}$$

and $|u_i - u_j| \leq 1 \ \forall i, j$*.*

Additive consistency is not the only type of consistency which can be applied to reciprocal relations. Multiplicative consistency is an alternative, but it can be presented in a similar fashion.

Definition 4 (Multiplicatively consistent reciprocal relations [104])**.** A reciprocal relation is multiplicatively consistent if the following condition holds

$$\frac{r_{ik}}{r_{ki}} = \frac{r_{ij}}{r_{ji}}\frac{r_{jk}}{r_{kj}} \quad \forall i, j, k. \tag{3.3}$$

We call $\mathcal{R}^\times$ the set of all the multiplicatively consistent reciprocal relations.

Proposition 2 ([112])**.** *If and only if* $\mathbf{R}$ *is multiplicatively consistent, i.e.* $\mathbf{R} \in \mathcal{R}^\times$*, then there exists a vector* $\mathbf{v} = (v_1, \ldots, v_n)$ *such that*

$$r_{ij} = \frac{v_i}{v_i + v_j} \quad \forall i, j \tag{3.4}$$

Once again, an example is proposed in order to clarify what has been stated so far.

Example 6. Let us consider the following example of reciprocal relation

$$\mathbf{R} = \begin{pmatrix} 0.5 & 0.55 & 0.65 & 0.85 \\ 0.45 & 0.5 & 0.6 & 0.8 \\ 0.35 & 0.4 & 0.5 & 0.7 \\ 0.15 & 0.2 & 0.3 & 0.5 \end{pmatrix}. \tag{3.5}$$

It can be checked that $\mathbf{R} \in \mathcal{R}^+$ because (3.1) holds. Its associated priority vector is $\mathbf{u} = (1.275, 1.175, 0.975, 0.575)$. At the same time one can verify that $\mathbf{R} \notin \mathcal{R}^\times$ and that consequently a vector $\mathbf{v}$ satisfying (3.4) does not exist. ⧫

3.3 Pairwise comparison matrices

The inception of pairwise comparison matrices, often also called multiplicative preference relations, is often dated back to the pioneering work of Saaty [95]. Nevertheless, and quite curiously, they had already been employed by Saaty himself [94] as a tool for estimating the membership function of finite sets.

Definition 5 (Pairwise comparison matrix [95])**.** A pairwise comparison matrix is a matrix $\mathbf{A} = (a_{ij})_{n\times n} \in \mathbb{R}_{>}^{n\times n}$ such that $a_{ij} = 1\ \forall i = j$ and $a_{ij}a_{ji} = 1\ \forall i, j$.

Using notational conventions as previously done for reciprocal relations, a pairwise comparison matrix, is then a representation of a preference relation such that $\mu_A(x_i, x_j) \in \mathbb{R}_{>}\ \forall i, j$. Some consistency condition can be posed for pairwise comparison matrices too.

Definition 6 (Consistent pairwise comparison matrices [95])**.** A pairwise comparison matrix is consistent if the following condition holds

$$a_{ik} = a_{ij}a_{jk} \quad \forall i, j, k. \tag{3.6}$$

We call $\mathcal{A}^*$ the class of all the consistent pairwise comparison matrices.

Proposition 3 ([95])**.** *If and only if* $\mathbf{A}$ *is consistent, i.e.* $\mathbf{A} \in \mathcal{A}^*$*, then there exists a vector* $\mathbf{w} = (w_1, \ldots, w_n)$ *such that*

$$a_{ij} = \frac{w_i}{w_j} \quad \forall i, j. \tag{3.7}$$

Let's also incidentally note that, if $\mathbf{A} \in \mathcal{A}^*$, then the vector $\mathbf{w}$ can be conveniently derived using the geometric mean method

$$w_i = \left(\prod_{j=1}^{n} a_{ij}\right)^{\frac{1}{n}} \quad \forall i. \tag{3.8}$$

It is the case to notice that, originally, Saaty based his studies on the work of Miller [77] who had stated that the human brain cannot handle more than 7 ± 2 alternatives at a time, and therefore he proposed to restrict the scale of admissible values to the integer numbers between 1 and 9 and their reciprocals. Besides respecting this approach, Saaty proposed to pairwise compare alternatives according to the semantic scale reported in table 3.1.

Example 7. Let us consider the following example of pairwise comparison matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 4 & 1/3 \\ 1/2 & 1 & 2 & 1 \\ 1/4 & 1/2 & 1 & 1/2 \\ 3 & 1 & 2 & 1 \end{pmatrix}. \tag{3.9}$$

where, if we stick to Saaty's intepretations, for instance, a_{41} states that x_4 is moderately more important (moderately preferred) than x_1. It can be checked that $\mathbf{A} \notin \mathcal{A}^*$ and that consequently $\nexists \mathbf{w} = (w_1, \ldots, w_n)$ such that (3.7) holds. If, instead, $a_{ij} = \frac{1}{a_{ji}} = 2$, then $\mathbf{A} \in \mathcal{A}^*$ and $\mathbf{w} = \left(\frac{4}{9}, \frac{2}{9}, \frac{1}{9}, \frac{2}{9}\right)$. ♦

Value	Degree	Description
1	Equal importance	Two activities contribute equally to the objective
3	Moderate importance	Experience and judgment slightly favor one activity over another
5	Strong importance	Experience and judgment strongly favor one activity over another
7	Very strong or demonstrated importance	An activity is strongly favored and its dominance demonstrated in practice
9	Extreme importance	The evidence favoring one activity over another is of the highest possible order of affirmation
2,4,6,8	Intermediate values	To reflect the compromise between two adjacent judgments
reciprocals		Used to fill the entries and to make the matrix reciprocal and non–negative

Table 3.1: Saaty's interpretation of the scale 1–9

3.4 Transformations

Reciprocal relations and pairwise comparison matrices are interchangeable tools for decision making. This is made evident by the similar structure based on reciprocity [88] and formalized by means of some consistency preserving functions which allow the switch from a framework to another one.

In this work we will consider two functions, f and g. First we can see the following transformation between pairwise comparison matrices and additive consistent reciprocal relations, $f : \left[\frac{1}{9}, 9\right] \to [0, 1]$.

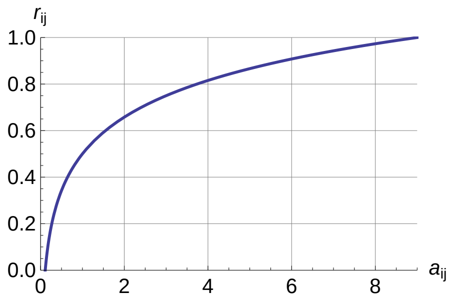
$$r_{ij} = f(a_{ij}) = \frac{1}{2}(1 + \log_9 a_{ij}), \tag{3.10}$$

and its inverse

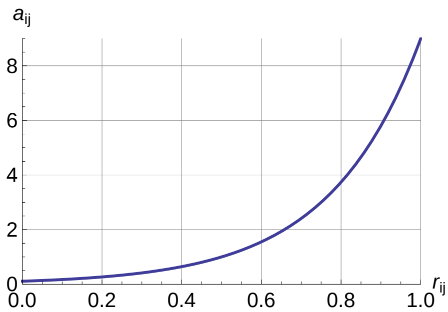
$$a_{ij} = f^{-1}(r_{ij}) = 9^{2r_{ij}-1}. \tag{3.11}$$

Function f is consistency preserving in the sense that, if it is applied to all the entries of a consistent pairwise comparison matrix, then it yields to an additively consistent reciprocal relation. Its inverse, f^{-1} is also consistency preserving, but in the other way round [43]. Both mappings f and f^{-1} are depicted in figure 3.1.

Conversely, transformation $g : \mathbb{R}_> \to]0, 1[$ is consistency preserving if $\mathbf{A} \in \mathcal{A}^*$ and we want to pass from the pairwise comparison matrices based approach to that based on multiplicatively consistent reciprocal relations.



(a) $r_{ij} = f(a_{ij}) = \frac{1}{2}(1 + \log_9 a_{ij})$



(b) $a_{ij} = f^{-1}(r_{ij}) = 9^{2(r_{ij}-0.5)}$

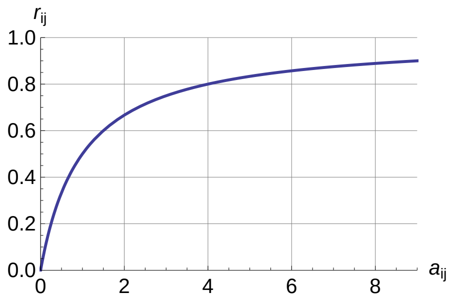
Figure 3.1: Mappings f and f^{-1}

$$r_{ij} = g(a_{ij}) = \frac{a_{ij}}{1 + a_{ij}}. \tag{3.12}$$

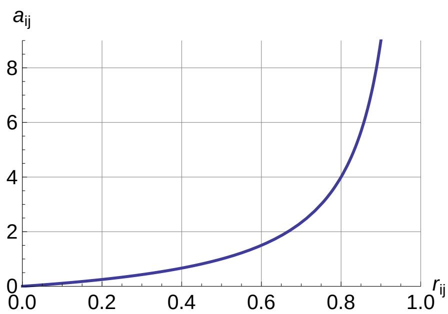
Its inverse is

$$a_{ij} = g^{-1}(r_{ij}) = \frac{r_{ij}}{1 - r_{ij}} \tag{3.13}$$

and both g and g^{-1} are represented in figure 3.2



(a) $r_{ij} = g(a_{ij}) = \frac{a_{ij}}{1+a_{ij}}$



(b) $a_{ij} = g^{-1}(r_{ij}) = \frac{r_{ij}}{1-r_{ij}}$

Figure 3.2: Mappings g and g^{-1}

Example 8. Consider the following, very easy, pairwise comparison matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 9 \\ 1/3 & 1 & 3 \\ 1/9 & 1/3 & 1 \end{pmatrix} \in \mathcal{A}^*.$$

Applying function f results in the following additively consistent reciprocal relation

$$\mathbf{R} = \begin{pmatrix} 0.5 & 0.75 & 1 \\ 0.25 & 0.5 & 0.75 \\ 0 & 0.25 & 0.5 \end{pmatrix} \in \mathcal{R}^{+}.$$

Similarly, if we used g instead of f, then we would have obtained

$$\mathbf{R} = \begin{pmatrix} 0.5 & 0.75 & 0.9 \\ 0.25 & 0.5 & 0.75 \\ 0.1 & 0.25 & 0.5 \end{pmatrix} \in \mathcal{R}^{\times}. \quad \blacklozenge$$

Function composition is an allowed operation and it can be used to pass from a type of consistency to the other within the framework of reciprocal relations. For instance, if we have an additively consistent reciprocal relation and we want to get to its associated multiplicatively consistent reciprocal relation, then we need to apply the function $(f^{-1} \circ g)(r_{ij})$.

Let us point out a fundamental difference between weight vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, which is going to be so important in the continuation of this dissertation to deserve a formal remark.

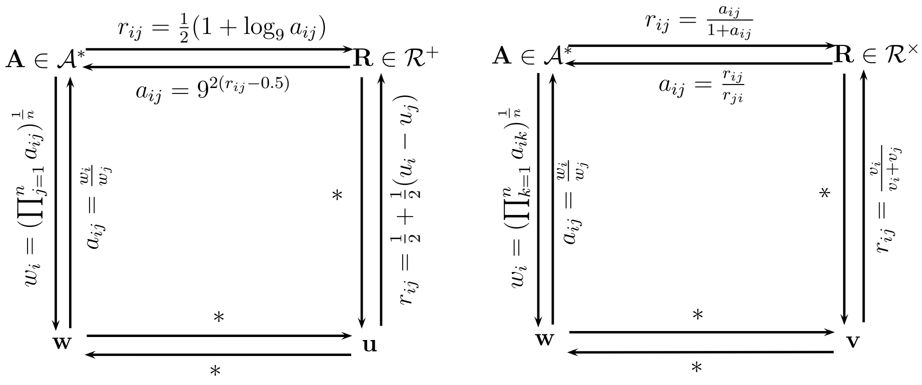
Remark 1. Vector $\mathbf{u}$ is unique up to *addition* of a positive scalar, whereas vectors $\mathbf{v}$ and $\mathbf{w}$ are unique up to *multiplication* of a non-zero scalar.

To clarify what has already been stated in literature, and simply recalled so far, figure 3.3 may be helpful.

Given an arbitrary consistent pairwise comparison matrix $\mathbf{A}$, the corresponding — through (3.10) — additively consistent reciprocal relation $\mathbf{R}$, their associated weight vectors, $\mathbf{w}$ and $\mathbf{u}$ respectively, and the various relationships are depicted in diagram (3.3a). Conversely, the corresponding — through (3.12) — multiplicatively consistent reciprocal relation $\mathbf{R}$, with $\mathbf{v}$ that is its associated weight vector and their relationships are illustrated in diagram (3.3b). In both commutative diagrams, the symbol $*$ indicates that the relation at issue has not been formalized in literature yet.

3.5 Formalizing some assumptions, or how to outline this thesis relaxing them

Preference relations are a convenient tool for representing preferences of a decision maker over a set of alternatives. As we have seen, the definition of preference relation is broad and a lot of different types of preference relations have been proposed to work in practice.



(a) Matrices $\mathbf{A} \in \mathcal{A}^*$, $\mathbf{R} \in \mathcal{R}^+$ and corresponding vectors $\mathbf{w}$ and $\mathbf{u}$

(b) Matrices $\mathbf{A} \in \mathcal{A}^*$, $\mathbf{R} \in \mathcal{R}^\times$ and corresponding vectors $\mathbf{w}$ and $\mathbf{v}$

Figure 3.3: Already known transformations and relationships

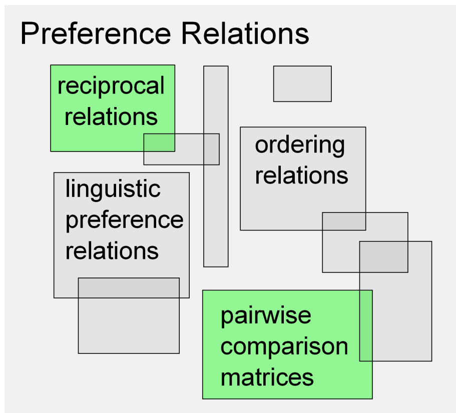


Figure 3.4: Preference relations

Figure 3.4 is nothing but a sketch which should, nevertheless, be helpful. In this maze, the two types of preference relations which have been highlighted in this section, as well as those in examples 5 and 4 are presented as special types on preference relations. So far, talking about pairwise comparison matrices and reciprocal relations, even if tacitly, some assumptions have been made and therefore only half of the story has been told. Actually a greater number of simplifications has been made, as for instance the fact that no uncertainty is present. Having said this, it is time to summarize the four important assumptions which will be the core of the next sections:

1. **Uniqueness of the decision maker** : There is only one decision maker;
2. **Completeness of the preferences** : The decision maker is able to compare all the pairs of alternatives;
3. **Consistency** : The decision maker is consistent and therefore totally rational in comparing alternatives;

4. **Priority vector** : The priority vector exists if and only if the preference relation is consistent.

We are going to see that all these four assumptions can be, to some extent, relaxed, so that we can tap all the potential of preference relations and make them useful and more real-world oriented. It will be clear that, if the assumptions hold, then we are facing a special, very rare and also very lucky case, in a much wider and complex framework. This section will also offer the opportunity to outline the contents of this thesis.

Group decision making

The case of the single decision maker is the starting point of the analysis and it is the best one to introduce the main results. However, in many cases, a *group of subjects* is supposed to make a decision. For example, an organization has a number of shareholders, stakeholders and the like, and each of them represents an individual subject. Examples are indeed very numerous and can be easily found in everyday life.

Example 9. We have a set of decision makers $D = \{d_1, d_2\}$ and a set of alternatives $X = \{x_1, x_2, x_3\}$. In this case, the decision makers d_1 and d_2 can express their opinions by means of the following two pairwise comparison matrices

$$\mathbf{A}_1 = \begin{pmatrix} 1 & 2 & 6 \\ 1/2 & 1 & 3 \\ 1/6 & 1/3 & 1 \end{pmatrix} \quad \mathbf{A}_2 = \begin{pmatrix} 1 & 1 & 2 \\ 1/1 & 1 & 2 \\ 1/2 & 1/2 & 1 \end{pmatrix},$$

respectively. ⧫

It should be clear that the case of the single decision maker is equivalent to the case of group decision making with D being a singleton, and hence it can be seen as a special case. This thesis does not focus on how to determine the final outcome of the decision process and how to aggregate different opinions, but it will show how social network analysis and its tools can be helpful for group decision analysis. Given a starting model, it is totally normal to refine it by means of a number of successive improvements, but I do believe that a lot has already been done in this subfield and there is not a lot of space left for further, and relevant, improvements of already existing models. Conversely, the power of social network analysis has not been exploited for decision making yet. In

- Brunelli M. and Fedrizzi M. (2009): A fuzzy logic approach to social network analysis. *ASONAM 2009*, (pp. 225–230), IEEE Computer Society

- Brunelli M., Fedrizzi M and Fedrizzi M. (2011): OWA-based fuzzy m-ary adjacency relations in Social Network Analysis. Yager R.R., Kacprzyk J. and Beliakov G. (Eds.): *Recent Developments in the Ordered Weighted Averaging Operators: Theory and Practice*, in: Studies in Fuzziness and Soft Computing, Vol. 265, (pp. 255–267) Springer-Verlag, Berlin Heidelberg

the potential of valued adjacency relations in representing compatibility between decision makers is explored. In doing so, a model for deriving m-ary fuzzy adjacency relations is proposed. Such model is grounded on the definition of ρ-characterization, which possibly represents the kernel of the contribution.

Incomplete preference relations

It can happen that a decision maker cannot pairwise compare some of the alternatives. This may be due to several reasons such as lack of time, resources, or competence. In all these cases, some entries of the preference relation are missing and the preference relation is then *incomplete*.

Example 10. Taking into account the following pairwise comparison matrix where $\dot{a}_{ij}$ indicates that the entry is missing

$$\dot{\mathbf{A}} = \begin{pmatrix} 1 & 2 & \dot{a}_{13} \\ 1/2 & 1 & 2 \\ 1/\dot{a}_{13} & 1/2 & 1 \end{pmatrix} \tag{3.14}$$

one can guess that the consistent, and consequently most rational, value for the missing entry is $\dot{a}_{13} = a_{12}a_{23} = 2 \cdot 2 = 4$ and that filling the matrix reduces to a trivial operation. However, looking further than this toy example, it is possible to see, as in the following case, that solving this problem can be extremely cumbersome.

$$\dot{\mathbf{A}} = \begin{pmatrix} 1 & 2 & \dot{a}_{13} & 1/2 & 1 \\ 1/2 & 1 & 8 & 3 & \dot{a}_{25} \\ 1/\dot{a}_{13} & 1/8 & 1 & 2 & 1/4 \\ 2 & 1/3 & 1/2 & 1 & 3 \\ 1 & 1/\dot{a}_{25} & 4 & 1/3 & 1 \end{pmatrix} \quad \blacklozenge$$

In literature there is a sufficient number of proposals aiming at dealing with incompleteness in pairwise comparison matrices and reciprocal relations. The paper

- Brunelli M., Fedrizzi M. and Giove S. (2007). Reconstruction methods for incomplete fuzzy preference relations: a numerical comparison. Masulli F. et al. (Eds.): *WILF 2007, LNAI 4578* (pp. 86–93). Springer-Verlag, Berlin Heidelberg presents a comparative study and investigates which method performs better given a specific situation.

Inconsistency

More or less implicitly we also assumed that preference relations should be consistent. The truth is that, although the consistent case is the most desirable one, it is not always possible to achieve such a goal. This is mainly due to the fact that human judgments are seldom transitive. Nevertheless, it is important to know how inconsistent a decision maker has been in expressing his opinion since it is commonly assumed that a low level of inconsistency is a signal that the decision maker has a good insight of the problem and put some effort in the analysis [95, 96]. *Consistency estimation* then becomes a crucial point in evaluating decisions.

The chapter devoted to this matter starts recalling some inconsistency indices already proposed in literature. Briefly recalling the main definitions is a necessary step because, just after that, an analysis, both theoretical and numerical, will be offered. From the theoretical point of view, in

- Brunelli M., Critch A. and Fedrizzi M. (2011): A note on the proportionality between some consistency indices in the AHP, submitted to *Applied Mathematics and Computation*

there are the proofs of the proportionality between some inconsistency indices. On the other hand, the numerical analysis proposed in

- Brunelli M., Canal L. and Fedrizzi M. (20xx): A comparative study on inconsistency indices, (working paper)

besides confirming the theoretical results on the proportionality, presents some statistics on how related different consistency indices are. Together with a short discussion of the main inconsistency indices, this chapter recalls the findings of

- Brunelli M. (2011): A note on the article "Inconsistency of pair-wise comparison matrix with fuzzy elements based on geometric mean" [Fuzzy Sets and Systems 161 (2010) 1604–1613], *Fuzzy Sets and Systems*, doi:10.1016/j.fss.2011.03.013

to highlight that one of the inconsistency indices at issue may fail to capture inconsistency. Moreover, the chapter contains a disquisition on what is called *preference strength effect* and a proposal to deal with it. Results draw heavily upon the following paper

- Brunelli M. and Fedrizzi M (2007). Fair consistency evaluation in fuzzy preference relations and in AHP. Apolloni B. et al. (Eds.): *KES 2007/ WIRN 2007, Part II, LNAI 4693* (pp. 612–618). Springer-Verlag, Berlin Heidelberg

and its extension

- Fedrizzi M. and Brunelli M. (2009). Fair consistency evaluation for reciprocal relations and in group decision making, *New Mathematics and Natural Computation*, 5(**2**), 407–420

Priority vector

Reciprocal relations and pairwise comparison matrices have been presented in such a way that only the consistent preference relations have a well-defined associated priority vector. In this case, the problem of finding such vector is trivially solved since it can be verified that, in the consistent case, the priority vector is represented by any column of the preference relation at issue. Unfortunately, if the matrix is not consistent identities characterizing priority vectors do no hold.
In spite of this, it is generally assumed, and in some cases it can be verified, that small perturbations in the entries of a matrix lead to small variations of the components of the weight vector. Hence, some methods for deriving a priority vector from an inconsistent matrix have been proposed, especially in the framework of pairwise comparison matrices. In contrast, the same large number of methods cannot be found for reciprocal relations. To solve this problem

- Fedrizzi M. and Brunelli M. (2010): On the priority vector associated with a reciprocal relation and with a pairwise comparison matrix, *Soft Computing*, 14(**6**), 639–645

exposes some new characterizations which can be used as straightforward methods for deriving the priority vector of a reciprocal relation. The very last paper

- Fedrizzi M. and Brunelli M. (2009). On the normalisation of a priority vector associated with a reciprocal relation, *International Journal of General Systems*, 38(**5**), 579–586

is a variation on the theme as it presents some arguments against a commonly used normalization constraint imposed in some optimization problems aiming at elicitating the priority vector of a reciprocal relation.

Chapter 4

Priority vector

> In character, in manner, in style, in all things, the supreme excellence is simplicity.
>
> Henry Wadsworth Longfellow

> The rule of accuracy: When working toward the solution of a problem it always helps if you know the answer
>
> John Peer

The priority vector is the final result which represents the rating of the alternatives obtained from the preference relation. Therefore, it is self-evident that the way to obtain it is a crucial step in decision making with preference matrices.

There exist several methods to estimate the priority vector of a pairwise comparison matrix and also some comparative studies have already been proposed in literature [25, 72]. As these studies report, there are at least eighteen such methods. The most common method is the eigenvector method, proposed by Saaty. According to the eigenvector method, the priority vector is the vector associated with the largest eigenvalue of the pairwise comparison matrix. Thus, the priority vector is obtained as one of the solutions of the following linear system

$$\mathbf{A}\mathbf{w} = \lambda_{\max}\mathbf{w} \tag{4.1}$$

with $\lambda_{\max}$ that is the maximum right eigenvalue of $\mathbf{A}$. Moreover, let us notice that the fact that $\lambda_{\max} \in \mathbb{R}$ and $\mathbf{w} \in \mathbb{R}^n$ is guaranteed by the theorem of

Perron-Frobenius [61]. Another very popular way for obtaining the priorities is the geometric mean method (3.8).

Certainly, there exist methods for estimating the priority vectors of reciprocal relations but they have not been examined in detail. The scope of this chapter is that of introducing some new characterizations for the priority vectors of reciprocal relations and, thanks to them, to propose some methods for estimating the vector itself. The last part is, instead, devoted to correct some models which contain an incompatible normalization constraint.

4.1 The priority vector of a reciprocal relation

This section starts with a brief presentation of some of the methods that have been proposed to estimate the weight vector in the framework of reciprocal relations. Descriptions are kept very short as it would be impossible to show the proposals of each paper in their entirety. What is important is to get the flavor of their complexity.

Fan et al. [41] established some quadratic programming problems to find the priority vector of a reciprocal relation. Fan et al. [39] constructed a multi-objective problem which they eventually formulate as a goal programming problem in n^2 variables and $n(n-1)/2$ constraints involving the deviations from the optimal solution. Their work was extended to the case where a plurality of decision makers are involved and they express their preferences by means of different preference formats [40]. Lipovetsky and Conklin [74] proposed an eigenproblem to estimate the weight vector as the solution of a system of linear equations. Their proposal was extended [124] to perform with incomplete reciprocal relations. Wang and Parkan [121] also based their optimization problems on the eigenproblem introduced in [74]. Xu [123] solved a multi-objective optimization problem using the goal programming methodology. Such a proposal is coherent with those already proposed in literature but it is more general as it deals with incomplete reciprocal relations and with multiple decision makers. An improved method which works under the same conditions, i.e. group decision making with lack of information, was proposed in [135]. Xu and Da [136] solved an optimization problem by means of a convergent iterative algorithm. Wang and Fan [119] proposed to solve a logarithmic least squares problem to find the priority vector. This approach and its objective function are then extended to the more general framework where some entries of the reciprocal relation are missing [54]. Wang et al. [120] proposed a rather elaborated chi-square method. Two other papers [128, 132] suggested to use some very flexible models for deriving the priority vector in a very general context. However, the validity of both approaches [128, 132] will be questioned in the rest of this chapter. Chiclana et al. [22] proposed the Quantifier-Guided Dominance Degree (QGDD) method and further analyzed [23] and developed it such that it can perform in contexts where a number of experts express their incomplete preferences [57, 58]. Although it does not aim at finding priority vector $\mathbf{u}$ as characterized by (3.2), in the special case where the quantifier is the identity function, i.e. $Q(x) = x$, it leads to a vector which is proportional to $\mathbf{u}$. To be more precise, it can be proven, and it will be clear later on in this section, that they are in the proportion 1:2 [46]. Some of the methods mentioned above share one of the following two desirable properties: (i) the weight vector $\mathbf{u}$ calculated from an additively consistent $\mathbf{R}$ satisfies (3.2); (ii) the weight vector $\mathbf{v}$ calculated from a multiplicatively consistent $\mathbf{R}$ satisfies (3.4). In spite of the large number of proposed methods, they still remain rather complex to be implemented and there is not a method which leads to such vectors $\mathbf{u}$ and $\mathbf{v}$, respectively, with a simple and easily interpretable formula. Their complexity is sometimes justified by the fact that some among these proposals can be applied to special cases, e.g. group decisions and incomplete reciprocal relations. Nevertheless, when the single decision maker deals with a complete reciprocal relation, this complexity does not seem to be justified and this is why, we aim at finding a simpler approach.

The intuition leading to some new characterizations, and therefore to some agile formulas to estimate vectors $\mathbf{u}$ and $\mathbf{v}$, comes naturally when looking at figure 3.3. Can we find a characterization that, given a consistent reciprocal relation, either additively or multiplicatively consistent, leads to its priority vector? The following propositions prove it positively and provide the analytic expression of the components of the priority vectors $\mathbf{u}$ and $\mathbf{v}$, respectively.

Proposition 4. *Given an additively consistent reciprocal relation,* $\mathbf{R} \in \mathcal{R}^+$, *the weight vector* $\mathbf{u} = (u_1 \ldots, u_n)$ *defined by*

$$u_i = \frac{2}{n} \sum_{j=1}^{n} r_{ij} \tag{4.2}$$

is the unique vector, up to an additive constant, that satisfies characterization (3.2).

Proof. By substituting (4.2) in the right hand side of (3.2), it is

$$0.5 + 0.5 \left(\frac{2}{n} \sum_{k=1}^{n} r_{ik} - \frac{2}{n} \sum_{k=1}^{n} r_{jk} \right) = 0.5 + \frac{1}{n} \sum_{k=1}^{n} (r_{ik} - r_{jk}) \,.$$

From additive consistency condition (3.1), it is $(r_{ik} - r_{jk}) = (r_{ij} - 0.5)$. Then,

$$0.5 + \frac{1}{n}\sum_{k=1}^{n}(r_{ij} - 0.5) = 0.5 + \frac{1}{n}(r_{ij} - 0.5)n = r_{ij} \ ,$$

proving that (4.2) satisfies (3.2).
To prove uniqueness, let us assume (3.2) and rewrite it in the form

$$2r_{ij} - 1 = u_i - u_j .$$

Then, by summing with respect to j,

$$2\sum_{j=1}^{n} r_{ij} - n = nu_i - \sum_{j=1}^{n} u_j$$

$$u_i = \frac{2}{n}\sum_{j=1}^{n} r_{ij} - 1 + \frac{1}{n}\sum_{j=1}^{n} u_j \ .$$

Since $c = -1 + \frac{1}{n}\sum_{j=1}^{n} u_j$ is constant with respect to i, it is

$$u_i = \frac{2}{n}\sum_{j=1}^{n} r_{ij} + c \ .$$

□

It may be noted that u_i is nothing other than the arithmetic mean of the entries on the i-th row of $\mathbf{R}$ multiplied by 2. Due to the uniqueness of this characterization, we can state that the simple arithmetic mean does *not* satisfy Tanino's characterization (3.2), as it can also be directly checked. Ma et al. [76] proposed a consistency improving method which is coherent with (4.2).
The following proposition is similar to Proposition 4 but it refers to multiplicatively consistent reciprocal relations.

Proposition 5. *Given a multiplicatively consistent reciprocal relation,* $\mathbf{R} \in \mathcal{R}^{\times}$*, the weight vector* $\mathbf{v} = (v_1 \ldots, v_n)$ *defined by*

$$v_i = \left(\prod_{j=1}^{n} \frac{r_{ij}}{r_{ji}}\right)^{\frac{1}{n}} \qquad (4.3)$$

is the unique vector, up to a multiplicative constant, that satisfies characterization (3.4).

Proof. Analogously to the proof of Proposition 4, we substitute (4.3) into the right hand side of (3.4)

$$\frac{v_i}{v_i + v_j} = \frac{1}{1 + \frac{v_j}{v_i}} = \frac{1}{1 + \frac{\left(\prod_{k=1}^{n} \frac{r_{jk}}{r_{kj}}\right)^{\frac{1}{n}}}{\left(\prod_{k=1}^{n} \frac{r_{ik}}{r_{ki}}\right)^{\frac{1}{n}}}} = \frac{1}{1 + \left(\prod_{k=1}^{n} \frac{r_{jk} r_{ki}}{r_{kj} r_{ik}}\right)^{\frac{1}{n}}} ,$$

then from (3.3)

$$\frac{1}{1 + \left(\prod_{k=1}^{n} \frac{r_{ji}}{r_{ij}}\right)^{\frac{1}{n}}} = \frac{1}{1 + \frac{r_{ji}}{r_{ij}}} = \frac{r_{ij}}{r_{ij} + r_{ji}} = r_{ij} ,$$

and (3.4) is satisfied.

To prove uniqueness, let us assume (3.4) and rewrite it in the form

$$v_i \left(\frac{1 - r_{ij}}{r_{ij}} \right) = v_j .$$

Then, by multiplying with respect to j and exploiting the additive reciprocity,

$$\prod_{j=1}^{n} \left(v_i \frac{r_{ji}}{r_{ij}} \right) = \prod_{j=1}^{n} v_j$$

$$v_i = \left(\prod_{j=1}^{n} \frac{r_{ij}}{r_{ji}} \right)^{\frac{1}{n}} \times \left(\prod_{j=1}^{n} v_j \right)^{\frac{1}{n}} = \left(\prod_{j=1}^{n} \frac{r_{ij}}{r_{ji}} \right)^{\frac{1}{n}} \times c$$

where $c = \left(\prod_{j=1}^{n} v_j\right)^{\frac{1}{n}}$ is constant with respect to i. □

Let us only highlight that we have just derived the explicit forms of the priority vectors involved in Tanino's two characterization theorems. While Tanino's characterizations (3.2) and (3.4) provide interpretations of the weights, formulas (4.2) and (4.3) give simple expressions of those weights. Furthermore, they can clearly be applied in the non-consistent case, as it is common practice with the geometric mean method for pairwise comparison matrices (3.8). The analogy with (3.8) will be better clarified in the following.

Let us now investigate the *relationships between weight vectors* **u**, **v** and **w** given by (4.2), (4.3) and (3.8), respectively. Bearing in mind that **u** is unique up to addition of a constant, while **w** and **v** are unique up to a multiplication by a constant (see remark 1), it possible to explore the relationships between vectors.

Proposition 6. *Let* $\mathbf{A} \in \mathcal{A}^*$*, and* $\mathbf{R} \in \mathcal{R}^+$ *its corresponding reciprocal relation obtained by applying (3.10) to* $\mathbf{A}$*. If* $\mathbf{u}$ *and* $\mathbf{w}$ *are given by (4.2) and (3.8) respectively, then, up to addition of a constant,*

$$u_i = \log_9 w_i \quad i = 1, \ldots, n \; . \tag{4.4}$$

Proof. Let us consider r_{ij} given by (3.10) and substitute it in (4.2). We obtain

$$u_i = \frac{2}{n} \sum_{j=1}^{n} \frac{1}{2}(1 + \log_9 a_{ij}) = 1 + \frac{1}{n} \log_9 \prod_{j=1}^{n} a_{ij} = 1 + \log_9 \left(\prod_{j=1}^{n} a_{ij} \right)^{\frac{1}{n}} \tag{4.5}$$

which, according to (3.8), can be rewritten

$$u_i = 1 + \log_9 w_i \; .$$

Finally, since $\mathbf{u}$ is unique up to addition of a constant, (4.4) holds. □

Proposition 7. *Let* $\mathbf{A} \in \mathcal{A}^*$*, and* $\mathbf{R} \in \mathcal{R}^\times$ *its corresponding reciprocal relation obtained by applying (3.12) to* $\mathbf{A}$*. If* $\mathbf{v}$ *and* $\mathbf{w}$ *are given by (4.3) and (3.8) respectively, then they are equal up to a multiplication by a constant,*

$$v_i = w_i \; , \quad i = 1, \ldots, n \; . \tag{4.6}$$

Proof. Thanks to (3.13), let us replace a_{ij} with r_{ij}/r_{ji} in (4.3). We obtain the right hand side of (3.8), thus proving (4.6). □

Corollary 1. According to propositions 6 and 7, one also obtains $u_i = \log_9 v_i$ and $u_i = 2g(w_i)$.

Example 11. We start by considering the following additively consistent reciprocal relation

$$\mathbf{R} = \begin{pmatrix} 0.5 & 0.55 & 0.65 & 0.85 \\ 0.45 & 0.5 & 0.6 & 0.8 \\ 0.35 & 0.4 & 0.5 & 0.7 \\ 0.15 & 0.2 & 0.3 & 0.5 \end{pmatrix} \in \mathcal{R}^+. \tag{4.7}$$

We can derive the priority vector with the aid of (4.2) and it is easy to verify that relation (3.2) is satisfied

$$\mathbf{u} = \begin{pmatrix} 1.275 \\ 1.175 \\ 0.975 \\ 0.575 \end{pmatrix} . \tag{4.8}$$

At this point we proceed using the inverse of (4.4), that is $w_i = 9^{u_i}$. After a multiplication by a proper scalar, and taking into account propositions 6 and 7, we derive the following two normalized vectors

$$\mathbf{w} = \mathbf{v} = \begin{pmatrix} 0.394505 \\ 0.316686 \\ 0.204070 \\ 0.084739 \end{pmatrix}, \tag{4.9}$$

which are associated to the following two matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 1.24573 & 1.93318 & 4.65554 \\ 0.802742 & 1 & 1.55185 & 3.73719 \\ 0.517282 & 0.644394 & 1 & 2.40822 \\ 0.214798 & 0.267581 & 0.415244 & 1 \end{pmatrix} \in \mathcal{A}^*, \tag{4.10}$$

$$\mathbf{R} = \begin{pmatrix} 0.5 & 0.554711 & 0.659073 & 0.823182 \\ 0.445289 & 0.5 & 0.608127 & 0.788905 \\ 0.340927 & 0.391873 & 0.5 & 0.706592 \\ 0.176818 & 0.211095 & 0.293408 & 0.5 \end{pmatrix} \in \mathcal{R}^\times, \tag{4.11}$$

respectively. ♦

Clearly, (4.10) is a consistent pairwise comparison matrix, as it satisfies (3.6), while (4.11) is a multiplicatively consistent reciprocal relation, as it satisfies (3.3). Moreover, it can be verified that $\mathbf{v}$ can also be derived directly from $\mathbf{R} \in \mathcal{R}^\times$ by using (4.3) and $\mathbf{w}$ from $\mathbf{A}$ by using (3.8). To summarize, matrices $\mathbf{R}$ and $\mathbf{A}$ are equivalent ways to express the consistent preferences of a decision maker, as well as $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are equivalent priority vectors.

However, although the matter appears trivial, it is crucial to correctly distinguish between priority vectors, since ratings $\mathbf{u}$ and $\mathbf{w}$ have different characterizations and meanings. Namely, $\mathbf{u}$ represents priority values on an *interval* scale while $\mathbf{w}$ does the same but on a *ratio* scale. That is, differences $u_i - u_j$ between components play a pivotal role in $\mathbf{u}$ (see (3.2)) while ratios w_i/w_j do the same in $\mathbf{w}$ (see (3.7)). Hence, constructing models that do not consider the different nature of $\mathbf{u}$ and $\mathbf{w}$ can lead to undesirable results. As an example, let us consider matrices (4.7) and (4.10) as representing the preferences of two decision makers in model (M11) in [128], where vectors $\mathbf{u}$ and $\mathbf{w}$ are treated without making any distinction (see (30) and (33) in [128]). Since the objective function J_8^* of (M11) is a conjoint estimation of both disagreement between decision makers and inconsistency, in this case its value should be null. Nevertheless, since vector (4.8), which is associated with (4.7), is different from vector (4.9), which is associated with

(3.9), it can be verified that one obtains $J_8^* > 0$ [133, 134]. Furthermore, this conclusion implies that the obtained priority vector cannot be representative of the preferences expressed by the decision makers as it represent another preference configuration.

The acknowledgment of this notable difference between priority vectors is also the fundamental starting point of the next section.

4.2 Normalization of the priority vector

Vector normalization is a widespread technique used in many fields of mathematics, physics, economics, etc. in order to obtain uniqueness from an infinite set of vectors. A well-known example is given by the eigenvectors of a square matrix. Usually, normalization is obtained by dividing every component w_i of a vector $\mathbf{w}$ by a suitable value k. Frequently used values of k are $k = \|\mathbf{w}\|$, i.e. the norm of $\mathbf{w}$, and $k = \sum_{i=1}^{n} w_i$. In the first case a unit-norm vector is obtained, $\|\mathbf{w}\| = 1$, while in the second case the components of the obtained vector sum up to one,

$$\sum_{i=1}^{n} w_i = 1. \tag{4.12}$$

Clearly, normalization is meaningful only if all the vectors of the infinite set we are dealing with are equivalent for our purpose, so that the normalized vector can correctly represent the whole vector set. Eigenvectors corresponding to a single eigenvalue are again a suitable example. As said before, in the AHP [95], as well as in other similar methods, the decision maker's judgments a_{ij} estimate the ratios of priorities w_i/w_j. Therefore, priorities (or weights) w_i can be multiplied or divided by the same arbitrarily chosen positive real number without changing ratios w_i/w_j. In this framework, normalization (4.12) is plainly justified and thus usually applied.

Nevertheless, careful attention must be paid in order to avoid misleading applications of (4.12) in problems where applying this constraint leads to unsatisfactory results. More precisely, it can be shown [45] that, as long as reciprocal relations are concerned, constraint (4.12) is incompatible with additive consistency. Since in many papers on reciprocal relations constraint (4.12) is imposed, it is important, in our opinion, that researchers are aware of this incompatibility. Several papers can be cited as examples [7, 68, 69, 70, 71, 87, 86, 123, 128, 129, 125, 127, 126, 130, 131, 132, 133, 134, 135], but possibly they are not the only ones.

Note 1. Given an additively consistent reciprocal relation $\mathbf{R} = (r_{ij})$, a vector $\mathbf{u}$ is called 'associated' with $\mathbf{R}$ if and only if it satisfies (3.2) as well as the assumptions of Proposition 1. Vector $\mathbf{u}$ is said to 'represent' the associated reciprocal relation.

Proposition 8. *For every positive integer $n \geq 3$, there exists at least an additively consistent reciprocal relation such that none of its associated weight vectors satisfies the constraint*

$$\sum_{i=1}^{n} u_i < n - 1. \tag{4.13}$$

Proof. Let us consider the following additively consistent reciprocal relation

$$\hat{\mathbf{R}} = (r_{ij})_{n \times n} = \begin{pmatrix} 0.5 & \cdots & 0.5 & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 0.5 & \cdots & 0.5 & 1 \\ 0 & \cdots & 0 & 0.5 \end{pmatrix} \tag{4.14}$$

We prove that every vector $\mathbf{u}$ associated with (4.14) cannot satisfy (4.13). By substituting $r_{in} = 1$ in (3.2) for $i = 1, ..., n-1$, one obtains

$$u_i = u_n + 1 \quad i = 1, ..., n-1,$$

and therefore

$$\sum_{i=1}^{n} u_i = (n-1)(u_n + 1) + u_n = nu_n + n - 1.$$

Since $u_n \geq 0$, inequality (4.13) is violated and the proposition is proved. □

Proposition 8 can clearly be equivalently reformulated in the following way,

Proposition 9. *For every positive integer $n \geq 3$, condition*

$$\sum_{i=1}^{n} u_i \geq n - 1. \tag{4.15}$$

is necessary, in order to represent every additively consistent reciprocal relation by means of a weight vector $\mathbf{u} = (u_1, \ldots, u_n)$.

The following proposition shows that the bound $n - 1$ is tight.

Proposition 10. *For every positive integer $n \geq 3$, every additively consistent reciprocal relation can be represented by means of a weight vector $\mathbf{u}$ satisfying*

$$\sum_{i=1}^{n} u_i \leq n - 1\ . \tag{4.16}$$

Proof. Let us consider an arbitrary additively consistent reciprocal relation $\mathbf{R} = (r_{ij})_{n\times n}$. Proposition 1 guarantees the existence of a vector $\mathbf{u} = (u_1, ..., u_n)$ representing $\mathbf{R}$, i.e. satisfying (3.1). Let us assume, without loss of generality, $u_n \leq u_{n-1} \leq \cdots \leq u_1$. Since components of $\mathbf{u}$ are unique up to addition of a real constant k (Proposition 1), by choosing $k = -u_n$, it is always possible to represent $\mathbf{R} \in \mathcal{R}^+$ by a vector $\mathbf{u}$ with $u_n = 0$, obtaining $\mathbf{u} = (u_1, \cdots, u_{n-1}, 0)$. From $u_n = 0$ and proposition 1, it follows $0 \leq u_i \leq 1$. Then it is $\sum_{i=1}^n u_i \leq n-1$. □

Note that $(1, 1, \cdots, 1, 0)$ is the priority vector representing (4.14) with the minimum value of the sum of its components and it is $\sum_{i=1}^n u_i = n-1$.

One might argue that (4.14) is a borderline and implausible example, as it corresponds to the case where the first $n-1$ alternatives are strongly preferred to the last one. Let us then briefly consider a very common case, where the preferences on the alternatives are uniformly distributed from the most preferred alternative x_1 to the less preferred x_n. This is perhaps the most simple and frequent reference case and, for $n = 4$, it is represented by the additively consistent reciprocal relation

$$\bar{\mathbf{R}} = (\bar{r}_{ij})_{n\times n} = \begin{pmatrix} 3/6 & 4/6 & 5/6 & 6/6 \\ 2/6 & 3/6 & 4/6 & 5/6 \\ 1/6 & 2/6 & 3/6 & 4/6 \\ 0/6 & 1/6 & 2/6 & 3/6 \end{pmatrix} \in \mathcal{R}^+. \tag{4.17}$$

As it can be easily verified by means of (3.2), $\bar{\mathbf{u}} = \left(1, \frac{2}{3}, \frac{1}{3}, 0\right)$ represents (4.17) and has the minimum value of the sum of the components, $\bar{u}_1 + \bar{u}_2 + \bar{u}_3 + \bar{u}_4 = 2$. Note that the priority vector $\bar{\mathbf{u}}$ indicates that $x_1 \succ x_2 \succ x_3 \succ x_4$ with uniformly spaced (as the preferences $\bar{r}_{ij}$ are) priority weights.

Example (4.17) can be extended to the general n–dimensional case,

$$\bar{\mathbf{R}} = (\bar{r}_{ij})_{n\times n} = \left(\frac{n-1+j-i}{2n-2}\right)_{n\times n} \tag{4.18}$$

where the priority vector satisfying (3.2) and representing (4.18) with the minimum value of the sum of the components is

$$\bar{\mathbf{u}} = \left(1, \frac{n-2}{n-1}, \cdots, \frac{2}{n-1}, \frac{1}{n-1}, 0\right),$$

with $\sum_{i=1}^n \bar{u}_i = \frac{n}{2}$. Let us sketch the proof. First, by substituting $\bar{u}_i$ and $\bar{u}_j$ in (3.2) it can be verified that $\bar{\mathbf{u}}$ represents (4.18). Then, by summing the components of $\bar{\mathbf{u}}$ one obtains $\sum_{i=1}^n \bar{u}_i = \sum_{i=1}^n \frac{n-i}{n-1} = \frac{1}{n-1}\left(2n - \frac{n(n+1)}{2}\right)$ $= \frac{n}{2}$. All other priority vectors associated to (4.18) have component sum larger than $\frac{n}{2}$, since they are obtained by adding a positive constant to each component of $\bar{\mathbf{u}}$. Therefore, also in this case, condition (4.12) cannot be satisfied and the larger n, the larger the spread between left and right hand side of (4.12). A priority vector satisfying (3.4) can be normalized using (4.12), since the ratio in (3.4) remains unchanged, as it is in w_i/w_j for Saaty's case. To conclude, normalization (4.12) can be properly applied in the framework of pairwise comparison matrices as well as in the framework of *multiplicatively* consistent reciprocal relations. In the following section we propose a normalization condition compatible with *additive* consistency for reciprocal relations.

Uniqueness of priority vector satisfying (3.2) can be achieved simply by adding the constant $k = -\min\{u_1, ..., u_n\}$ to each component u_i, thus obtaining a vector with the minimum component equal to zero. Assuming $u_n \leq u_{n-1} \leq \cdots \leq u_1$, it is $k = -u_n$ and the normalized vector becomes

$$\mathbf{u} = (u_1, \cdots, u_{n-1}, 0). \tag{4.19}$$

Contrary to (4.12), this alternative normalization procedure is compatible with proposition 1 and, as proved above, it guarantees that all the priorities u_i are in the interval $[0, 1]$. This is a good standard result that also allows an easier and more familiar understanding of the obtained priorities. To summarize, the normalization constraint we propose is

$$\begin{array}{l} \min\{u_1, ..., u_n\} = 0 \\ 0 \leq u_i \leq 1 \quad i = 1, ..., n \end{array} \tag{4.20}$$

Up to now, we have considered the case of additively consistent reciprocal relations. Let us now consider the case in which additive consistency is not *a priori* satisfied, but it is the goal of a proposed optimization model. Xu [123], for instance, considered incomplete reciprocal relations and proposed some goal programming models to obtain the priority vector. He [123] referred to proposition 1 to construct the following multi-objective programming model

$$\begin{array}{ll} \text{(MOP1)} & \min \varepsilon_{ij} = \delta_{ij} | \, r_{ij} - 0.5(u_i - u_j + 1) | \quad i, j = 1, ..., n \\ & \text{s.t. } \; u_i \geq 0, \;\; i = 1, .., n, \quad \displaystyle\sum_{i=1}^{n} u_i = 1. \end{array}$$

To solve (MOP1), the author introduced the following (linear) goal programming model,

$$
\text{(LOP2)} \qquad \min J = \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} (d_{ij}^{+} + d_{ij}^{-})
$$

$$
\text{s.t. } \delta_{ij}[r_{ij} - 0.5(u_i - u_j + 1)] - d_{ij}^{+} + d_{ij}^{-} = 0, \ \ i, j \in N, \, i \neq j
$$

$$
u_i \geq 0, \ \ i \in N, \ \sum_{i=1}^{n} u_i = 1
$$

$$
d_{ij}^{+} \geq 0, \ d_{ij}^{-} \geq 0, \ \ i, j \in N, \, i \neq j \, .
$$

where $N = \{1, \ldots, n\}$. Optimization models (MOP1) and (LOP2) are clearly based on the idea of moving as close as possible to satisfying (3.2). The proposal is appropriate and effective but, as proved in the previous section, the normalization constraint (4.12) required in both (MOP1) and (LOP2) conflicts with the goal. This should not happen as the constraint to add the weights up to one is needed to find uniqueness of solution and it should not affect the value of the objective function. For obvious reasons, this kind of constraints have sometimes been called cosmetic constraints.

Example 12. Let us consider the incomplete reciprocal relation obtained from the reciprocal relation in (4.17) by considering r_{14} (and therefore also r_{41}) as missing. Following definition 2.5 of [123], this reciprocal relation is called an *additively consistent incomplete fuzzy preference relation.* By applying (LOP2), vector $\mathbf{u}^* = \left(\frac{2}{3}, \frac{1}{3}, 0, 0\right)$ is obtained, and its corresponding value of the objective function is $J(\mathbf{u}^*) = \frac{2}{3}$, evidencing that (3.2) has not been completely fulfilled. Conversely, if the constraint (4.12) was substituted by (4.20) in (LOP2), one would obtain the vector $\bar{\mathbf{u}} = \left(1, \frac{2}{3}, \frac{1}{3}, 0\right)$, with $J(\bar{\mathbf{u}}) = 0$, so that (3.2) is completely fulfilled. Note that $\mathbf{u}^*$ does not even respect preference ordering, as it is $u_3^* = u_4^*$ with $r_{34} > 0.5$. Moreover, while $\bar{\mathbf{u}}$ is associated to (4.17), vector $\mathbf{u}^*$ is associated to a different consistent reciprocal relation, more precisely to

$$
\mathbf{R}^* = \begin{pmatrix} 3/6 & 4/6 & 5/6 & 5/6 \\ 2/6 & 3/6 & 4/6 & 4/6 \\ 1/6 & 2/6 & 3/6 & 3/6 \\ 1/6 & 2/6 & 3/6 & 3/6 \end{pmatrix} \in \mathcal{R}^{+}. \tag{4.21}
$$

Analogous results are obtained if the goal of the optimization models is still additive consistency for a reciprocal relation, but this goal is not fully achievable. ⧫

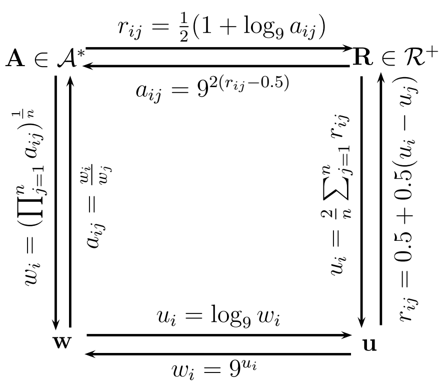
Xu and Chen [133] also considered *interval* reciprocal relations, represented by square matrices whose entries are real intervals. This approach generalizes the former [123], as each preference is quantified by using an interval $[r_{ij}^{-}, r_{ij}^{+}]$, instead of a single value r_{ij}. Their optimization models [133] denoted by (M–1), (M–2), (M–3), (M–4) and (M–5) are still based on the objective of best fulfillment of Tanino's condition (3.2), but they contain constraint (4.12). Therefore, all the arguments exposed above can be repeated also in this case and therefore is not not the case to report a detailed discussion with examples.

Nevertheless, it is necessary to draw the attention on the consequences of imposing (4.12) in definitions 3 and 4 given by [133] for an 'additive consistent interval fuzzy preference relation' (or 'additive consistent interval reciprocal relation', following our terminology). These definitions extend the well-known case of additively consistent reciprocal relation by requiring that in each entry of the interval matrix a single value $r_{ij} \in [r_{ij}^-, r_{ij}^+]$ can be chosen to form an additively consistent reciprocal relation, i.e. satisfying (3.2). In other words, an interval reciprocal relation is called additively consistent if it 'contains' an additively consistent reciprocal relation. By including (4.12) in definitions 3 and 4, it is implicitly required that Tanino condition (3.2) must be associated to (4.12) in order to obtain additive consistency. As we stated above with proposition 9, the two of them are incompatible. Coherence with the definition of additively consistent reciprocal relation can be achieved only by removing (4.12) or by substituting it with a suitable normalization, e.g. (4.20). Otherwise, it is easy to check that an interval reciprocal relation obtained simply by adding a small spread to the entries of an additively consistent reciprocal relation could not satisfy the previous definitions and should be classified as inconsistent. This is clearly unacceptable and an example can be constructed by means of (4.17). It can be verified that the interval reciprocal relation whose entries, for $i \neq j$, are intervals centered in $\bar{r}_{ij}$, i.e. $[r_{ij}^-, r_{ij}^+] = [\bar{r}_{ij} - \varepsilon, \bar{r}_{ij} + \varepsilon]$, does not satisfy the definitions 3 and 4 if $\varepsilon < 0.166$. To be more precise, since all the considered values must remain in the interval [0, 1], we should better define $r_{ij}^- = \max(0, \bar{r}_{ij} - \varepsilon)$ and $r_{ij}^+ = \min(1, \bar{r}_{ij} + \varepsilon)$, but this does not change our conclusion. Definition 3 of [133] is also reported in another work on the same issue [134] and in a survey of preference relations [128], where it is referred to as definition 10.

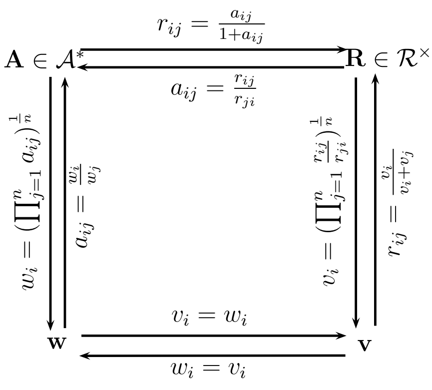
4.3 Discussion

This chapter has shown some results regarding the priority vector of a reciprocal relation and served to shed more light on the relation between a reciprocal relation and its priority vector in both consistent and inconsistent cases (see section 3.5). In summary, the scope of this section was at least fourfold.

First, propositions 4, 5, 6 and 7 completed the set of transformations and relationships between consistent pairwise comparison matrices, the two types of corresponding consistent reciprocal relations and their associated weight vectors. Figure 4.1 shows the same diagrams presented in Figure 3.3 but completed with the relationships that we have introduced in this chapter. While Tanino [112] proved the existence of vectors **u** and **v** satisfying (3.2) and (3.4) respectively, characterizations (4.2) and (4.3) provide the simplest representations of such vectors.



(a) Matrices $\mathbf{A} \in \mathcal{A}^*$, $\mathbf{R} \in \mathcal{R}^+$ and corresponding vectors **w** and **u**



(b) Matrices $\mathbf{A} \in \mathcal{A}^*$, $\mathbf{R} \in \mathcal{R}^\times$ and corresponding vectors **w** and **v**

Figure 4.1: Complete diagrams of transformations and relationships

Thanks to formulas (4.2) and (4.3), which can be considered to be the counterparts of (3.8) for reciprocal relations satisfying (3.1) and (3.3) respectively, we propose two methods to derive vectors **u** and **v** from inconsistent reciprocal relations too. A natural consequence of this analogy is that formulas (4.2) and (4.3) share, and introduce in the framework of reciprocal relations, the same good properties that make the geometric mean method one of the best estimation methods for the weight vector of an inconsistent pairwise comparison matrix [72].

The third scope of this chapter has been that of remarking the difference of characterization and interpretation between ratio cardinal rankings **v** and **w** and interval cardinal rankings **u** in order to avoid misunderstandings.

Fourth, and last, given the very frequent use of vector normalization, it is important that researchers are warned not to consider it as a risk-free routine when dealing with reciprocal relation. It was shown how some interesting proposals can become useless, due to an inadequate choice of the normalization constraint.

Chapter 5

Consistency

> Many complain of their memory,
> few of their judgment.
>
> Benjamin Franklin

Although some studies claimed the contrary [73, 113], it is generally assumed that the reliability of the weight vector goes arm-in-arm with the consistency of the judgments expressed by the decision maker [1]. In fact, the more consistent the judgments are, the more likely it is that the decision maker is a good expert with a deep insight into the problem and paid the due attention in eliciting his/her preferences. Conversely, if judgments are far from consistency, it is likely that the he/she gave them with scarce competence and care. In fact, everyone would be able to provide a strongly inconsistent preference relation, perhaps by randomly generate it, but, in practice, only a sufficiently well-informed decision maker is likely to provide a consistent one.

Therefore, in several applications it is crucial that the consistency evaluation process is carried out in a fair way. In group decision making, for instance, the aggregation of the individual preferences can be performed taking into account the consistency of the judgments [24, 44]. The importance of consistency evaluation, as well as the reason behind the previous claims, is easily deduced if we reckon that the property of consistency is the extension to the cardinal framework of the property of transitivity of relations. It is just the case to remember that transitivity is one of the axioms proposed by von Neumann and Morgenstern [117] and it is fundamental for the whole microeconomics theory [62].

[1] Or, at least, it is the best proxy measure find so far to estimate the reliability of an expert.

As aforementioned, consistent preferences denote coherence and absence of contradiction in the decision maker's judgments. Although the consistent case is certainly the most desirable one, it is hardly ever possible for a decision maker to be fully consistent, especially if the number of alternatives is large enough. Hence, a certain degree of inconsistency should be tolerated. Nevertheless, as the inconsistency increase, judgments become less and less reliable. These aspects contribute to make the consistency evaluation a crucial matter in the decision making process. In this context, hereafter, the word inconsistency means an evaluation of the degree of incoherence of the expressed preferences of a given decision maker.

In the next section the best known consistency indices are briefly recalled by means of some self-contained descriptions. After having done this, some theoretical advances are presented. Such advances are two propositions stating the proportionality, and therefore the equivalence, between some indices. This will greatly simplify the numerical analysis of the indices. The analysis comprehends a section based on [14] which highlights a shortcoming of one of the inconsistency indices. The last part of this chapter, prior to some final remarks, will be an analysis of the preference strength effect and will contain a proposal to neutralize it.

5.1 (In)consistency Indices

Eleven indices for estimating (in)consistency have been chosen to be presented in this section and analyzed in the following ones. The description of the indices will be kept brief and unnecessary details be skipped, as their full illustration would go beyond the scope of this dissertation.

Relative Error

The relative error index [9] requires the construction of an auxiliary matrix $\mathbf{A}^+ = (a^+_{ij})_{n\times n} = (\log_2 a_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$ which is skew symmetric and often called 'additive' pairwise comparison matrix [9]. The second step consists in deriving a weight vector $\mathbf{w}^+ = (w^+_1, \ldots, w^+_n)$ with $w^+_i = \frac{1}{n}\sum_{j=1}^n a^+_{ij}$. Having done this, the *consistent part* of $\mathbf{A}^+$ is obtained as $\mathbf{C} = (c_{ij})_{n\times n} = (w^+_i - w^+_j)_{n\times n}$. Another matrix, $\mathbf{E} = (e_{ij})_{n\times n} = (a_{ij} - c_{ij})_{n\times n}$, is also obtained to represent the error part of $\mathbf{A}$, such that $\mathbf{C} + \mathbf{E} = \mathbf{A}$. At this point the relative error index is derived as

$$RE = \frac{\sum_{i=1}^n \sum_{j=1}^n e^2_{ij}}{\sum_{i=1}^n \sum_{j=1}^n a^2_{ij}} .$$

CI and CR

Following the result that the maximum eigenvalue, $\lambda_{\max}$, of a pairwise comparison matrix $\mathbf{A}$, equals n if and only if the matrix is consistent (and is greater then n otherwise), Saaty [95] proposed a consistency index

$$CI = \frac{\lambda_{\max} - n}{n-1}. \tag{5.1}$$

However, empirical tests showed that the expected value of CI of a random matrix of size $n+1$ is, on average, higher than the expected value of CI of a random matrix of order n. Consequently, CI is not reliable in comparing matrices of different size and it needs to be normalized.
CR, which stands for *Consistency Ratio*, is the normalized version of CI. Given a matrix of order n, CR can be obtained by dividing CI by a real number RI (*Random Index*) which is the average CI obtained from a large enough set of randomly generated matrices of size n. Hence,

$$CR = \frac{CI}{RI} \tag{5.2}$$

This normalization process should have been implemented for the other indices too, but, often, this step was not considered in the original papers.

Squared differences index

The definition of this index [26] is based on (3.7) and it assumes that each deviation from the situation of perfect consistency should be considered a symptom of inconsistency. Thus, the sum of the squares of the deviations $(a_{ij} - \frac{w_i}{w_j})\ \forall i \neq j$ is considered a fair and global quantification of inconsistency

$$LS = \min_{w_1,\ldots,w_n} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \left(a_{ij} - \frac{w_i}{w_j} \right)^2 \quad \text{s.t.} \quad \sum_{i=1}^{n} w_i = 1, \quad w_i > 0. \tag{5.3}$$

Index LS, which stands for least squares, is also easy to be standardized since the number of (non-diagonal) terms of the sum, as noted above, is $n(n-1)$. Let us note that the argument minimizing (5.3) is the priority vector $\mathbf{w}^* = (w_1^*, \ldots, w_n^*)$ associated to the pairwise comparison matrix $\mathbf{W}^* = (w_i^*/w_j^*)_{n\times n}$ which minimizes the Frobenius norm $||\mathbf{A} - \mathbf{W}||_2$ with $\mathbf{W} = (w_i/w_j)_{n\times n}$. Despite the elegant formulation, optimization problem (5.3) is difficult to solve numerically, multiple solutions can exist and at least so far, no analytic solution has been found. Bozoki [12] built an equation system whose roots yield to the optimal components of $\mathbf{w}$. However, this method suffers of a huge computational complexity. In order to overcome this problem, other authors [5] proposed some simplifications which are — though — based on some uncertain assumptions.

Cavallo - D'Apuzzo

In their papers [19, 20], Cavallo and D'Apuzzo, besides proposing a general framework based on Abelian linearly ordered groups for some representations of cardinal preferences, introduced an approach based on some new metrics and a normalized index. Given a pairwise comparison matrix $\mathbf{A}$, their index is

$$I = \prod_{i=1}^{n} \prod_{j>i}^{n} \prod_{k>j}^{n} \left(\frac{a_{ik}}{a_{ij}a_{jk}} \vee \frac{a_{ij}a_{jk}}{a_{ik}} \right)^{\frac{1}{\binom{n}{3}}} \in [1, +\infty[. \tag{5.4}$$

Index of determinants

This index [85] is based on the following property of pairwise comparison matrices of order three. Expanding the determinant of a real matrix of order 3 one obtains

$$\det(\mathbf{A}) = \frac{a_{ik}}{a_{ij}a_{jk}} + \frac{a_{ij}a_{jk}}{a_{ik}} - 2.$$

If the matrix is not consistent, then $\det(\mathbf{A}) > 0$, because $\frac{a}{b} + \frac{b}{a} - 2 > 0 \ \forall \, a \neq b, \ a, b > 0$.
It is possible to generalize the approach for matrices of order greater than three and define such inconsistency index as the arithmetic mean of the determinants of all the possible submatrices $\mathbf{T}_{ijk}$ of a given pairwise comparison matrix, constructed in a way so that they respect the following formulation

$$\mathbf{T}_{ijk} = \begin{pmatrix} 1 & a_{ij} & a_{ik} \\ a_{ji} & 1 & a_{jk} \\ a_{ki} & a_{kj} & 1 \end{pmatrix}, \forall i < j < k.$$

The number of so constructed submatrices is $\binom{n}{3} = \frac{n!}{3!(n-3)!}$. The result is a normalized index and its value is the average inconsistency computed for all the submatrices $\mathbf{T}_{ijk}$ $(i < j < k)$

$$CI^* = \sum_{i=1}^{n} \sum_{j>i}^{n} \sum_{k>j}^{n} \left(\frac{a_{ik}}{a_{ij}a_{jk}} + \frac{a_{ij}a_{jk}}{a_{ik}} - 2 \right) \Big/ \binom{n}{3}. \tag{5.5}$$

Golden-Wang index

The Golden-Wang index [53] assumes that the priority vector is normalized such that its components sum up to one. They call the so obtained vector $\mathbf{w}^* = (w_1^*, \dots, w_n^*)$ The same operation is similarly repeated on the columns of $\mathbf{A}$. Namely, entries on the j-th column are divided by a constant $k_j = \sum_{i=1}^{n} a_{ij}$ and the new matrix can then be called $\mathbf{A}^* = (a_{ij}^*)$. The standardized index proposed in [53] is

$$GW = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} |\, a_{ij}^* - w_i^* |.$$

Geometric consistency index

This method was first implicitly introduced by Crawford and Williams [27], then it was reexamined by other authors in [3]. It considers the priority vector to be estimated by means of the geometric mean method (3.8). With the so estimated weights it is possible to build a local estimator of inconsistency,

$$e_{ij} = a_{ij} \frac{w_j}{w_i}, \quad i, j = 1, \dots, n. \tag{5.6}$$

For consistent matrices the value of e_{ij} is equal to 1 because it is the result of a multiplication of an entry times its reciprocal. Therefore, since $a_{ij} = \frac{w_i}{w_j} \Rightarrow \ln e_{ij} = 0$, it is then possible to define a global inconsistency index, i.e. the Geometric Consistency Index (GCI), that is

$$GCI = \frac{2}{(n-1)(n-2)} \sum_{i=1}^{n} \sum_{j>i}^{n} \ln^2 e_{ij}. \tag{5.7}$$

Harmonic consistency index

If and only if $\mathbf{A}$ is a consistent pairwise comparison matrix, then its columns are proportional and $\text{rank}(\mathbf{A}) = 1$. Therefore, it is fair to suppose that the less proportional are the columns, the less consistent is the matrix. A new consistency index, loosely based on the proportionality between columns, was then proposed by Stein and Mizzi [110]. Given a matrix $\mathbf{A}$, they proposed to construct an auxiliary vector $\mathbf{s} = (s_1, \dots, s_n)$ with $s_j = \sum_{i=1}^{n} a_{ij} \; \forall j$. It was proven that $\sum_{j=1}^{n} s_j^{-1} = n$ if and only if $\mathbf{A}$ is consistent and smaller than n, otherwise. The harmonic mean of the components of vector $\mathbf{s}$ is then the result of the following

$$\text{HM} = \frac{n}{\sum_{j=1}^{n} \frac{1}{s_j}}. \tag{5.8}$$

HM itself is an index of inconsistency, but Stein and Mizzi [110], according to computational experiments, proposed a normalization in order to align the behavior of their index with that of CI. The Harmonic Consistency Index is then

$$\mathrm{HCI} = \frac{(\mathrm{HM} - n)(n+1)}{n(n-1)}. \tag{5.9}$$

The index c_3

The index c_3 of the characteristic polynomial of a pairwise comparison matrix was also suggested as an inconsistency index [105, 106, 107]. In fact, by definition, the characteristic polynomial of a real valued matrix has always the following form

$$P_{\mathbf{A}}(\lambda) = \lambda^n + c_1\lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n,$$

with $c_1, \ldots, c_n$ that are real numbers and λ the unknown. Shiraishi et al. prove that, if and only if $c_3 < 0$ the matrix at issue is not consistent. In fact, this is evident if we consider that the only possible formulation of the characteristic polynomial that returns $\lambda_{\max} = n$, is

$$P_{\mathbf{A}}(\lambda) = \lambda^{n-1}(\lambda - n). \tag{5.10}$$

Thus, the presence of c_3 is certainly a symptom of inconsistency. Moreover, Shiraishi et al. also proved that c_3 has the following analytic expression

$$c_3 = \sum_{i=1}^{n}\sum_{j>i}^{n}\sum_{k>j}^{n}\left(2 - \frac{a_{ik}}{a_{ij}a_{jk}} - \frac{a_{ij}a_{jk}}{a_{ik}}\right). \tag{5.11}$$

Ramík-Korviny Index

Ramík and Korviny [90] and Ramík and Perzina [91] presented a consistency index for pairwise comparison matrices whose entries are triangular fuzzy numbers. However, as they treat it as a more general case, their index can be adapted to work with pairwise comparison matrices with crisp entries in the interval $[1/\sigma, \sigma]$. They formulate the inconsistency index as follows:

$$NI_n^{\sigma} = \gamma_n^{\sigma} \max_{i,j}\left\{\left|a_{ij} - \frac{w_i}{w_j}\right|\right\}, \tag{5.12}$$

where the weights should be obtained by means of the geometric mean method and

$$\gamma_n^{\sigma} = \begin{cases} \dfrac{1}{\max\left\{\sigma - \sigma^{\frac{2-2n}{n}}, \sigma^2\left(\left(\frac{2}{n}\right)^{\frac{2}{n-2}} - \left(\frac{2}{n}\right)^{\frac{n}{n-2}}\right)\right\}}, & \text{if } \sigma < \left(\frac{n}{2}\right)^{\frac{n}{n-2}} \\ \dfrac{1}{\max\left\{\sigma - \sigma^{\frac{2-2n}{n}}, \sigma^{\frac{2n-2}{n}} - \sigma\right\}}, & \text{if } \sigma \geq \frac{n}{2}^{\frac{n}{n-2}} \end{cases}$$

is a positive normalization factor.

The index ρ

The index ρ, originally introduced by Fedrizzi et al. [42] and then studied by Fedrizzi and Giove [47], is based on the definition of consistency of a single triple of pairwise comparisons of a reciprocal relation. The papers at issue, recalled the condition of additive consistency for reciprocal relations (3.1) and defined the local inconsistency associated with the three alternatives x_i, x_j, x_k as

$$(r_{ik} + 0.5 - r_{kj} - r_{ij})^2 \ . \tag{5.13}$$

Fedrizzi and Giove [47] gave the definition of an index of global inconsistency,

$$\rho = \sum_{i<j<k}^{n} (r_{ik} + 0.5 - r_{kj} - r_{ij})^2 \Big/ \binom{n}{3}. \tag{5.14}$$

It is interesting to see that ρ can be reformulated, and equivalently introduced, in the framework of pairwise comparison matrices. We can do it thanks to the following proposition.

Proposition 11. *The index*

$$\rho_{\mathbf{A}} := \frac{1}{4} \sum_{i<j<k}^{n} (\log_9 a_{ik} a_{ji} a_{kj})^2 \Big/ \binom{n}{3} \tag{5.15}$$

computed on a pairwise comparison matrix $\mathbf{A}$ *is equal to* ρ *computed on its associated reciprocal relation* $\mathbf{R}$ *by means of the function (3.10).*

Proof. First, let us note that, thanks to reciprocity $r_{ij} + r_{ji} = 1$, (5.13) can be rewritten as $\left(r_{ik} + r_{jk} + r_{ji} - \frac{3}{2}\right)^2$. Substituting this expression in (5.14) and then applying the consistency preserving mapping $r_{ij} = \frac{1}{2}(1 + \log_9 a_{ij})$, we obtain

$$\sum_{i<j<k}^{n} \left(\frac{1}{2} \log_9 a_{ik} + \frac{1}{2} \log_9 a_{ij} + \frac{1}{2} \log_9 a_{jk} \right)^2 \Big/ \binom{n}{3}$$

then, with a succession of elementary steps we get to

$$\frac{1}{4} \sum_{i<j<k}^{n} (\log_9 a_{ik} a_{ji} a_{kj})^2 \Big/ \binom{n}{3}$$

which completes the proof. □

Koczkodaj

Koczkodaj [67] proposed to estimate the inconsistency of a pairwise comparison matrix of order three as

$$K(x_i, x_j, x_k) = \min\left\{ \frac{1}{a_{ij}}\left|a_{ij} - \frac{a_{ik}}{a_{jk}}\right|, \frac{1}{a_{ik}}\left|a_{ik} - a_{ij}a_{jk}\right|, \frac{1}{a_{jk}}\left|a_{jk} - \frac{a_{ik}}{a_{ij}}\right| \right\} \tag{5.16}$$

where x_i, x_j, x_k are the only alternatives for such a preference relation. This method was then generalized [35] for $n \geq 3$

$$K = \max\{K(x_i, x_j, x_k) | 1 \leq i < j < k \leq n\} \tag{5.17}$$

so that inconsistencies of several transitivities are aggregated by means of the max function. Note that, for sake of simplicity, it was proven [35] that (5.16) collapses into the following

$$K(x_i, x_j, x_k) = \min\left\{ \left|1 - \frac{a_{ik}}{a_{ij}a_{jk}}\right|, \left|1 - \frac{a_{ij}a_{jk}}{a_{ik}}\right| \right\} \tag{5.18}$$

5.2 Theoretical results

It is interesting to note that some seemingly different indices, are instead proportional. Moreover, it is important to be aware of their proportionality for two reasons. From an empirical point of view, they should not be considered as contributing independent evidence for the consistency of a subject's preferences. Besides, from a pure mathematical perspective, their equivalence may be taken to suggest that they represent an important quantity. Hereafter, with two propositions we will justify our claims that c_3 is proportional to CI^*, and ρ is proportional to GCI.

Proposition 12. *Given a pairwise comparison matrix* $\mathbf{A} = (a_{ij})_{n \times n}$ *with* $n \geq 3$*, the consistency indices* c_3 *and* CI^* *satisfy the equality*

$$c_3 = -\binom{n}{3} CI^*. \tag{5.19}$$

Proof. Consistency index CI^* was defined as

$$CI^* = \sum_{i=1}^{n}\sum_{j>i}^{n}\sum_{k>j}^{n} \left(\frac{a_{ik}}{a_{ij}a_{jk}} + \frac{a_{ij}a_{jk}}{a_{ik}} - 2 \right) \Big/ \binom{n}{3}. \tag{5.20}$$

Furthermore, since $\mathbf{A}$ is positive and reciprocal, Shiraishi et al. [106] proved that

$$c_3 = \sum_{i=1}^{n}\sum_{j>i}^{n}\sum_{k>j}^{n} \left(2 - \frac{a_{ik}}{a_{ij}a_{jk}} - \frac{a_{ij}a_{jk}}{a_{ik}} \right). \tag{5.21}$$

Said this, equality (5.19) follows from (5.20) and (5.21). □

If in this case the similarity between the two indices was quite clear, then the same cannot be said about the next two. For this reason, if the previous proof was rather straightforward, the next involves more computations.

Proposition 13. *Given a reciprocal relation* $\mathbf{R} = (r_{ij})_{n\times n}$ *and its associated — by means of (3.11) — pairwise comparision matrix* $\mathbf{A} = (a_{ij})_{n\times n}$*, the consistency indices* ρ *and* GCI *satisfy the equality*

$$\rho = \frac{3}{4\ln^2(9)} GCI \tag{5.22}$$

for every $n \geq 3$*.*

Proof. For later convenience, letting $q_{ij} = r_{ij} - 0.5$ allows us to write $r_{ij} + r_{ji} = 1$ property as $q_{ij} = -q_{ji}$. Then, (3.11) becomes $a_{ij} = 9^{2q_{ij}}$. Now, write $t_{ijk} = r_{ij} - r_{ik} - r_{kj} + 0.5 = q_{ij} + q_{jk} + q_{ki}$ so that, from (5.14), the index ρ can be reformulated as (see [47])

$$\begin{aligned} \rho &= \sum_{ijk}^{n} (r_{ij} - r_{ik} - r_{kj} + 0.5)^2 \Big/ 6\binom{n}{3} \\ &= \sum_{ijk} t_{ijk}^2 \Big/ 6\binom{n}{3} . \end{aligned}$$

Let us rewrite the Geometric Consistency Index (5.7) for reciprocal relations by applying (3.10). From (3.8),

$$\log_9 w_i = \frac{2}{n} \sum_k q_{ik}$$

and thus, from the definition of local inconsistency $e_{ij} := a_{ij} \frac{w_j}{w_i}$ in (5.7),

$$\begin{aligned} n \log_9(e_{ij}) &= 2nq_{ij} + 2\sum_k (q_{jk} - q_{ik}) \\ &= 2\sum_k (q_{ij} + q_{jk} + q_{ki}) \\ &= 2\sum_k t_{ijk} \end{aligned}$$

so the Geometric Consistency Index equals

$$\begin{aligned}
GCI &= \frac{2}{(n-1)(n-2)} \sum_i \sum_{j>i} \ln^2 e_{ij} \\
&= \frac{1}{(n-1)(n-2)} \sum_{ij} \ln^2 e_{ij} \\
&= \frac{\ln^2(9)}{(n-1)(n-2)} \sum_{ij} \left(\frac{2}{n} \sum_k t_{ijk} \right)^2 \\
&= \frac{4 \ln^2(9)}{n^2(n-1)(n-2)} \sum_{ij} \left(\sum_k t_{ijk} \right)^2
\end{aligned}$$

At this point, the proportionality claim $\rho \propto GCI$ is equivalent to

$$\sum_{ijk} t_{ijk}^2 \quad \propto \quad \sum_{ij} \left(\sum_k t_{ijk} \right)^2$$

(where the constant of proportionality could depend on n). First, let us compute the LHS:

$$t_{ijk}^2 \quad = q_{ij}^2 + q_{jk}^2 + q_{ki}^2 + 2(q_{ij}q_{jk} + q_{jk}q_{ki} + q_{ki}q_{ij})$$

Let $S = \sum_{ij} q_{ij}^2$ and $C = \sum_{ijk} q_{ij}q_{jk}$. Summing the expansion of t_{ijk}^2 one term at a time,

$$\sum_{ijk} q_{ij}^2 = \sum_k \sum_{ij} q_{ij}^2 = nS$$

and by symmetry,

$$\sum_{ijk} q_{jk}^2 = \sum_{ijk} q_{ki}^2 = nS.$$

Similarly,

$$\sum_{ijk} q_{ij}q_{jk} = \sum_{ijk} q_{jk}q_{ki} = \sum_{ijk} q_{ki}q_{ij} = C.$$

Hence,

$$\text{LHS} = \sum_{ijk} t_{ijk}^2 = nS + nS + nS + 2(C + C + C) = 3(nS + 2C).$$

Next let us compute the RHS, first by rewriting:

$$\text{RHS} = \sum_{ij} \left(\sum_k t_{ijk} \right)^2 = \sum_{ij} \left(\sum_{kl} t_{ijk}t_{ijl} \right) = \sum_{ijkl} t_{ijk}t_{ijl}$$

$$\begin{aligned} t_{ijk}t_{ijl} &= (q_{ij} + q_{jk} + q_{ki})(q_{ij} + q_{jl} + q_{li}) \\ &= q_{ij}^2 + q_{ij}q_{jl} + q_{ij}q_{li} + q_{jk}q_{ij} + q_{jk}q_{jl} + q_{jk}q_{li} + q_{ki}q_{ij} + q_{ki}q_{jl} + q_{ki}q_{li} \end{aligned}$$

The 1st term sums to

$$\sum_{ijkl} q_{ij}^2 = \sum_{kl}\sum_{ij} q_{ij}^2 = n^2 S.$$

The 2nd term sums to

$$\sum_{ijkl} q_{ij}q_{jl} = \sum_{k}\sum_{ijl} q_{ij}q_{jl} = nC.$$

Similarly, the 3rd, 4th, and 7th terms respectively sum to

$$\sum_{ijkl} q_{li}q_{ij} = \sum_{ijkl} q_{ij}q_{jk} = \sum_{ijkl} q_{ki}q_{ij} = nC,$$

whereas the 5th and 9th terms each sum to

$$\sum_{ijkl} -q_{kj}q_{jl} = \sum_{ijkl} -q_{ki}q_{il} = -nC.$$

The 6th term sums to

$$\sum_{ijkl} q_{jk}q_{li} = \left(\sum_{jk} q_{jk}\right)\left(\sum_{li} q_{li}\right) = (0)(0) = 0,$$

and similarly the 8th term sums to 0. Hence, the total sum is

$$\begin{aligned} \text{RHS} &= n^2S + nC + nC + nC - nC + 0 + nC + 0 - nC \\ &= n^2S + 2nC \\ &= n(nS + 2C) \end{aligned}$$

so we obtain the proportionality

$$\frac{\text{LHS}}{\text{RHS}} = \frac{3(nS+2C)}{n(nS+2C)} = \frac{3}{n},$$

and also recover the proportionality factor

$$\begin{aligned} \frac{\rho}{GCI} &= \frac{\text{LHS}}{\text{RHS}} \cdot \frac{n^2(n-1)(n-2)}{4\ln^2(9)} \cdot \frac{1}{6\binom{n}{3}} \\ &= \frac{3n(n-1)(n-2)}{4\ln^2(9)} \cdot \frac{1}{n(n-1)(n-2)} \\ &= \frac{3}{4\ln^2(9)}. \end{aligned}$$

□

In order to stress that the quantity expressed by these two seemingly different indices may be interpreted as an important one, let us formulate the following corollary to show how it comes from two seemingly different formulas.

Corollary 2. As a natural consequence of propositions 11 and 13, given a pairwise comparison matrix $\mathbf{A} = (a_{ij})_{n\times n}$ and given a priority vector $\mathbf{w}$ estimated by means of the geometric mean method, then

$$\sum_{i=1}^{n}\sum_{j>i}^{n} \ln^2 a_{ij}\frac{w_j}{w_i} \propto \sum_{i<j<k}^{n} \left(\log_9 a_{ik}a_{ji}a_{kj}\right)^2 .$$

Consequently, GCI can be formulated as an index of deviation from both consistency conditions (3.6) and (3.7). The same applies to index ρ too.

To conclude the section, let us note that the constant of proportionality between c_3 and CI^* depends on the number n of alternatives, whereas the one between ρ and GCI does not. Propositions 12 and 13 can also be represented graphically. A large number of randomly generated pairwise comparison matrices (or, equivalently, reciprocal relations) was created, and to each of them was associated a point on the Cartesian plane having as coordinates the corresponding values of the two consistency indices involved in proposition 12. As expected, all the points lie on a straight line. The same holds for proposition 13.

5.3 Numerical results

Despite the large number of consistency indices, there is not any comparative study in literature. The main question that such a study should answer is how different they are. The theoretical results derived in the previous section proved that some indices are proportional but we still know nothing about the others. Therefore, in this section, we are going to investigate the degree of agreement between consistency indices by means of numerical simulations. In doing so, we created a large sample of 10,000 randomly generated pairwise comparison matrices and, for each, we estimated the inconsistency by means different consistency indices. Graphical results, which are more than a simple curiosity (Anscombe's quartet *docet* [6]), on a smaller sample of 500 pairwise comparison matrices, are collected in figure 5.1. An expanded representation of the same scatter plots is reported in appendix A.

The numerical analysis was based on the Spearman index [108]. Therefore, it is a good idea to shortly recall its formulation and its importance. Imagine that the scores of two consistency indices are collected into two

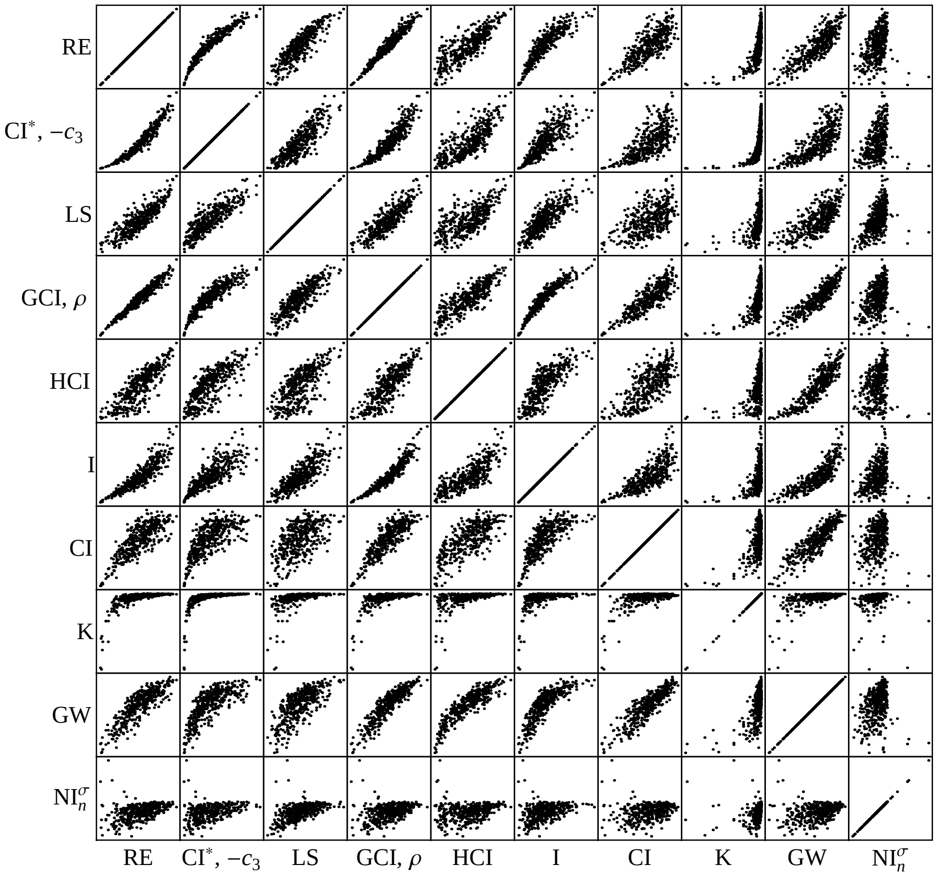


Figure 5.1: Pairwise scatterplots of consistency indices

Index	CI	CI^*, $-c_3$	LS	GCI, ρ	HCI	I	RE	K	GW	NI_n^σ
CI	1.	0.975	0.872	0.973	0.826	0.918	0.765	0.802	0.845	0.437
CI^*, $-c_3$	-	1.	0.854	0.935	0.789	0.848	0.724	0.874	0.803	0.440
LS	-	-	1.	0.853	0.690	0.828	0.566	0.734	0.735	0.563
GCI, ρ	-	-	-	1.	0.837	0.967	0.841	0.708	0.906	0.377
HCI	-	-	-	-	1.	0.805	0.707	0.598	0.893	0.290
I	-	-	-	-	-	1.	0.811	0.618	0.884	0.346
RE	-	-	-	-	-	-	1.	0.501	0.886	0.228
K	-	-	-	-	-	-	-	1.	0.585	0.462
GW	-	-	-	-	-	-	-	-	1.	0.303
NI_n^σ	-	-	-	-	-	-	-	-	-	1

Table 5.1: Spearman index computed on 10,000 randomly generated pairwise comparison matrices of order 6

sequences $\mathbf{y} = (y_1, \ldots, y_q)$ and $\mathbf{z} = (z_1, \ldots, z_q)$ where q is the number of preference relations that we are considering. Furthermore, given that each sequence can be associated to a vector of ranks, then we represent these latter vectors as $\bar{\mathbf{y}} = (\bar{y}_1, \ldots, \bar{y}_q)$ and $\bar{\mathbf{z}} = (\bar{z}_1, \ldots, \bar{z}_q)$. At this point, the Spearman index is calculated as

$$s(\mathbf{y}, \mathbf{z}) = 1 - \frac{6 \sum_{i=1}^{q} d_i^2}{n(n^2-1)} \in [-1, 1]$$

where $d_i = \bar{x}_i - \bar{y}_i$. However, as in the case of teaching a kid how to tie the shoestrings, perhaps it is more efficient to illustrate the Spearman index by means of an example than with words.

Example 13. Given two sequences of observations

$$\mathbf{y} = (1.2, 1.1, 1.9, 0.7, 1.3)$$
$$\mathbf{z} = (243, 451, 459, 12, 250)$$

we obtain the following two auxiliary vectors, containing the ranks

$$\bar{\mathbf{y}} = (3, 2, 5, 1, 4), \quad \bar{\mathbf{z}} = (2, 4, 5, 1, 3),$$

One can see that the highest rank, q, is given to the position associated with the maximum element of the original sequence. Analogously, the second highest rank, $q-1$, is given to the component of the auxiliary associated vector with the second largest component of the sequence, and so forth. Having done this, one obtains $s(\mathbf{y}, \mathbf{z}) = 7/10$. ♦

The main advantage of the Spearman index over the common correlation coefficient is that it estimates the co-monotonicity between variables regardless the nature of the correlation itself. The results are reported in Table 5.1.

Index	CI	CI^*, $-c_3$	LS	GCI, ρ	HCI	I	RE	K	GW	NI_n^σ
CI	1.	0.950	0.875	0.976	0.823	0.900	0.795	0.604	0.848	0.266
CI^*, $-c_3$	-	1.	0.864	0.917	0.779	0.832	0.695	0.504	0.754	0.287
LS	-	-	1.	0.861	0.693	0.835	0.577	0.463	0.717	0.448
GCI, ρ	-	-	-	1.	0.841	0.949	0.855	0.572	0.894	0.227
HCI	-	-	-	-	1.	0.798	0.724	0.464	0.889	0.143
I	-	-	-	-	-	1.	0.777	0.440	0.827	0.224
RE	-	-	-	-	-	-	1.	0.551	0.892	0.062
K	-	-	-	-	-	-	-	1.	0.573	0.026
GW	-	-	-	-	-	-	-	-	1.	0.133
NI_n^σ	-	-	-	-	-	-	-	-	-	1

Table 5.2: Linear correlation computed on 10,000 randomly generated pairwise comparison matrices of order 6

Nevertheless, for comparison, results obtained with the linear correlation are reported as well, in table 5.2.

Although the numerical results are easily interpretable, some remarks are indeed necessary. The first case regards the index of least squares LS, whose value achieved with both the Spearman index and the linear correlation could be interpreted as a lower bound of the true value. As said before, seemingly all the feasible optimization algorithms can fail to find an optimal solution in reasonable time and in the simulations, due to the number of matrices taken into account, a metaheuristic was used. Hence, it is reasonable to assume that its degree of agreement with the other indices, as computed here with the Spearman index, but also with the linear correlation, should be taken as a lower bound for the real agreement level, as, in some cases, the value of the index could not be the optimal one.

Another remark is about indices K and NI_n^σ which scored 'poorly'. First, the fact that they scored 'poorly' does not mean that they are bad indices, but just that they are likely to be very different from the others. The main difference is that they focus on the maximal local inconsistency and therefore they result to be invariant with respect to the variations of some elements of the pairwise comparison matrix. Particularly in the case of NI_n^σ, it has been shown [14] that this index can lead to undesired results [2].

5.4 Shortcoming

Seen from another perspective, some of the already cited indices [3, 27, 47, 53, 35, 85, 95, 106, 110] do not fairly estimate inconsistency. In fact, as already suggested in [17], it can be noted that, the further the judgments are from the indifference, the more difficult it is for the decision maker

[2] Such results were published in [14] after this dissertation had been submitted for pre-evaluation, and thus they cannot be included in this manuscript.

to reach a good consistency level. All along this paper, we will call this phenomenon *preference strength effect*. To go straight to the point, let us now consider table 5.3 where the results obtained from a sample of 10,000 randomly generated reciprocal relations of order 6

$$\bar{\mathbf{R}} = (\bar{r}_{ij})_{6\times 6} \quad \text{s.t. } \bar{r}_{ij} \in [r_{ij}^{-}, r_{ij}^{+}] \tag{5.23}$$

are summarized. The second column of the table indicates the interval of the real numbers in which the entries are randomly generated. The last column reports the percentage of matrices which would be considered inconsistent if Saaty's threshold $CR < 0.10$ [95] was applied (transformation (3.11) was employed to pass from reciprocal relations to pairwise comparison matrices).

#	r_{ij}^{-}	r_{ij}^{+}	%
1	0.4	0.6	0
2	0.3	0.7	14.82
3	0.2	0.8	87.69
4	0.1	0.9	98.44
5	0	1	99.75

Table 5.3: Percentage of inconsistent matrices

Therefore, if we take case number 1, the decision maker is always considered as acceptably consistent. Consequently, in a competitive framework he/she has no incentives to be rational.

To summarize, the closer the judgments are to the indifference ($r_{ij} \approx 0.5,\ a_{ij} \approx 1$), the less demanding it is to achieve a good level of consistency. Conversely, a decision maker with strong preferences is unfairly penalized. Although we use the CR to estimate inconsistency, such drawback is shared by most of the indices mentioned above. If these remarks were accepted, a direct consequence would be that inconsistency indices which do not take into account the preference strength effect are unable to fairly estimate the real reliability of a decision maker. If we want to overcome this shortcoming, we need a different approach to inconsistency evaluation. To this aim, in the next section we propose to partition the set of pairwise comparison matrices into equivalence classes taking into account the preference strength effect. The same operation will then be naturally extended to the case of reciprocal relations too.

5.5 Consistency equivalence classes

For notational convenience, let us denote by $\mathcal{A}_n$ the set of $n \times n$ pairwise comparison matrices. First of all, let us find a consistency preserving transformation f to be applied to the entries of a pairwise comparison matrix. More precisely, if $\mathbf{A} = (a_{ij})$ is consistent, we want to find a function $f : [1/9, 9] \to \mathbb{R}$ such that $f(\mathbf{A}) := (f(a_{ij}))$ is still a consistent pairwise comparison matrix. The consistency condition (3.6) for the pairwise comparison matrix $(f(a_{ij}))$ is

$$f(a_{ij}) = f(a_{ik})f(a_{kj}). \tag{5.24}$$

Since $\mathbf{A} = (a_{ij})$ is consistent, it is $a_{ij} = a_{ik}a_{kj}$ and from (5.24) the following Cauchy's functional equation is obtained [1]

$$f(a_{ik}a_{kj}) = f(a_{ik})f(a_{kj}) \tag{5.25}$$

Excluding the trivial solution and assuming continuity, since $a_{ij} > 0$, the general solution of (5.25) is

$$f(a_{ij}) = (a_{ij})^{\xi} \quad \xi \in \mathbb{R}_{>}. \tag{5.26}$$

As a consequence, if $\mathbf{A} = (a_{ij})$ is a consistent pairwise comparison matrix, then every $\hat{\mathbf{A}} = (\hat{a}_{ij})$ obtained from $\mathbf{A}$ by means of (5.26), $\hat{a}_{ij} = f(a_{ij}) = (a_{ij})^{\xi}$, is also consistent for every real value of ξ.

The general result stated above clearly requires that the scale $\frac{1}{9}, \ldots, 9$ originally proposed by Saaty is extended to the set of positive real numbers. If it is required that the entries of the pairwise comparison matrix remain in the interval $[\frac{1}{9}, 9]$, it is sufficient to conveniently bound the value of ξ in (5.26).

So far, the argumentation has been that an unbiased method for consistency evaluation should take into account only the *mutual coherence* of the judgments and thus should be, in a suitable way, independent from the preference strength effect. Starting from the consistency preserving transformation (5.26), let us define an equivalence relation on $\mathcal{A}_n$.

Definition 7 (Consistency–Equivalence for pairwise comparison matrix)**.** Let $\mathbf{A}, \mathbf{B} \in \mathcal{A}_n$, $\mathbf{A} = (a_{ij})$, $\mathbf{B} = (b_{ij})$. $\mathbf{A}$ is said to be consistency-equivalent to $\mathbf{B}$, $\mathbf{A} \sim \mathbf{B}$, if and only if $\exists \xi > 0$ [3] s.t. $a_{ij} = b_{ij}^{\xi}\ \forall i, j$.

Proposition 14. *Consistency–Equivalence $\sim$ is an equivalence relation.*

[3] Positivity of ξ is required only to avoid preference reversal

Proof.

1. *Reflexivity.* Clearly, with $\xi = 1$, it is $\mathbf{A} \sim \mathbf{A}$.

2. *Symmetry.* From $a_{ij} = b_{ij}^{\xi}$, it is $b_{ij} = a_{ij}^{\frac{1}{\xi}}$; then $\mathbf{A} \sim \mathbf{B} \Rightarrow \mathbf{B} \sim \mathbf{A}$.

3. *Transitivity.* Let $\mathbf{C} \in \mathcal{A}_n$, $\mathbf{C} = (c_{ij})$. If $a_{ij} = b_{ij}^{\xi}$ and $b_{ij} = c_{ij}^{\tau}$, then $a_{ij} = c_{ij}^{\xi\tau}$. Therefore $(\mathbf{A} \sim \mathbf{B}$ and $\mathbf{B} \sim \mathbf{C}) \Rightarrow \mathbf{A} \sim \mathbf{C}$.

□

As a consequence of Proposition 14, set $\mathcal{A}_n$ is partitioned by $\sim$ into equivalence classes. Let $\mathcal{A}_n / \sim$ be the quotient set. To overcome the above mentioned drawback, we propose to consider equivalent, from the point of view of consistency, all the pairwise comparison matrices in the same equivalence class $\Theta \in \mathcal{A}_n / \sim$. Therefore, we assign to all the pairwise comparison matrices $\mathbf{A} \in \Theta$ the same numerical value to quantify their consistency. We denote this value by $C(\mathbf{A})$ and we will call it *inconsistency level* of $\mathbf{A}$. Since all the pairwise comparison matrices in Θ share the same inconsistency level, we can denote it by C_Θ. Using this notation, it is $\mathbf{A} \in \Theta \Rightarrow C(\mathbf{A}) = C_\Theta$.

We propose to define C_Θ in the following way. We choose a particular pairwise comparison matrix $\hat{\mathbf{A}} \in \Theta$ as representative for the entire equivalence class, we compute its consistency ratio $CR(\hat{\mathbf{A}})$ and we then assign to all the matrices in Θ the inconsistency level

$$C_\Theta = CR(\hat{\mathbf{A}}) \ . \tag{5.27}$$

We also want to propose a natural method to find the matrix $\hat{\mathbf{A}}$ representing its whole class. We start defining an index for the preference strength

$$PS_{\mathbf{A}} = \frac{\sum^* a_{ij}}{\frac{n(n-1)}{2}} \in [0,1] \ , \tag{5.28}$$

where the sum $\sum^*$ is extended to the $n(n-1)/2$ values $a_{ij} \geq 1$ corresponding to all the pairwise comparisons with $i \neq j$ and where the possible value $a_{ij} = a_{ji} = 1$ is taken only once.

Example 14. Let us consider the following pairwise comparison matrix with the entries taken into account in $\sum^*$ that are circled

$$\mathbf{A} = \begin{pmatrix} 1 & \textcircled{3} & \textcircled{1} & 1/2 \\ 1/3 & 1 & \textcircled{4} & \textcircled{5} \\ 1 & 1/4 & 1 & 1/3 \\ \textcircled{2} & 1/5 & \textcircled{3} & 1 \end{pmatrix} .$$

We can therefore obtain $PS_{\mathbf{A}} = (3+1+4+5+2+3)/6 = 3$. ♦

We propose to choose $\hat{\mathbf{A}}$ with average preference strength (5.28) as the matrix representing its equivalence class Θ. In Saaty's approach, the average value of $PS_{\mathbf{A}}$ is the average value of the interval $[1, 9]$. Thus,

$$(\overline{PS}_{\mathbf{A}}) = 5. \tag{5.29}$$

The fact that the matrix representative of each class was chosen in this particular way has a justification. Often, to discriminate between matrices of sufficient consistency and matrices which are not consistent enough, a threshold is imposed. For instance, according to Saaty's approach, the value of CR must be smaller than 0.10 to allow a matrix to pass the consistency test. The value of RI in the formula of CR is usually computed on a very large set of random matrices whose average preference strength tends to 5, as the sample size grows. Therefore, it seems fair that the representative matrix share the same preference strength with the matrices to which it is compared. The preference strength effect is so neutralized and preferences expressed in a pairwise comparison matrices can be fairly compared to some randomly generated ones in order to test their consistency. Throughout this section, we shall consider the value $PS_{\mathbf{A}} \in [1, 9]$ as an estimator of the preference strength of a decision maker on the set of alternatives X. Obviously, if $PS_{\mathbf{A}} = 1$ the decision maker expresses full indifference while, if $PS_{\mathbf{A}} = 9$, he/she has the strongest possible degree of preference between the alternatives. Furthermore, in light of what we have shown up to this point, we suggest that the matrix $\hat{\mathbf{A}}$ with $PS_{\hat{\mathbf{A}}} = 5$ represents the equivalence class to which it belongs.

The same findings exposed above can be extended to the case of reciprocal relations. Let $\mathbf{R}$ be an additively consistent reciprocal relation, i.e. satisfying (3.1). The simplest way to obtain the consistency preserving transformation $f : [0, 1] \to \mathbb{R}$ suitable for reciprocal relations is to apply (3.10) to the consistency preserving transformation (5.26) previously derived for pairwise comparison matrices. It results

$$f(r_{ij}) = 0.5 + \xi(r_{ij} - 0.5). \tag{5.30}$$

If $\mathbf{R} = (r_{ij})$ is an additively consistent reciprocal relation, then every $\hat{\mathbf{R}} = (\hat{r}_{ij})$ obtained from $\mathbf{R}$ by means of (5.30), $\hat{r}_{ij} = f(r_{ij}) = 0.5 + \xi(r_{ij} - 0.5)$, is also an additively consistent reciprocal relation for every real value of ξ.

Clearly, function (5.30) could also be derived analogously to (5.26), i.e. by solving Cauchy's functional equation corresponding to additive consistency (3.1),

$$f\left((r_{ik} - 0.5) + (r_{kj} - 0.5)\right) = f(r_{ik} - 0.5) + f(r_{kj} - 0.5)\ . \tag{5.31}$$

The extension to additively consistent reciprocal relations of the results exposed above for pairwise comparison matrices is presented very briefly in the following, as they correspond to the already described ones.

Definition 8 (Consistency-Equivalence for reciprocal relations)**.** Let $\mathcal{R}_n$ denote the set of the n-dimensional reciprocal relations. Let $\mathbf{R}, \mathbf{S} \in \mathcal{R}_n$, $\mathbf{R} = (r_{ij})$, $\mathbf{S} = (s_{ij})$. $\mathbf{R}$ is said consistency-equivalent to $\mathbf{S}$, $\mathbf{R} \sim \mathbf{S}$, if and only if $\exists \xi > 0$ s.t. $r_{ij} = 0.5 + \xi(s_{ij} - 0.5)\ \forall i, j$.

Proposition 15. *Consistency-Equivalence for reciprocal relations $\sim$ is an equivalence relation.*

Proof. Similar to proof of Proposition 14. □

The set or reciprocal relations or order n, $\mathcal{R}_n$, is partitioned by $\sim$ in equivalence classes, and $\mathcal{R}_n/\sim$ is the quotient set. All the reciprocal relations in the same equivalence class Φ share the same inconsistency level C_Φ.

Instead, if we consider a multiplicatively consistent reciprocal relation, the consistency preserving transformation is the following,

$$\hat{r}_{ij} = \frac{\left(\frac{r_{ij}}{1-r_{ij}}\right)^{\xi}}{1 + \left(\frac{r_{ij}}{1-r_{ij}}\right)^{\xi}} \quad i, j = 1, \ldots, n, \quad \xi \in \mathbb{R}_{>}. \tag{5.32}$$

As observed for (5.26), the general validity of the results presented above requires the use of an open scale, as it was assumed in [9]. Nevertheless, if it is required that the entries of $\hat{\mathbf{R}}$ remain in the interval $[0, 1]$, it is sufficient to conveniently bound the value of ξ in (5.30).

In [59] a related problem is addressed: the function

$$\varphi(x) = \frac{1}{1 + 2a} \cdot x + \frac{a}{1 + 2a} \tag{5.33}$$

was used to rescale into the interval $[0, 1]$ an additively consistent reciprocal relation with entries in the interval $[-a, 1 + a]$. Note that function (5.33) is a special case of (5.30), obtained for $\xi = \frac{1}{2a+1}$.

Let us end the section noting that, analogously to what proposed for pairwise comparison matrices, it is possible to define the preference strength of a reciprocal relation,

$$PS_{\mathbf{R}} = \frac{\sum^{*} r_{ij}}{\frac{n(n-1)}{2}} \in [0.5, 1] , \tag{5.34}$$

where the sum $\sum^{*}$ is extended to the $n(n-1)/2$ values $r_{ij} \geq 0.5$ corresponding to all the pairwise comparisons with $i \neq j$ and where the possible value $r_{ij} = r_{ji} = 0.5$ is taken only once. Consequently, the average value, analogous to (5.29), is $\overline{PS}_{\mathbf{R}} = 0.75$.

Example 15. Let us consider a simple and merely illustrative example in which two decision makers are involved and express their preferences by means of the following reciprocal relations

$$\mathbf{R}_1 = \begin{pmatrix} 0.5 & 0.8 & 1 & 1 \\ 0.2 & 0.5 & 0.9 & 0.9 \\ 0 & 0.1 & 0.5 & 0.8 \\ 0 & 0.1 & 0.2 & 0.5 \end{pmatrix} \quad \mathbf{R}_2 = \begin{pmatrix} 0.5 & 0.6 & 0.4 & 0.6 \\ 0.4 & 0.5 & 0.6 & 0.4 \\ 0.6 & 0.4 & 0.5 & 0.5 \\ 0.4 & 0.6 & 0.5 & 0.5 \end{pmatrix}.$$

Applying (3.11) to $\mathbf{R}_1$ and $\mathbf{R}_2$ we obtain the corresponding pairwise comparison matrices $\mathbf{A}_1$ and $\mathbf{A}_2$, respectively. At this point, using the traditional approach suggested by Saaty, we calculate the consistency ratios $CR(\mathbf{A}_1) \approx 0.119665$ and $CR(\mathbf{A}_2) \approx 0.0824676$, which entail that the second decision maker has been sufficiently consistent while the first one has been not. Using our approach these results are inverted because, using (5.27), we obtain $C(\mathbf{A}_1) \approx 0.0936079$ and $C(\mathbf{A}_2) \approx 1.75276$. Let us conclude the example saying that there is no wonder to see that, according to the consistency levels, $\mathbf{A}_2$ has such very high inconsistency level, because the preferences, as expressed by the second decision maker, contain two cycles, i.e. $x_1 \succ x_2 \succ x_3 \succ x_1$ and $x_2 \succ x_3 \sim x_4 \succ x_2$ (thus revealing intransitivity), while those provided by the first decision maker are strictly acyclical. ♦

5.6 An excursus on preference aggregation

Pairwise comparison matrices and reciprocal relations are appealing tools for group decision making too. Let us consider a group decision problem with the usual set of n alternatives $X = \{x_1, \ldots, x_n\}$ ($n \geq 2$) and a set of m decision makers $D = \{d_1, \ldots, d_m\}$ ($m \geq 2$). If each decision maker expresses his/her preferences by means of pairwise comparisons on the alternatives, then the preferences of the m decision makers must be taken into account in order to obtain an outcome that can be either a collective matrix or a collective weight vector. The process yielding to a collective matrix/vector and having as input m individual matrices/vectors is called *preference aggregation*.

A large number of methods have been proposed for aggregating preferences, e.g. [2, 50, 89, 116], but few take into account the reliability/consistency of the decision maker's preferences [24, 42]. This section introduces a method which weighs the importance of decision makers according to their inconsistency levels as defined in the previous section. It is assumed that the more inconsistent his/her judgments are, the more irrational the decision maker is. Therefore, reliability of expressed preferences is strictly related with their consistency. According to this assumption, we are going to aggregate preferences using the information provided by the inconsistency estimations. Namely, if the estimations of the reliability (then the consistency) of the decision makers is partially affected by the strength of their preferences, then every aggregation process based on them yields to a solution which is closer to the indifference than it should reasonably be. In fact, decision makers providing matrices with entries close to the indifference would be considered more rational than they really are. This is why inconsistency levels (5.27), instead of inconsistency indices, should be taken into account in the aggregation process.

First of all, it must be said that it is possible to aggregate preferences either by aggregating matrices or priority vectors. A work on this issue is that by Forman and Peniwati [50] in which they distinguish between (i) aggregation of matrices, which is considered as an aggregation of individual judgments (AIJ), and (ii) aggregation of vectors which, in contrast, is seen as an aggregation of individual priorities (AIP). Our idea applies to preferences expressed as preference relations or priority vectors.

Given m pairwise comparison matrices $\mathbf{A}_1, \ldots, \mathbf{A}_m$, each associated to a decision maker, we derive their inconsistency levels $C(\mathbf{A}_1), \ldots, C(\mathbf{A}_m)$ referring to (5.27). Then we define the weight p_k to be assigned to the k-th decision maker as a function of his/her inconsistency level $C(\mathbf{A}_k)$,

$$p_k = \frac{1}{(\phi \cdot C(\mathbf{A}_k)) + 1}, \tag{5.35}$$

where $\phi \in [0, +\infty[$ is a parameter which stress the role of the inconsistency level. Weights can finally be normalized,

$$\sum_{k=1}^{m} p_k = 1. \tag{5.36}$$

Matrix aggregation

As already mentioned, preferences can be aggregated alternatively at two stages. Here we propose to aggregate the m pairwise comparison matrices $\mathbf{A}_1, \ldots, \mathbf{A}_m$ by means of a weighted geometric mean,

$$a^{\star}_{ij} = \prod_{k=1}^{m} a_{ij}^{p_k} \quad i, j = 1, \ldots, n, \tag{5.37}$$

where $\mathbf{A}^{\star} = (a^{\star}_{ij})$ is the collective pairwise comparison matrix resulting after the aggregation.

Vector aggregation

It is also possible to elicitate m priority vectors thanks to one of the several methods proposed in literature and then aggregate them. In this second case, we define $\mathbf{w}^\star = (w_1^\star, \dots, w_n^\star)$ to be the priority vector which synthesize all the m priority vectors of the m decision makers. According to [50], we can proceed in two different ways. The first method is the weighted geometrical mean

$$w_i^\star = \prod_{k=1}^{m} w_i^{p_k} \quad i = 1, \dots, n, \tag{5.38}$$

and the second is the weighted arithmetical mean

$$w_i^\star = \sum_{k=1}^{m} p_k w_i \quad i = 1, \dots, n. \tag{5.39}$$

Although (5.38) and (5.39) lead to different results, in [50] they are both considered reasonable and reliable. However, it does not seem that the aggregation of preferences under the form of priority vector should be restricted to these two methods. Namely, arithmetic and geometric means are the most popular aggregation methodologies but several others can indeed be employed.

Example 16. Let us assume that four decision makers provide the following pairwise comparison matrices

$$\mathbf{A}_1 = \begin{pmatrix} 1 & 3 & 6 & 9 \\ 1/3 & 1 & 5 & 8 \\ 1/6 & 1/5 & 1 & 3 \\ 1/9 & 1/8 & 1/3 & 1 \end{pmatrix} \quad \mathbf{A}_2 = \begin{pmatrix} 1 & 3 & 2 & 2 \\ 1/3 & 1 & 1/4 & 1 \\ 1/2 & 4 & 1 & 2 \\ 1/2 & 1 & 1/2 & 1 \end{pmatrix}$$

$$\mathbf{A}_3 = \begin{pmatrix} 1 & 1/2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 1/3 & 1/4 & 1 & 2 \\ 1/4 & 1/3 & 1/2 & 1 \end{pmatrix} \quad \mathbf{A}_4 = \begin{pmatrix} 1 & 1/2 & 3 & 6 \\ 2 & 1 & 2 & 1/2 \\ 1/3 & 1/2 & 1 & 2 \\ 1/6 & 2 & 1/2 & 1 \end{pmatrix}$$

According to the method exposed above, the algorithm leading to a collective pairwise comparison matrix is the following,

step 1 To each pairwise comparison matrix $\mathbf{A}_k$ we associate, by means of (5.26), the pairwise comparison matrix $\hat{\mathbf{A}}_k$ representative of its equivalence class.

step 2 For $k = 1, 2, 3, 4$, we compute the CR of $\hat{\mathbf{A}}_k$, thus obtaining the inconsistency levels $C(\mathbf{A}_k)$ for $\mathbf{A}_k$,

$C(\mathbf{A}_1) = 0.0544351$, $C(\mathbf{A}_2) = 0.158733$, $C(\mathbf{A}_3) = 0.112451$, $C(\mathbf{A}_4) = 1.0876$.

step 3 By means of (5.35) and (5.36), we map inconsistency levels into normalized weights p_k. In this example, we assume $\phi = 1$ and we obtain $p_1 = 0.297359$, $p_2 = 0.270594$, $p_3 = 0.281852$, $p_4 = 0.150195$.

step 4 We aggregate $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4$ with the aid of (5.37). The so obtained collective pairwise comparison matrix is

$$\mathbf{A}^\star \approx \begin{pmatrix} 1 & 1.38332 & 3.30359 & 4.48514 \\ 0.722897 & 1 & 1.63917 & 2.27933 \\ 0.302701 & 0.610066 & 1 & 2.25628 \\ 0.222959 & 0.438726 & 0.443208 & 1 \end{pmatrix}. \quad \blacklozenge$$

5.7 Discussion

When making use of the various indices observed and proven proportional in this chapter, it is important that the applied mathematician be aware of their equivalence. This avoids redundancy in the consideration of evidence for consistent preferences, and allows any existing results proven for one index to apply directly to other indices which are proportional to it.

Furthermore, if two indices are, instead, not proportional it is still relevant to approximately know how much co-monotone they are. It is very difficult to state if, in general, an index is better than another; more likely some instances can be brought up in order to show that, in some very special cases one index is more reasonable than another. All in all, it cannot be hidden that the choice of an inconsistency index is a 'religious-like' decision, i.e. everyone chooses the index in which he/she believes the most. Nevertheless, having a good insight to the inconsistency indices may help the decision maker to figure out how far different 'religions' are.

Finally, the method proposed in the last part of this chapter for evaluating the consistency/reliability of a decision maker aims to overcome what could be consider a commonly shared shortcoming of most of the known consistency indices, i.e. they do not take into account the *preference strength effect*. It is interesting to observe that, seemingly, the elegant index proposed by Barzilai in [9] is the only one which is invariant with respect to (5.26), thus avoiding the above mentioned shortcoming.

All in all, this chapter has possibly cleared some of the fog in the maze of consistency indices and the relevance of its results can naturally be related to what originally stated in section 3.5, especially to the part in which the assumption that the decision maker is fully rational is dropped.

Chapter 6

Incomplete preference relations

> We can try to avoid making choices by doing nothing, but even that is a decision.
>
> Gary Collins

Having, and manipulating, a complete and consistent preference relation means dealing with rich and reliable information and therefore, it represents the most desirable situation in a decision making problem with preference relations. However, sometimes, it is not possible for the decision maker to elicitate all his/her pairwise preferences under the form of pairwise comparisons and therefore, it is nowadays common practice to accept that some entries of a pairwise comparison matrix be missing [18].

As a matter of fact, in complex problems, it may happen that the decision maker cannot complete a preference relation due to the nature of the problem, his/her incapacity in comparing two alternatives of different nature, and so forth [16].

All in all, the range of reasons is wide and the main problem is how to derive a reliable priority vector when there is not full information about the preferences on alternatives. Several methods have been implemented to face this problem and, despite their diversities, and bearing in mind the fact that the final scope of preference relations is that of allowing an estimating a priority vector, they can be classified into two main families, according to the following diagram, where, for sake of simplicity, only the case of pairwise comparison matrices is taken into account, and where $\mathbf{A}$ and $\dot{\mathbf{A}}$ are a complete and an incomplete pairwise comparison matrix respectively. In words, given an incomplete pairwise comparison matrix $\dot{\mathbf{A}}$, the decision maker can proceed in one of the two following alternative ways (see diagram in figure 6.1):

- Complete the matrix by means of the information provided by the existing comparisons, ①. This operation is usually carried out following some principles of consistence, in the sense that the missing comparisons should be as coherent as possible with the existing ones. Having done this, it is possible to estimate the priority vector by means of one of the methods proposed in literature [72], ②

- Estimate directly the priority vector by means of some especially modified algorithms which work even is some comparisons are missing, i.e. ③.

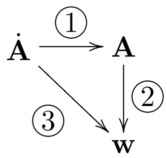


Figure 6.1: Obtaining $\mathbf{w}$ from $\dot{\mathbf{A}}$

Given an incomplete preference relation, it is possible to compute the corresponding priority vector for the alternatives in two ways. The first one is to directly use one of the few methods proposed in the literature for incomplete reciprocal relation [47, 59, 123, 124]. The second one is to use, as explained in section 3, one of the methods proposed for the same problem in the framework of pairwise comparison matrices [55, 56, 106, 115]. For the numerical simulations presented in this section we have chosen three methods of the first kind and four of the second one. The scope of the simulations is that of estimating how good different methods are.

6.1 Reconstruction of incomplete preference relations

The simulations presented in section 6.2 considered three methods, denoted in the following by M5–M7, proposed in the literature to compute the missing entries $\dot{r}_{ij}$ of an incomplete reciprocal relation, as well as four methods, M1–M4, proposed to compute the missing entries $\dot{a}_{ij}$ of an incomplete pairwise comparison matrix. The methods considered for the analysis are listed below. For easy reference, a tag and a name are assigned to each method,

M1 Least Squares method 1

M2 Least Squares method 2

M3 Harker

M4 Shiraishi et al. c_3

M5 Xu goal programming

M6 Xu eigenproblem

M7 Fedrizzi-Giove ρ

In the following, methods M1–M7 are briefly described, each time referring to the original papers.

Method M1 – Least squares 1

The priorities w_i of a complete pairwise comparison matrix $\mathbf{A}$ can be computed by solving, with respect to $w_1, ..., w_n$, the problem (see [26])

$$\underset{w_1,\ldots,w_n}{\arg\min} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \left(a_{ij} - \frac{w_i}{w_j} \right)^2 \quad \text{s.t.} \quad \sum_{i=1}^{n} w_i = 1, \quad w_i > 0. \tag{6.1}$$

If $\dot{\mathbf{A}}$ is incomplete, the only change needed in (6.1) is to skip, in the objective function, the terms $\dot{a}_{ij}$ corresponding to the missing comparisons [115].

Method M2 – Least squares 2

In some cases, method M1 can have serious problems in numerical computation, due to the presence of the variable w_j in the denominator. Therefore it was modified, obtaining

$$\underset{w_1,\ldots,w_n}{\arg\min} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} (a_{ij} w_j - w_i)^2 \quad \text{s.t.} \quad \sum_{i=1}^{n} w_i = 1, \quad w_i > 0. \tag{6.2}$$

Details can be found in [26, 115].

Method M3 – Harker

This method, proposed by Harker [56], is not based on the optimization of an objective function, but refers to the eigenvector approach of Saaty. Practically, it extends Saaty's approach to non-negative quasi-reciprocal matrices, in order to apply it to the case of incomplete preferences. The procedure requires to construct the auxiliary matrix $\mathbf{C} = (c_{ij})_{n\times n}$ as follows

$$c_{ij} = \begin{cases} 1 + m_i, & \forall i = j \\ \dot{a}_{ij}, & \forall i \neq j \text{ and } \dot{a}_{ij} \text{ is not missing} \\ 0, & \dot{a}_{ij} \text{ is missing} \end{cases}$$

where m_i is the number of missing comparisons on row i. Having done this, the priority method can be estimated by means of the eigenvector method.

Method M4 – Shiraishi et al. c_3

The name c_3 refers to the coefficient of λ^{n-3} of the characteristic polynomial of the matrix $\mathbf{A}$. Shiraishi et al. [107] observed that c_3 can be considered an index of consistency for a pairwise comparison matrix (see chapter 5). Then, in order to maximize the consistency of $\dot{\mathbf{A}}$, the authors considered the m missing comparisons as variables $\alpha_1, \dots, \alpha_m$ and proposed to maximize c_3 as a function of these variables, thus obtaining the optimal values as the solution of

$$\underset{\alpha_1,\dots,\alpha_m}{\arg\max}\, c_3(\alpha_1, \dots, \alpha_m) \quad \text{s.t. } \alpha_1, \dots, \alpha_m > 0 \tag{6.3}$$

Example 17. An example is here presented in order to make the explanation as clear as possible. First, we present a preference relation $\dot{\mathbf{A}}$

$$\dot{\mathbf{A}} = \begin{pmatrix} 1 & 4 & 5 & \dot{a}_{14} \\ 1/4 & 1 & 1/3 & 1/6 \\ 1/5 & 3 & 1 & 2 \\ 1/\dot{a}_{14} & 6 & 1/2 & 1 \end{pmatrix}.$$

Its missing comparison is estimated by (6.3), which returns $\max(c_3) \approx -51.1742$ with $\dot{a}_{14} \approx 2.58199$. The plot in figure 6.2 shows the relation between the value of $\dot{a}_{14}$ and index c_3. ♦

Method M5 – Xu goal programming

In [123], Xu proposed a model, based on goal programming to calculate the priority vector of an incomplete reciprocal relation. This method was already presented as LOP2 in (4.21). However, to obtain satisfactory results, it was necessary to remove, from the proposed model, a normalization constraint, which conflicted with the optimization problem [45].

Method M6 – Xu eigenproblem

In his second proposal, Xu [124] developed a method for incomplete reciprocal relations, similar to M3. In [124] the priority vector $\mathbf{v}$ is calculated by solving a system of equations which resembles the auxiliary eigenproblem developed by Harker in the framework of pairwise comparison matrices.

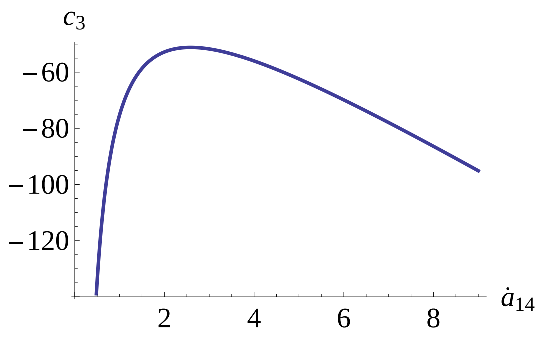


Figure 6.2: c_3

Method M7 – Fedrizzi–Giove ρ

The method proposed in [47] for incomplete reciprocal relations, considers, as in M4, the m missing comparisons as variables $\alpha_1, \dots, \alpha_m$ and computes their optimal values by minimizing the (in)consistency index $\rho(\alpha_1, \dots, \alpha_m)$ based on the condition of additive consistency for reciprocal relations (3.1),

$$\underset{\alpha_1,\dots,\alpha_m}{\arg\min} \sum_{i<j<k=1}^{n} (r_{ik} + r_{kj} - r_{ij} - 0.5)^2 \Big/ \binom{n}{3} \quad \text{s.t. } 0 \leq x_j \leq 1. \tag{6.4}$$

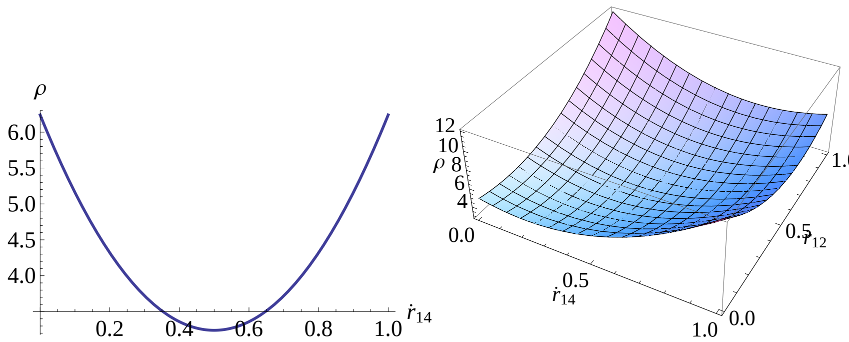
Example 18. Let's take into account the following incomplete reciprocal relation

$$\dot{\mathbf{R}} = \begin{pmatrix} 0.5 & 0.6 & 0.3 & \dot{r}_{14} \\ 0.4 & 0.5 & 0.2 & 0.7 \\ 0.7 & 0.8 & 0.5 & 0.4 \\ 1 - \dot{r}_{14} & 0.3 & 0.6 & 0.5 \end{pmatrix}.$$

We can elicitate the optimal value of $\dot{r}_{14}$ by applying minimization (6.4) and obtain $\rho = 3.24$ with $\dot{r}_{14} = 0.5$. A plot of the function is here presented Figure 6.3a. To go further, let us imagine that entries r_{12} and r_{21} were also missing. Then we would have to solve a quadratic programming problem in two variables whose plot is showed in figure 6.3b. The optimal value of the objective function is $\rho = 2.88$ with $\dot{r}_{12} = 0.4$ and $\dot{r}_{14} = 0.4$. ♦

6.2 Numerical simulations

The objective of the numerical simulations presented in this section is to study how well the seven methods mentioned above are able to reconstruct an incomplete preference relation and to compare the results obtained in the considered cases.



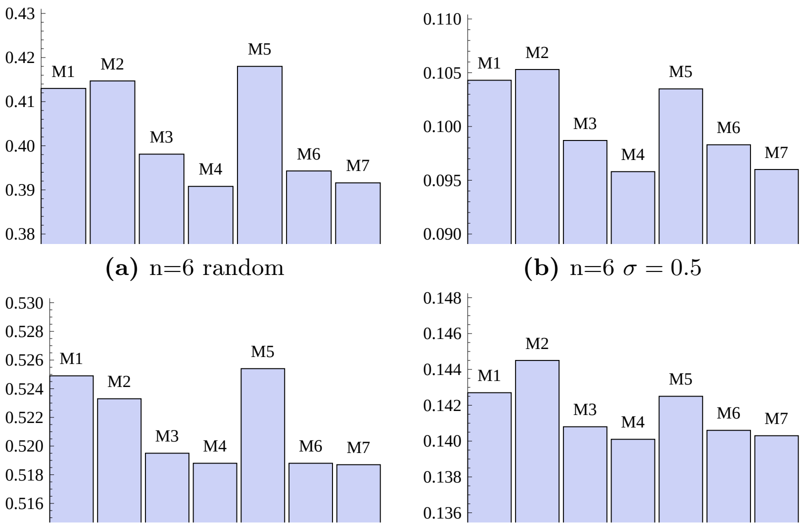
(a) one missing comparison **(b)** two missing comparisons

Figure 6.3: ρ

It should be preliminarily noted that methods M1, M2, M3, M5 and M6 give, as a result, not directly the missing comparisons, but a priority vector. Nevertheless, each missing comparison can be estimated by means of characterizations (3.2), (3.4), and (3.7). Thus, for every considered method, the final result is a complete (reconstructed) preference matrix.

The results of the various methods are compared on the basis of the consistency of the reconstructed preference relations. As seen and discussed before, many different methods have been proposed to measure the inconsistency level of a preference relation. We chose to use the most old and popular: the consistency ratio, CR. The smaller is the CR, the more consistent is the preference relation, with $CR = 0$ only for fully consistent matrices. We assume that the more consistent is a reconstructed matrix, the better is the reconstruction method, as the computed missing comparisons are coherent with the known entries. Since the CR can be calculated for pairwise comparison matrices only, functions f, g, and their inverses (see chapter 3) are used to pass from one approach to another.

In order to study the performances of the methods in different consistency situations, we use two classes of matrices: random matrices — which are very inconsistent — and consistent matrices slightly modified by a Gaussian noise. The results of the simulations are summarized in table 6.1 and in figures 6.4 (a)–(d). We proceeded as follows. First, we randomly generated 1,000 pairwise comparison matrices of order 6. For each matrix we randomly chose three comparisons to be considered missing; due to the reciprocity, six entries of the matrix are missing. We applied the seven methods, obtaining, for each matrix, seven different reconstructed matrices. We computed the CR of the obtained matrices. Finally, we computed, for each method, the average value of the CR on the 1,000 preference relations. We reported this average value in the first column of table 6.1. The values in the second col-



(a) n=6 random

(b) n=6 $\sigma = 0.5$

(c) n=10 random

(d) n=10 $\sigma = 0.5$

Figure 6.4: Consistency Ratio of the reconstructed preference relations

umn were obtained in the same way, with the only difference that, instead of random matrices, we used randomly generated consistent matrices which were then modified with Gaussian random noise. The parameter σ represents the standard deviation of the Gaussian distribution. Also in this case the reported average values are obtained from 1,000 simulations.

Columns three and four of table 6.1 are obtained with the same kind of simulations, but with 10×10 matrices. Column two is interesting because the CR values are close to Saaty's threshold 0.1 for acceptability and some methods succeeded in respecting it, while others did not.

The last row of table 6.1 reports the average CR of the original matrices, before having considered three entries (plus the reciprocals) as missing. Note that, as expected, all the considered methods improve in average the consistency of the original complete matrices.

The results of the simulations are shown in figure 6.4 (a)–(d) by means of bar charts.

It can be observed that the best results are obtained by using the optimization methods M4 and M7, where the missing entries are directly computed, followed by the methods where the priority weights are first computed. Good results are also obtained by using M3 and M6, which are methods based on the eigenvalue approach. The two least squares based methods M1 and M2 form the last group, together with M5.

Some more experiments, which are not reported here, showed that, by varying the order of the preference relations and the number of missing

	n = 6		n = 10	
	random	noise $\sigma = 0.5$	random	noise $\sigma = 0.5$
M1	0.4130	0.1043	0.5249	0.1427
M2	0.4147	0.1053	0.5233	0.1445
M3	0.3981	0.0987	0.5195	0.1408
M4	0.3908	0.0958	0.5188	0.1401
M5	0.4180	0.1035	0.5254	0.1425
M6	0.3943	0.0983	0.5188	0.1406
M7	0.3916	0.0960	0.5187	0.1403
original	0.5599	0.1395	0.5621	0.1531

Table 6.1: Consistency ratio of the reconstructed preference relations

comparisons, the relative performances of the methods do not significantly change. Given this stability with respect of these parameters, we have omitted to report other tables and bar charts.

Example 19 (Example of reconstruction)**.** Let's take into account an incomplete pairwise comparison matrix. We consider the matrix on page 14 in [102] with some entries missing so that, such a matrix has the following form with a_{12}, a_{28}, a_{37} and their reciprocals being missing entries too:

$$\dot{\mathbf{A}} = \begin{pmatrix} 1 & \dot{a}_{12} & 3 & 7 & 6 & 6 & 1/3 & 1/4 \\ 1/\dot{a}_{12} & 1 & 1/3 & 5 & 3 & 3 & 1/5 & \dot{a}_{28} \\ 1/3 & 3 & 1 & 6 & 3 & 4 & \dot{a}_{37} & 1/5 \\ 1/7 & 1/5 & 1/6 & 1 & 1/3 & 1/4 & 1/7 & 1/8 \\ 1/6 & 1/3 & 1/3 & 3 & 1 & 1/2 & 1/5 & 1/6 \\ 1/6 & 1/3 & 1/4 & 4 & 2 & 1 & 1/5 & 1/6 \\ 3 & 5 & 1/\dot{a}_{37} & 7 & 5 & 5 & 1 & 1/2 \\ 4 & 1/\dot{a}_{28} & 5 & 8 & 6 & 6 & 2 & 1 \end{pmatrix}$$

Different methods complete the matrix with different degrees of consistency. In details, They perform as follows: $M1 = 0.0785$, $M2 = 0.077$, $M3 = 0.0746$, $M4 = 0.07463$, $M5 = 0.0843$, $M6 = 0.0777$, $M7 = 0.07467$. ♦

6.3 Discussion

In this chapter we have considered and compared seven reconstruction methods, but they are not the only ones, and so the analysis could be extended to other methods.

Moreover, the CR could be substituted by any other index proposed in the literature and discussed in chapter 5. Here we have chosen the CR of

Saaty for two main reasons: the first one is that it is the oldest and most popular one; the second reason is its *neutrality*: some other possible consistency indices, as c_3 and ρ, are precisely the objective function of the corresponding reconstruction method, and would unfairly favorite the method itself. Note that each optimization-based method implicitly defines a consistency index and vice versa. Nevertheless, alternative ways for evaluating the various reconstruction methods could be taken into account.

With all the above mentioned limits, this chapter has been useful to see which reconstruction methods performs better in some given situation. Furthermore, it can be easily connected with the theory of preference relations whenever the assumptions that the preference is complete is dropped.

Chapter 7

Fuzzy adjacency relations

Social Network Analysis (SNA) is a relatively new and still developing subject that focuses on the study of social relationships [103, 122]. It can be seen as a branch of the broader discipline named network analysis [80] whose main goal is the study of the relationships between objects belonging to one or more reference sets. It is worth noting that, during the very recent period, the study of SNA has attracted the interest of scholars in fields of study such as decision making [109] and systems science [21].

The approach proposed in this chapter takes advantage of the ability of fuzzy relations [66, 140] to model imprecision permeating the relationships between the actors in the network, and of the OWA functions [137, 139] to move continuously from non-compensatory to full-compensatory situation and characterizing therefore the attitude of the actors to connect each other. The relevance of this chapter for the issue of group decision making, and thus its connection with with the rest of the thesis, shall be clarified as some optimization problems will be presented. This chapter is outlined as follows. Section 7.1 offers a presentation of SNA and introduces the adjacency matrix, which is the main tool to perform the analysis. In section 7.2 it is shown that adjacency relations can be valued (cardinal) relations and that fuzzy adjacency relations are simply a special case of valued relations. Having presented that, in section 7.3 fuzzy m-ary adjacency relations are defined and a method based on aggregation functions for estimating them is presented. It will be claimed that OWA functions satisfy some reasonable properties and that they can be employed as suitable aggregation functions to increase the dimension of the analysis. Namely, fuzzy m-ary adjacency relations are obtained by aggregating fuzzy binary adjacency relations by means of OWA functions. Section 7.4 proposes some optimization models, based on fuzzy m-ary adjacency relations, which can be interpreted in terms of finding the 'most consensual subset' of a set of decision makers. Section 7.5 contains a commented example and, finally, section 7.6 points out some concluding remarks. All in all, the scope of this chapter is that of bridging the gap between SNA and group decisions, and to show that these these two research themes are quite close to each other and. In fact some tools from SNA can be used to estimate the level of agreement between subsets of decision makers.

7.1 SNA and adjacency matrix

As already mentioned, SNA is the branch of network analysis devoted to studying and representing relationships between 'social' objects. Hence, hereafter, we consider a finite non-empty set $D = \{d_1, \ldots, d_n\}$ of social objects. The letter d has been chosen to remark the possible connection between objects and decision makers. The main tool to explore the relationships between the elements of D is called adjacency matrix. An adjacency matrix is a representation of an adjacency relation, $\tilde{T}_2 \subseteq D \times D$, whose characteristic function is $\mu_{\tilde{T}_2} : D \times D \to \{0, 1\}$ such that

$$\mu_{\tilde{T}_2}(d_i, d_j) = \begin{cases} 1, & \text{if } d_i \text{ is related to } d_j \\ 0, & \text{if } d_i \text{ is not related to } d_j \end{cases}$$

By definition [66], adjacency relations satisfy properties of reflexivity, $\mu_{\tilde{T}_2}(d_i, d_i) = 1 \ \forall i$, and symmetry, $\mu_{\tilde{T}_2}(d_i, d_j) = \mu_{\tilde{T}_2}(d_j, d_i) \ \forall i, j$. Note that, unlike for equivalence relations, no transitivity condition is required to hold.

If D is reasonably not too large, then the adjacency relation can be conveniently represented by a adjacency matrix $\tilde{\mathbf{T}} = (\tilde{t}_{ij})_{n \times n} \in \{0, 1\}^{n \times n}$ with $\tilde{t}_{ij} := \mu_{\tilde{T}_2}(d_i, d_j)$. One of the good characteristics of matrix $\tilde{\mathbf{T}}$ is that it is a concise synthesis of the pairwise relationships between elements in D. On the other hand, it can be said that it does not take into account the strength of the relationships. Due to this drawback, it could happen that, by using such type of adjacency relation, very different cases are treated in the same way, without discriminating among situations where intensities of relationship may be very different. This can seriously weaken the analysis of a social network.

7.2 Valued and fuzzy adjacency relations

One natural proposal for overcoming the problem of taking into account different degrees of relationship and coherently representing them, is that of employing a scale of measurement. Each entry of the matrix would therefore indicate the degree of relationship between two objects and not only state whether they are related of not.

A binary fuzzy relation on a single set, hereafter only fuzzy relation if not differently stated, is a fuzzy subset of the Cartesian product, i.e. a relation $T_2 \subseteq D \times D$ defined through the following membership function

$$\mu_{T_2} : D \times D \to [0,1]. \tag{7.1}$$

Also in this case, putting $t_{ij} := \mu_{T_2}(d_i, d_j)$, a fuzzy relation can be suitably represented by a matrix $\mathbf{T} = (t_{ij})_{n\times n} \in [0,1]^{n\times n}$ where the value of each entry is the degree to which the relation between d_i and d_j holds. In other words, the value of $\mu_{T_2}(d_i, d_j)$ is the answer to the question: 'how strong is the relationship between d_i and d_j?'. Therefore, in the context of SNA

$$\mu_{T_2}(d_i, d_j) = \begin{cases} 1, & \text{if } d_i \text{ has the strongest possible} \\ & \text{degree of relationship with } d_j \\ \gamma \in]0,1[& \text{if } d_i \text{ is, to some extent, related to } d_j \\ 0, & \text{if } d_i \text{ is not related with } d_j \end{cases}$$

Moreover, let us remark that, in literature, the term adjacency relation is often considered interchangeable with tolerance, proximity [33], and compatibility [66].

Fuzzy adjacency relations, as well as crisp adjacency relations, are here assumed to be reflexive and symmetric. It is useful to spend some words about symmetry. A fuzzy binary relation is symmetric if and only if

$$\mu_{T_2}(d_i, d_j) = \mu_{T_2}(d_j, d_i) \quad i,j = 1,\ldots,n. \tag{7.2}$$

Although the assumption of symmetry is a simplification, it is of great help for the model because, thanks to it, such relations can be represented by means of undirected graphs and problems related with the so-called combinatorial explosion are partially avoided. Furthermore, in many real-world cases, symmetry is spontaneously satisfied by the nature of the relationship.

As for the case of crisp adjacency relations, fuzzy adjacency relations are not necessarily transitive. For this reason we remark the difference between them and similarity relations, i.e. fuzzy equivalence relations, which, conversely, are transitive [141] definite.

According to its definition, a fuzzy adjacency relation seems to be a special case of a valued adjacency relation. The case is not restrictive as the scale $[0,1]$, albeit usually part of the definition, is not a necessary condition. It has been remarked in literature [52] that the unit interval can consistently be substituted by any lattice, L, and therefore the membership function be generalized,

$$\mu_{T_2} : D \times D \to L\,. \tag{7.3}$$

Thus, valued relations may be easily meant to be fuzzy relations in their broader sense. In spite of this remark, whose aim was that of underlying the non-restrictive nature of the interval [0,1], for simplicity we are going to use the real unit interval in the following.

Let us end the paragraph stressing some further points:

- Fuzzy relations contain more information than crisp relations. This has been made clear by α-cuts. In fact, we can shift from the fuzzy approach to the crisp one thanks to the α-cuts. An α-cut, T_2^α, of a fuzzy relation T_2 is the crisp subset of $D \times D$ defined by its membership function:

$$\mu_{T_2^\alpha}(d_i, d_j) = \begin{cases} 1, & \text{if } \mu_{T_2}(d_i, d_j) \geq \alpha \\ 0, & \text{if } \mu_{T_2}(d_i, d_j) < \alpha \end{cases}$$

  For instance, considering a fuzzy binary relation in his matrix form

$$\mathbf{T} = \begin{pmatrix} 1 & 0.7 & 0.3 & 0.7 \\ 0.7 & 1 & 0.1 & 0.8 \\ 0.3 & 0.1 & 1 & 0.2 \\ 0.7 & 0.8 & 0.2 & 1 \end{pmatrix}, \tag{7.4}$$

  its α-cut with $\alpha = 0.5$ is

$$\mathbf{T}^{0.5} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \tag{7.5}$$

  It can be seen that a crisp relation appears as an approximation of a fuzzy relation. Therefore, it is always possible, given a fuzzy relation, to obtain a crisp relation (this operation is not reversible).

- Applying fuzzy relations to SNA, we can extend most of the techniques employed for analyzing crisp adjacency matrices. A significant example is the normalized index of local centrality [122], that is

$$C(d_i) = \frac{1}{n-1} \sum_{\substack{j=1 \\ j \neq i}}^{n} t_{ij} \ . \tag{7.6}$$

  If $c_i := C(d_i)$ and $\mathbf{c} = (c_1, \ldots, c_n)$, then we can refer to $\mathbf{T}$ in (7.4) and find that $\mathbf{c} = \left(\frac{17}{30}, \frac{8}{15}, \frac{1}{5}, \frac{17}{30}\right)$. Moreover, this result is more informative than the same index computed for $\mathbf{T}^{0.5}$ in (7.5), i.e. $\mathbf{c} = \left(\frac{2}{3}, \frac{2}{3}, 0, \frac{2}{3}\right)$.

- The structure of the problem can be addressed thanks to graph theory too. More precisely, a fuzzy binary relation can be represented by a complete weighted graph where the n nodes are the n social objects $d_1, \ldots, d_n$ and the weights of every edge (d_i, d_j) are the degrees of relationship $\mu_{T_2}(d_i, d_j)$ between objects d_i and d_j, $i, j = 1, \dots, n$. Therefore, the problem can also be addressed in a graphical way with $\mu_{T_2}(d_i, d_j)$ representing the 'thickness' of the edge between d_i and d_j. Furthermore, a fuzzy relation can be seen as a fuzzy graph [79] with the peculiarity that edges only are fuzzy, whereas nodes are crisp.

7.3 Fuzzy m-ary adjacency relations and OWA functions

Thus far, adjacency relations have been tackled in their binary form. Namely, they have been good at expressing relationships over pairs. In this section we propose an extension of the analysis involving m-dimensional relations with $m \in \{2, \dots, n\}$. In performing this generalization we consider the binary case as a special case in a more general approach. Therefore, considering an m-ary adjacency relation, the degree of membership on an m-tuple in the relation represents the degree to which the elements of the tuple are related. Analogously to the binary case (7.1), it is straightforward to define a fuzzy m-ary relation.

Definition 9. A fuzzy m-ary relation T_m on a single set D is a fuzzy subset of D^m defined by means of the membership function

$$\mu_{T_m} : D^m \to [0, 1]. \tag{7.7}$$

Therefore, the membership function characterizing fuzzy m-ary relations is the following

$$\mu_{T_m}(d_{p_1}, \dots, d_{p_m}) = \begin{cases} 1, & \text{if } d_{p_1}, \dots, d_{p_m} \text{ are definitely related} \\ & \text{one another} \\ \gamma \in]0,1[& \text{if } d_{p_1}, \dots, d_{p_m} \text{ are, to some extent,} \\ & \text{related one another} \\ 0, & \text{if } d_{p_1}, \dots, d_{p_m} \text{ are definitely not related} \\ & \text{one another} \end{cases}$$

Nevertheless, as the semantic underlying the membership function remains substantially unchanged, properties of reflexivity and symmetry are extended to the m-dimensional case in the following way. An m-ary relation is reflexive if and only if

$$\mu_{T_m}(d_{p_1}, \dots, d_{p_m}) = 1$$

for all $p_1 = \dots = p_m$. An m-ary relation is symmetric if and only if

$$\mu_{T\,m}(d_{p_1}, \dots, d_{p_m}) = \mu_{T\,m}(d_{\pi(p_1)}, \dots, d_{\pi(p_m)})$$

where π is any permutation map of the multiset $\langle p_1, \ldots, p_m \rangle$. A multiset is a generalization of the idea of set, characterized by the fact that multiple instances of the same member can occur.
At this point, having defined fuzzy m-ary adjacency relations, it is the case to highlight the difference between an element of a fuzzy m-ary adjacency relation and a clique [75, 84, 122]. Namely, a clique of a graph is a maximum complete subgraph whereas, if we deal with m-ary relations and the contrary is not made explicit, the value $\mu_{T_m}(d_{p_1}, \ldots, d_{p_m})$ simply states, by means of the bounded unipolar scale $[0, 1]$, the degree to which the relationship holds, without taking into account any maximality condition.

Although we can easily define an m-ary relation under the theoretical point of view, it might be very difficult to represent it in a direct way. In the following part of the paper we aim at introducing a method for representing m-ary relations starting from the (binary) fuzzy adjacency relation.
Let us first introduce the notation which will be used hereafter. Given m not necessarily different indices $p_1, \ldots, p_m \in \{1, \ldots, n\}$, we take into account the symmetric property of a fuzzy binary adjacency relation and, to avoid unnecessary repetitions, we consider the $k = \binom{m}{2} = m(m-1)/2$ ordered pairs (d_{p_i}, d_{p_j}) such that $i < j$. We can then define ρ-dependence.

Definition 10 (ρ-dependence [15])**.** Given a fuzzy binary adjacency relation T_2, a fuzzy m-ary adjacency relation T_m $(m > 2)$ is said to be ρ-dependent on T_2 if and only if there exists a function

$$\rho : [0, 1]^k \to [0, 1] \tag{7.8}$$

such that, for any choice of m indices $p_1, \ldots, p_m \in \{1, \ldots, n\}$ it is

$$\mu_{T_m}(d_{p_1}, \ldots, d_{p_m}) = \rho\big(\mu_{T_2}(d_{p_1}, d_{p_2}), \ldots, \mu_{T2}(d_{p_{m-1}}, d_{p_m})\big), \tag{7.9}$$

where the $k = \binom{m}{2}$ arguments of function ρ are the values $\mu_{T_2}(d_{p_i}, d_{p_j})$ of the fuzzy binary adjacency relation T_2, with $i < j$.

Example 20. Imagine we have a fuzzy adjacency relation T_2 on the set $\{d_1, d_2, d_3, d_4\}$ and, given ρ, we want to verify if a relation T_3 is ρ-dependent on T_2. This means that $n = 4$ and $m = 3$. Therefore, T_3 is ρ-dependent on T_2 if and only if the following equalities are simultaneously satisfied

$$\begin{aligned}\mu_{T_m}(d_{p_1}, d_{p_2}, d_{p_3}) = \rho\big((\mu_{T_2}(d_{p_1}, d_{p_2}), (\mu_{T_2}(d_{p_1}, d_{p_3}),\\ (\mu_{T_2}(d_{p_2}, d_{p_3})\big), \quad \forall\, p_1, p_2, p_3 \in \{1, 2, 3, 4\}.\end{aligned}$$

In addition, it can be checked that the number of arguments of ρ is $k = \binom{m}{2} = \binom{3}{2} = 3$.

We propose to represent m-ary adjacency relations by means of ρ-dependence. Nevertheless, it is necessary to define a suitable function ρ.

Hereafter until the end of the section, some assumptions regarding ρ will be formalized. First of all, coherently with the spirit of a process of aggregation of information, ρ is required to be an aggregation function. One reaches this conclusion by considering that, in order to estimate a coherent m-ary relation, ρ should somehow weigh the contribution brought by each involved pair.

Definition 11 (Aggregation function [10]). An aggregation function is a function of $k > 1$ arguments that maps the (k-dimensional) unit cube onto the unit interval, $f : [0,1]^k \to [0,1]$, with the properties

- $f(0,\ldots,0) = 0$ and $f(1,\ldots,1) = 1$ (extremal conditions)
- $\mathbf{a} \leq \mathbf{b}$ implies $f(\mathbf{a}) \leq f(\mathbf{b})$ for all $\mathbf{a}, \mathbf{b} \in [0,1]^k$ (monotonicity)

where $\mathbf{a} \leq \mathbf{b}$ means that each component of $\mathbf{a}$ is no greater than the corresponding component of $\mathbf{b}$.

Moreover, we are going to propose and justify some other properties which ought to be satisfied by every ρ.

1. idempotency: $\rho(a,\ldots,a) = a$. Therefore, if m elements of some set are pairwise related with degree a, then we assume that the intensity of relationship computed on the k-tuple containing those objects has value a as well. We will see later that idempotency is implicitly embedded in property 4, i.e. averaging behavior.

2. commutativity: $\rho(a_1,\ldots,a_k) = \rho(a_{\pi(1)},\ldots,a_{\pi(k)})$ where π is any permutation map of the index set $\{1,\ldots,k\}$. This property is required to hold because fuzzy adjacency relations are symmetrically defined for all $m = 2,\ldots,n$.

3. strict monotonicity: $\rho(a_1,\ldots,a_k) > \rho(b_1,\ldots,b_k)$ if $a_i \geq b_i$ $\forall i$ and there exists at least one j such that $a_j > b_j$. Strict monotonicity is asked to hold in order to overcome some evaluation problems which would arise if we used non-strictly monotonically increasing functions as, for instance, the geometric mean $g(\cdot)$. To give an example, substituting g to ρ we would have $g(1,\ldots,1,0) = g(0,\ldots,0)$, which is not a desirable result from the social analysis point of view.

4. strictly averaging behavior: $\min(\mathbf{a}) < \rho(\mathbf{a}) < \max(\mathbf{a})$. This requirement implies that conjunctive and disjunctive aggregation function are excluded. This property is actually a consequence of strict monotonicity ([10] pg. 11).

These four more assumptions lead us to choose within a restricted class of aggregation functions. It is easy to check that a large number of common aggregation functions are excluded. For instance, as already mentioned, the geometric mean is excluded because it is not strictly monotone. Whenever it does not collapse into the simple arithmetic mean, the weighted arithmetic mean is also excluded because it is not commutative.

We propose to use as function ρ an OWA function with weights $w_i \in]0,1[$, since such type of aggregation function satisfies all the listed properties and can be characterized by some relevant and well interpretable indices, as those of orness and entropy. OWA functions were introduced by Yager in [137] and further on studied by many other authors as, for instance, in [139]. One of the reasons behind the success of OWA functions is that they are sufficiently flexible to cover a range of some well-known aggregation functions.

As said before, OWA functions with $w_i \in]0,1[$ are strictly increasing functions in all the arguments. Therefore, their definition is the following.

Definition 12 (Strictly monotone OWA function)**.** A strictly monotone OWA function of dimension k is a mapping $F : \mathbb{R}^k \to \mathbb{R}$, that has an associated weighting vector $\mathbf{w} = (w_1, \ldots, w_k)$ such that $w_i \in]0,1[$ and $\sum_{i=1}^{k} w_i = 1$. Furthermore,

$$F(a_1, \ldots, a_k) = w_1 b_1 + \cdots + w_k b_k = \sum_{j=1}^{k} w_j b_j$$

where b_j is the j-th largest element of the multiset $A = \langle a_1, \ldots, a_k \rangle$.

To better justify the use of OWA functions in this context let us take the following simple example into account. Suppose ρ is replaced by the arithmetic mean. Then we would get to the result that

$$\rho(\underbrace{1, \ldots 1}_{\text{half arguments}}, \underbrace{0, \ldots, 0}_{\text{half arguments}}) = \rho(1/2, \ldots, 1/2). \tag{7.10}$$

which might be acceptable but, the social analyst, according to the context, may also prefer to consider the situation described in the left hand side of (7.10) to be more/less expressive of a high level of relationship than the situation considered in the right hand side. For instance, we can assume that the accomplishment of a task requires, as a necessary condition, the cooperation of all the subjects involved. In this case it is likely that the set of objects represented by binary relations on the right hand side of (7.10) is preferred to the one described on the left hand side of the same equality.

Therefore, it is possible to obtain the most suitable outcome of the aggregation with respect to the faced problem by assigning different weights to the terms of the aggregation function according to their values. OWA functions are particularly appealing for our proposal, thanks to some of their properties and the fact that they are easily interpretable in terms of a trade-off between the two operators min (and) and max (or). Also a recently published work by Yager [138] is supporting the natural connection between granular computing, OWA and SNA thanks to a new paradigm for intelligent social network analysis (PISNA).

To each OWA is associated a measure of orness [51, 82] which estimates how much the aggregation process tends to be an 'orlike' operation

$$orness(\mathbf{w}) = \frac{1}{k-1}\sum_{i=1}^{k}(k-i)w_i \ . \tag{7.11}$$

It is possible to determine a complementary index, $andness(\mathbf{w}) = 1 - orness(\mathbf{w})$, that is its mirror image. For clarity, let us briefly recall some special cases

$$orness(\mathbf{w}) = \begin{cases} 1, & \text{if } \mathbf{w} = (1, 0, \dots, 0) \\ 0, & \text{if } \mathbf{w} = (0, \dots, 0, 1) \\ 0.5, & \text{if } \mathbf{w} = (1/k, \dots, 1/k) \end{cases} \tag{7.12}$$

In the first case, with $\mathbf{w} = (1, 0, \dots, 0)$, the OWA operator becomes the max (or) operator, in the second case, with $\mathbf{w} = (0, \dots, 0, 1)$, the OWA function becomes the min (and) operator. Note that we exclude the extreme values 0 and 1 for the weights w_i, so that for strictly monotone OWA functions it is always $0 < orness(\mathbf{w}) < 1$. Thus, in light of the preceding remarks, we suggest that OWA operators can be considered a suitable choice for function ρ in (7.9). In this case, the elements $a_1, \dots, a_k$ of definition 12 are the k values, $\mu_{T_2}(d_{p_i}, d_{p_j})$, $i, j = 1, \dots, m, \ i < j$.

7.4 Optimization models for m-ary relations

In the previous section we defined fuzzy m-ary relations $\mu_{T_m}(d_{p_1}, \dots, d_{p_m})$ under the general assumption that the indices $p_1, \dots, p_m \in \{1, \dots, n\}$ are not necessarily distinct. This assumption was necessary in order to extend the well established approach of binary adjacency relation where reflexivity ($\mu_{T_2}(d_i, d_i) = 1$) is assumed and full matrix representation is used. In spite of it, in the optimization problems we are considering in this section, it is convenient to restrict the study to the case where the m indices $p_1, \dots, p_m \in \{1, \dots, n\}$ are distinct indices. In fact, by this assumption it is possible to interpret each m-tuple $(d_{p_1}, \dots, d_{p_m})$ as a set of social objects, for instance decision makers, containing precisely m distinct elements, thus allowing a better understanding of the suggested application models. Therefore, in the following, by considering $\mu_{T_m}(d_{p_1}, \ldots, d_{p_m})$, we assume $i \neq j \Rightarrow p_i \neq p_j$, $i, j = 1, \ldots, m$. Moreover, for sake of simplicity, and thanks to symmetry, we will assume $p_1 < \cdots < p_m$. Let us consider some optimization problems that can be of interest in applying fuzzy m-ary adjacency relations to SNA.

The first model is useful to find the subset with cardinality m whose elements are the most related between each other.

$$\text{(OPT1)} \quad \max\{\mu_{T_m}(d_{p_1}, \ldots, d_{p_m}) | p_1 < p_2 < \cdots < p_m\} \tag{7.13}$$

This proposal can be refined by assuming that each element $d_i \in D$ has a specific weight ω_i denoting its relative importance. Let us consider the weight vector

$$\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n) \quad \text{such that} \sum_{i=1}^{n} \omega_i = 1, \ \omega_i \geq 0 \ \forall i. \tag{7.14}$$

Then, we can perform an analysis similar to that described above by assuming that parameter m is free, and by requiring that the sum of the weights associated to the considered m elements is equal or greater than a given majority threshold $0 < \tau < 1$. Thus, the optimization problem is

$$\begin{aligned} \text{(OPT2)} \quad \max \Bigg\{ &\mu_{T_m}(d_{p_1}, \ldots, d_{p_m}) \mid p_1, \ldots, p_m \in \{1, \ldots, n\}, \\ & p_1 < \cdots < p_m, \sum_{i=1}^{m} \omega_{p_i} > \tau, \ m = 2, \ldots, n-1 \Bigg\} \end{aligned} \tag{7.15}$$

Some comments on (OPT2) could be useful to better understand the involved optimization. In (OPT2) we are still interested in the most related subset, but the constraint of having a fixed number of elements is replaced by a constraint on a majority threshold τ to be satisfied by the sum of the weights of the elements in the subset. That is, subsets with different number m of elements are taken into account, provided that they fulfill threshold τ. Note that large values of μ_{T_m} can be easily achieved if the number m of elements is small, while the constraint $\sum_{i=1}^{m} \omega_{p_i} > \tau$ is satisfied by the subsets with a sufficiently large number of strong elements. Therefore, the optimal solution of (OPT2) arises by taking into account the two conflicting criteria: power of the subset and degree of relationship among the elements of the subset. We stress again that the number m of the elements is optimally determined only after having solved (OPT2). Note that vector $\boldsymbol{\omega}$ defining the relative importance of each $d_i \in D$ must not be confused with vector $\mathbf{w}$ of an OWA function, which is used in this paper to assign weights to degrees of relationships among elements in D.

Maximization problem (OPT2) can bear the addition of some constraints. For example, let us assume that we are interested in solving it with the constraint that some objects are included in the m-tuple which optimize the model. In this case, we can assume the indices of those objects that we want to be part of the solution are the only elements whose index belongs to an index set $I \subset \{1, \dots, n\}$. This said, the new constrained model is

$$\text{(OPT3)} \quad \max \Big\{ \mu_{T_m}(d_{p_1}, \dots, d_{p_m}) \mid \{p_1, \dots, p_m\} \supseteq I, \\ p_1 < \dots < p_m, \; \sum_{i=1}^{m} \omega_{p_i} > t, \; m = 2, \dots, n-1 \Big\} . \tag{7.16}$$

Another problem that can be addressed is that of maximizing the number m of elements in a subset satisfying a fixed majority threshold. Namely, let us fix a threshold $\delta \in [0, 1]$ such that $\mu_{T_m}(d_{p_1}, \dots, d_{p_m}) > \delta$ and leave the dimension m of our analysis free. In this way, progressively increasing m and calculating $\mu_{T_m}(d_{p_1}, \dots, d_{p_m})$ at every stage, we can detect the largest $B \subseteq D$ such that $\mu_{T_m}(B) > \delta$. Let $\hat{m}$ denote this maximal cardinality.

$$\text{(OPT4)} \quad \hat{m} = \max\{m \mid \exists B \subseteq D, \; m = |B|, \; \mu_{T_m}(B) > \delta\} .$$

It may occur that set B is not unique, since there exist ν different subsets $B_j, \;\; j = 1, \dots, \nu$ satisfying inequality $\mu_{T_m}(B_j) > \delta$ with the same maximal cardinality $\hat{m}$. In this case, it is possible to define a winner as the subset B_i with the strongest degree of relationship, $\mu_{T_m}(B_i) \geq \mu_{T_m}(B_j), \; j = 1, \dots, \nu$. If again the solution is not unique, the multiple solutions are considered equivalent for our analysis.

7.5 Example

At this point it is worthwhile to present a numerical example based on the following fuzzy adjacency matrix

$$\mathbf{T} = \begin{pmatrix} 1 & 0.4 & 0.5 & 0.5 & 0.8 & 0.4 & 0.3 \\ 0.4 & 1 & 0.6 & 0.7 & 0.7 & 0.7 & 0.8 \\ 0.5 & 0.6 & 1 & 0.2 & 0.9 & 1 & 1 \\ 0.5 & 0.7 & 0.2 & 1 & 0.4 & 0.7 & 0.2 \\ 0.8 & 0.7 & 0.9 & 0.4 & 1 & 0 & 0.5 \\ 0.4 & 0.7 & 1 & 0.7 & 0 & 1 & 0.1 \\ 0.3 & 0.8 & 1 & 0.2 & 0.5 & 0.1 & 1 \end{pmatrix} \tag{7.17}$$

To make this simple example more consistent with real-world cases, it is possible to imagine, for instance, that entries of $\mathbf{T}$ are estimations of the degrees of consensus between some decision makers. Let us suppose that we are interested in estimating the ternary fuzzy adjacency relation on D for all $(d_{p_1}, d_{p_2}, d_{p_3})$ such that $p_1 \neq p_2 \neq p_3$. A suitable OWA function has to be chosen. All along the example, we suppose that the analyst desires to use a vector $\mathbf{w}$ with $orness(\mathbf{w}) = 0.5$ with maximal entropy [51, 82]. In this case, such vector represents the arithmetic mean, i.e. all the weights are equal. Going ahead, it is possible to derive the ternary relations on d by means of (7.9).

$$\begin{aligned} \mu_{T_3}(d_1, d_2, d_3) &= 0.5 \\ \mu_{T_3}(d_1, d_2, d_4) &= 0.533 \\ \vdots &= \vdots \\ \mu_{T_3}(d_4, d_6, d_7) &= 0.33 \\ \mu_{T_3}(d_5, d_6, d_7) &= 0.2 \end{aligned}$$

Note that, as specified in section 7.4, we only take into account the relationship degrees $\mu_{T_3}(d_{p_1}, d_{p_2}, d_{p_3})$ with $p_1 < p_2 < p_3$. Let us now imagine that we are seeking for the three objects that are the most related. For example, if the objects of our analysis were decision makers, then we would be seeking for the most 'consensual' subset with cardinality equal to three. In this case, the set of the three elements which we are looking for is the solution of model (OPT1) with fixed length of the m-tuple $(d_{p_1}, \dots, d_{p_m})$. In our example, in light of the results obtained before, the three objects at issue are d_2, d_3, d_7 with $\mu_{T_3}(d_2, d_3, d_7) = 0.8$.

It is particularly interesting to set the analysis with $m = m^\star$ where $m^\star$ is the integer part of $n/2 + 1$,

$$m^\star = \lfloor n/2 + 1 \rfloor = \begin{cases} \frac{n}{2} + 1, & \text{if } n \text{ is an even integer} \\ \frac{n}{2} + \frac{1}{2}, & \text{if } n \text{ is an odd integer} \end{cases} \tag{7.18}$$

Namely, $m^\star = \min\{z \in \mathbb{N} | z > n/2\}$, where $\mathbb{N}$ is the set of the natural numbers.

The result of (OPT1) with $m = m^\star$ can be seen as minimum winning coalition with the strongest degree of relationship. If we realistically assume that the larger $\mu_{T_m}(d_{p_1}, \dots, d_{p_m})$, the more likely to arise a coalition among $d_{p_1}, \dots, d_{p_m}$ is, then the value $m^\star$ becomes very useful. In our example, $m^\star = 4$ and the elements at issue are d_2, d_3, d_5, d_7 with $\mu_{T_4}(d_2, d_3, d_5, d_7) = 0.75$.

As said before, (OPT2) is a generalization of (OPT1). Let us go through it by means of our example. Referring to matrix (7.17), if we apply model

(OPT2) with $\boldsymbol{\omega} = (0.23, 0.14, 0.07, 0.12, 0.16, 0.05, 0.23)$ and $\tau = 1/2$, then we would find that the solution is (d_2, d_5, d_7) with

$$\begin{aligned} \mu_{T_3}(d_2, d_5, d_7) &= \frac{1}{3}\mu_{T_2}(d_2, d_7) + \frac{1}{3}\mu_{T_2}(d_2, d_5) + \frac{1}{3}\mu_{T_2}(d_5, d_7) \\ &= \frac{1}{3}0.8 + \frac{1}{3}0.7 + \frac{1}{3}0.5 = \frac{2}{3} \approx 0.66667. \end{aligned} \tag{7.19}$$

We can further check that the threshold is respected; in fact, it is $\omega_2 + \omega_5 + \omega_7 = 0.14 + 0.16 + 0.23 = 0.53 > \tau$

Although the example proposed here is not based on a real world case, the problem solved in (OPT2) could be applied to economics and political sciences. In fact, it is possible to see vector $\boldsymbol{\omega}$ as a collection of weights for experts or political parties. In this latter case, if we are able to establish a index of proximity between any two decision makers d_i and d_j i.e. estimating entries of matrix $\mathbf{T}$, then we can apply (OPT2) and find the strongest winning coalition.

If we want to find a coalition and impose some constrains on what elements we want to be part of it, then (OPT3) might be used. Again, to formulate an example, suppose $I = \{1, 2\}$. In this case the argument maximizing (OPT3) is (d_1, d_2, d_5) with

$$\begin{aligned} \mu_{T_3}(d_1, d_2, d_5) &= \frac{1}{3}\mu_{T_2}(d_1, d_2) + \frac{1}{3}\mu_{T_2}(d_1, d_5) + \frac{1}{3}\mu_{T_2}(d_2, d_5) \\ &= \frac{1}{3}0.4 + \frac{1}{3}0.8 + \frac{1}{3}0.7 = \frac{1.9}{3} \approx 0.63333, \end{aligned} \tag{7.20}$$

with $\omega_1 + \omega_2 + \omega_5 = 0.23 + 0.14 + 0.16 = 0.53 > \tau$.

To show (OPT4), suppose $\delta = 0.6$. A good way to find B is that of starting with $m = 7$ and decrease m step by step until we find a m such that there exists $\mu_{T_m}(\cdot) > \delta$. At this point we should verify its uniqueness. If it is not unique we seek for the maximum and if we still cannot achieve uniqueness we consider the solutions to be equivalent.

In our case, $\hat{m} = 5$ and there are two $\hat{m}$-tuples with membership value greater than 0.6. They are $(d_1, d_2, d_3, d_5, d_7)$ and $(d_2, d_3, d_5, d_6, d_7)$. As

$$\mu_{T_5}(d_1, d_2, d_3, d_5, d_7) = 0.65 > \mu_{T_5}(d_2, d_3, d_5, d_6, d_7) = 0.63$$

we find that $(d_1, d_2, d_3, d_5, d_7)$ is the solution of (OPT4).

The very last observation concerns the dimension of the analysis. If $m = n$, then the degree to which this particular relationship holds is a measure of how strong the relationship among all the $d_i \in D$ is. It can be interpreted as the degree of social relationship computed on the entire network.

7.6 Discussion

In the first part of this chapter the natural connection between fuzzy relations, social network analysis, and group decision analysis was shortly discussed and justified. After the introductory part, attention was drawn on the natural extension of relations to the m-dimensional case. Thus, fuzzy m-adjacency relations were introduced and their link with fuzzy binary relations is formalized. To this aim, OWA functions appear to be valuable tools in order to estimate m-ary relations. Future work shall focus on the structural properties of the already mentioned connection between the binary and the m-ary case. Moreover, we have discussed an approach which takes into account imprecision and so far we have not dealt with uncertainty. Thus, extending the analysis of m-ary relations and the ρ-dependence in order to consider uncertainty could be a promising future direction.

Chapter 8

Discussion

A number of problems were posed in the first part of this thesis and each section has tried to clear the fog in the maze of preference relations by means of proving the equivalence between consistency indices, numerically comparing methods, and finding new and better algorithms. The research questions have focused on a very wide spectrum of issues in the methodology of preference relations, and their answers have been arranged in this thesis, or, at least, this was my goal. Rather than following the tradition and summarize, once again, this dissertation, I prefer to discuss how future research can improve some of the results. Naturally, suggesting developments and improvements will serve to highlight the limitations of this work.

8.1 Future research

If something has been achieved in this thesis, then a lot more is still unexplored. Hereafter, by means of a list, some hot topics will be pointed out and briefly explained.

- A wide number of inconsistency indices have been proposed and a comparative study, under both the theoretical and numerical point of view, is reported in this dissertation. Despite this, there is a need for a formal definition of inconsistency index. Some future efforts should aim at clarifying what a consistency index is and perhaps trying to formalize some characterizing properties. Afterwards, with a widely agreed definition of inconsistency index, it would be possible to check whether or not, the already proposed indices satisfy such definition.
- The numerical study on inconsistency indices presented in this thesis gives an idea of the behavior of indices but it is far from being en exhaustive statistical study. For instance, only randomly generated

preference relations have been taken into account whereas the most plausible case is that of slightly perturbed consistent matrices. Moreover, the Spearman index is only one (and perhaps one of the least sophisticated) among the tools for analyzing agreement between observations on two quantities. Future research should try to achieve more robustness and statistical significance.

- Some attempts have already been made in order to bring the concept of preference relation to a higher mathematical level. Recently, Cavallo and D'Apuzzo [19, 20] employed Abelian linearly ordered groups in order to develop a common framework for some different types of preference relations. Barzilai [9], even earlier, tried to give a more formal structure to some preference relations. The importance of this kind of research does not reduce to mere aesthetics, since establishing some such connections between different approaches would prove different approaches isomorphic. Thanks to such abstract relations, we know that the algebraic and topological properties of one approach could be automatically transfered to those which are isomorphic to it. Another attempt to use very abstract mathematical instruments is that proposed by Eklund et al. [38] where the authors tried to describe social choice by means of category theory. These are possibly promising, and still unexplored, areas of study which connects the theory of preference relations with abstract algebra.

- The difference between open and closed scales for preference relations should be formalized and studied, also empirically. Each condition of consistency for preference relation always embeds a not necessarily total binary operator $* : S \times S \to S$ such that the preference relation is consistent if and only if

$$\mu_P(x_i, x_j) * \mu_P(x_j, x_k) = \mu_P(x_i, x_k) \qquad \forall i, j, k$$

  Example 21. Given reciprocal relations and multiplicative consistency, it can be verified that the operator $*$ is the following uninorm $* :]0,1[\times]0,1[\to]0,1[$

$$r_{ik} = \frac{r_{ij} r_{jk}}{r_{ij} r_{jk} - (1 - r_{ij})(1 - r_{jk})},$$

  which, in fact, is nothing else but another way of posing the condition of multiplicative consistency for reciprocal relations. ⧫

  A scale S, with respect to an operator $*$, is open if and only if the set S is closed under that same operator. Equivalently, it can be said that a given scale S is open under a binary operator $*$ if and only if that operator is total on the set S.

Example 22. In the pairwise comparison matrices case, the domain of representation of the preferences can either be the interval $[1/9, 9]$ or the set of positive reals, $\mathbb{R}_>$. In the first case, the set is not closed under the operation of multiplication and therefore the scale is closed. In the latter case, the contrary is true; hence, the scale is open. ♦

Seemingly, open scales are more elegant and allow a greater mathematical abstraction. However, due to psychological reasons, closed scales seem to be more suitable in practice. This issue has not been studied in details and therefore future research should try to bridge the gap or at least show drawbacks and good points in using one approach rather than the other.

- The study of incomplete preference relations presented in chapter 6 was supposed to be the starting point of a more extensive research, since it takes into account only a limited number of methods and it presents the results without statistical analysis. Further studies could aim at comparing different methods by means of statistical tools. However, it appears that such a study would always share some of the limitations of the original one. Namely, the scope of some of the studied methods is that of estimating a priority vector from an incomplete preference relation rather than completing the preference relation according to some criterion of consistency. Thus, any comparative study would remain questionable.
- The approach to social network analysis can be improved and studied in many ways:
  - This dissertation has presented fuzzy binary adjacency relations, but it has not proposed a method for obtaining them. However, it is realistic to assume that they can be derived from preference relations of decision makers. Future research should investigate the connection between preference relations and relations of adjacency. It would be quite interesting to build a bridge between preference relations and adjacency relations such that, given a number of preference relations, each associated to a decision maker, then we can estimate an adjacency relation on the set of decision makers. In doing so, all the tools of social network analysis could be naturally introduced into group decision making theory.
  - The algorithms for constructing m-ary adjacency relations have not been studied from the point of view of computational complexity. The same applies to the combinatorial optimization problems proposed in the same chapter. My feeling is that, given the

particular configuration of the underlying algorithms, complexity can be estimated rather precisely. These answers are necessary to understand if, and at what point, the combinatorial explosion can make the model impractical for large networks.

- If it is true that the proof of a pudding is in the eating, then the ρ-characterization and its associated optimization problems should be applied to at least one real-world case, e.g. elections, in order to be validated.

Bibliography

[1] J. Aczél, *Lectures on Functional Equations and their Applications*. Academic Press, 1966.

[2] J. Aczél and T. L. Saaty, "Procedures for synthesizing ratio judgments," *Journal of Mathematical Psyshology*, vol. 27, no. 1, pp. 93–102, 1983.

[3] J. Aguarón and J. M. Moreno-Jiménez, "The geometric consistency index: Approximated thresholds," *European Journal of Operational Research*, vol. 147, no. 1, pp. 137–145, 2003.

[4] S. Alonso, F. Chiclana, F. Herrera, and E. Herrera-Viedma, "A procedure for learning missing values in fuzzy preference relations based on additive consistency," in *Lecture Notes in Computer Science*, vol. 3131, 2004, p. 227238.

[5] M. Anholcer, V. Babiy, S. Bózoki, and W. W. Koczkodaj, "A simplified implementation of the least squares solution for pairwise comparisons matrices," *Central European Journal of Operations Research*, in press.

[6] F. J. Anscombe, "Graphs in statistical analysis," *The American Statistician*, vol. 17, no. 1, pp. 17–21, 1973.

[7] B. Bai, L. Cao, and J. Lin, "Mining weight information from a triangular fuzzy number preference relation," in *Fifth International Conference on Fuzzy Systems and Knowledge Discovery*, J. Ma, Y. Yin, J. Yu, and S. Zhou, Eds., vol. 1. IEEE Computer Society, 2008, pp. 329–334.

[8] C. Bana e Costa and J. Vansnick, "MACBETH - an interactive path towards the construction of cardinal value functions," *International Transactions in Operational Research*, vol. 1, pp. 489–500, 1994.

[9] J. Barzilai, "Consistency measures for pairwise comparison matrices," *Journal of Multi-Criteria Decision Analysis*, vol. 7, no. 3, pp. 123–132, 1998.

[10] G. Beliakov, A. Pradera, and T. Calvo, *Aggregation functions: A guide for practicioners*, ser. Studies in Fuzziness and Soft Computing. Springer-Verlag, 2007, vol. 221.

[11] D. Bernoulli, "Exposition of a new theory on the measurement of risk," *Econometrica*, vol. 22, no. 1, pp. 23–36, 1954.

[12] S. Bózoki, "Solution of the least squares method problem of pairwise comparison matrices," *Central European Journal of Operations Research*, vol. 16, no. 4, pp. 345–358, 2008.

[13] J. P. Brans and P. Vincke, “A preference ranking organisation method: The PROMETHEE method for MCDM,” *Management Science*, vol. 31, no. 6, pp. 647–656, 1985.

[14] M. Brunelli, “A note on the article “Inconsistency of pair-wise comparison matrix with fuzzy elements based on geometric mean” [Fuzzy Sets and Systems 161 (2010) 1604–1613],” *Fuzzy Sets and Systems*, in press.

[15] M. Brunelli and M. Fedrizzi, “A fuzzy approach to social network analysis,” in *2009 International Conference on Advances in Social Network Analysis and Mining*, IEEE Computer Society, Ed., 2009, pp. 225–230.

[16] M. Brunelli, M. Fedrizzi, and S. Giove, “Reconstruction methods for incomplete fuzzy preference relations: A numerical comparison,” in *WILF*, 2007, pp. 86–93.

[17] C. Carlsson and P. Walden, “AHP in political group decisions: A study in the art of possibilities,” *Interfaces*, vol. 25, no. 4, pp. 14–29, 1995.

[18] F. J. Carmone Jr, A. Kara, and S. H. Zanakis, “A Monte Carlo investigation of incomplete pairwise comparison matrices in AHP,” *European Journal of Operational Research*, vol. 102, no. 3, pp. 538–553, 1997.

[19] B. Cavallo and L. D’Apuzzo, “A general unified framework for pairwise comparison matrices in multicriterial methods,” *International Journal of Intelligent Systems*, vol. 24, no. 4, pp. 377–398, 2009.

[20] ——, “Characterizations of consistent pairwise comparison matrices over Abelian linearly ordered groups,” *International Journal of Intelligent Systems*, vol. 25, no. 10, pp. 1035–1059, 2010.

[21] H. Chen and C. C. E. Yang, “Social media analytics: Understanding the pulse of the society,” *IEEE Transactions on systems, man and cybernetics: part A*, [special issue] 2010.

[22] F. Chiclana, E. Herrera-Viedma, and F. Herrera, “Integrating three representation models in fuzzy multipurpose decision making based on fuzzy preference relations,” *Fuzzy Sets and Systems*, vol. 97, no. 1, pp. 33–48, 1998.

[23] ——, “Integrating multiplicative preference relations in a multipurpose decision-making model based on fuzzy preference relations,” *Fuzzy Sets and Systems*, vol. 122, no. 2, pp. 277–291, 2001.

[24] F. Chiclana, E. Herrera-Viedma, F. Herrera, and S. Alonso, “Some induced ordered weighted averaging operators and their use for solving group decision-making problems based on fuzzy preference relations,” *European Journal of Operational Research*, vol. 182, no. 1, pp. 383–399, 2006.

[25] E. U. Choo and W. C. Wedley, “A common framework for deriving preference values from pairwise comparison matrices,” *Computers & Operations Research*, vol. 31, no. 6, pp. 893–908, 2004.

[26] A. T. W. Chu, R. E. Kalaba, and K. Springarn, “A comparison of two methods for determining the weights of belonging to fuzzy sets,” *Journal of Optimization Theory and Applications*, vol. 27, no. 4, pp. 531–538, 1979.

[27] G. Crawford and C. Williams, "A note on the analysis of subjective judgment matrices," *Journal of Mathematical Psychology*, vol. 29, no. 4, pp. 387–405, 1985.

[28] B. De Baets, H. De Meyer, and K. De Loof, "On the cycle-transitivity of the mutual rank probability relation of a poset," *Fuzzy Sets and Systems*, vol. 161, no. 20, pp. 2695–2708, 2010.

[29] B. De Baets, H. De Meyer, B. De Schuymer, and S. Jenei, "Cyclic evaluation of transitivity of reciprocal relations," *Social Choice and Welfare*, vol. 26, no. 2, pp. 217–238, 2006.

[30] B. De Baets and J. Fodor, "Preference modelling, a matter of degree," in *Trends in Multi Criterial Decision Analysis*, M. Ehrgott, J. R. Figueira, and S. Greco, Eds. Springer, 2010, vol. 142, pp. 123–156.

[31] J.-C. de Borda, "Sur la forme des elections au scrutin," *Histoire de l'Académie Royale des Sciences*, pp. 657–665, 1784.

[32] J. A. N. de Condorcet, *Essai sur l'Application de l'Analyse á la Probabilité des Décisions Rendues á la Pluralité des Voix.* L'Imprimerie Royale, 1785.

[33] D. Dubois and H. Prade, *Fuzzy Sets and Systems: Theory and Applications*, ser. Mathematics in Science and Engineering. Academic Press, 1980, vol. 144.

[34] ——, "An introduction to bipolar representations of information and preference," *International Journal of Intelligent Systems*, vol. 23, no. 8, pp. 866–877, 2008.

[35] Z. Duszak and W. W. Koczkodaj, "Generalization of a new definition of consistency for pairwise comparisons," *Information Processing Letters*, vol. 52, no. 5, pp. 273–276, 1994.

[36] M. Ehrgott, *Multicriteria optimization*, ser. Lecture Notes in Economics and Mathematical Systems. Springer-Verlag, 2000.

[37] S. Eilon, "OR is not mathematics," *Omega*, vol. 15, no. 2, pp. 87–92, 1987.

[38] P. Eklund, M. Fedrizzi, and H. Nurmi, "A categorical approach to the extension of social choice functions," in *IPMU*, vol. 2, 2010, pp. 261–270.

[39] Z.-P. Fan, G.-F. Hu, and S.-H. Xiao, "A method for multiple attribute decision-making with the fuzzy preference relation on alternatives," *Computers & Industrial Engineering*, vol. 46, no. 2, pp. 321–327, 2004.

[40] Z.-P. Fan, J. Ma, Y.-P. Jiang, J.-H. Sung, and L. Ma, "A goal programming approach to group decision making based on multiplicative preference relations and fuzzy preference relations," *European Journal of Operational Research*, vol. 174, no. 1, pp. 311–321, 2006.

[41] Z.-P. Fan, J. Ma, and Q. Zhang, "An approach to multiple attribute decision making based on fuzzy preference information on alternatives," *Fuzzy Sets and Systems*, vol. 131, no. 1, pp. 101–106, 2002.

[42] M. Fedrizzi, M. Fedrizzi, and R. A. Marques Pereira, "On the issue of consistency in dynamical consensual aggregation," in *Technologies for Constructing Intelligent Systems*. Heidelberg, Germany: Physica-Verlag GmbH, 2002, vol. 1.

[43] M. Fedrizzi, "On a consensus measure in a group MCDM problem," in *Multiperson Decision Making Models using Fuzzy Sets and Possibility Theory*, J. Kacprzyk and M. Fedrizzi, Eds. Kluwer Academic Publishers, 1990, p. 231241.

[44] M. Fedrizzi and M. Brunelli, "Fair consistency evaluation for reciprocal relations and in group decision making," *New Mathematics & Natural Computation*, vol. 5, no. 2, pp. 407–420, 2009.

[45] ——, "On the normalisation of a priority vector associated with a reciprocal relation," *International Journal of General Systems*, vol. 38, no. 5, pp. 579–586, 2009.

[46] ——, "On the priority vector associated with a reciprocal relation and with a pairwise comparison matrix," *Soft Computing*, vol. 14, no. 6, pp. 639–645, 2010.

[47] M. Fedrizzi and S. Giove, "Incomplete pairwise comparisons and consistency optimization," *European Journal of Operational Research*, vol. 183, no. 1, pp. 303–313, 2007.

[48] P. C. Fishburn, "Preference relations and their numerical representations," *Theoretical Computer Science*, vol. 217, no. 2, pp. 359–383, 1999.

[49] J. Fodor and M. Roubens, *Fuzzy Preference Modelling and Multicriteria Decision Support*, ser. Theory and decision library D: System Theory, Knowledge Engineering and Problem Solving. Kluwer Academic Publishers, 1994, vol. 14.

[50] E. Forman and K. Peniwati, "Aggregating individual judgments and priorities with the analytic hierarchy process," *European Journal of Operational Research*, vol. 108, no. 1, pp. 165–169, 1998.

[51] R. Fullér and P. Majlender, "An analytic approach for obtaining maximal entropy OWA operator weights," *Fuzzy Sets and Systems*, vol. 124, no. 1, pp. 53–57, 2001.

[52] J. A. Goguen, "*L*-fuzzy sets," *Journal of Mathematical Analysis and Applications*, vol. 18, no. 1, pp. 145–174, 1967.

[53] B. L. Golden and Q. Wang, "An alternate measure of consistency," in *The Analythic Hierarchy Process, Applications and studies*, B. L. Golden, E. A. Wasil, and P. T. Harker, Eds. Springer-Verlag, 1989, pp. 68–81.

[54] Z.-W. Gong, "Least-square method to priority of the fuzzy preference relations with incomplete information," *International Journal of Appoximate Reasoning*, vol. 47, no. 2, pp. 258–264, 2008.

[55] P. T. Harker, "Alternative modes of questioning in the analytic hierarchy process," *Mathematical Modelling*, vol. 9, pp. 353–360, 1984.

[56] ——, “Incomplete pairwise comparisons in the analytic hierarchy process,” *Mathematical Modelling*, vol. 9, no. 11, pp. 837–848, 1987.

[57] E. Herrera-Viedma, S. Alonso, F. Chiclana, and F. Herrera, “A consensus model for group decision making with incomplete fuzzy preference relations,” *IEEE Transactions on Fuzzy Systems*, vol. 15, no. 5, pp. 863–877, 2007.

[58] E. Herrera-Viedma, F. Chiclana, F. Herrera, and S. Alonso, “Group decision-making model with incomplete fuzzy preference relations based on additive consistency,” *IEEE Transactions on Systems, Man and Cybernetics, Part B*, vol. 37, no. 1, pp. 176–189, 2007.

[59] E. Herrera-Viedma, F. Herrera, F. Chiclana, and M. Luque, “Some issues on consistency of fuzzy preference relations,” *European Journal of Operational Research*, vol. 154, no. 1, pp. 98–109, 2004.

[60] F. S. Hillier and G. J. Lieberman, *Introduction to Operational Research.* Mc-Graw Hill, 2005.

[61] R. A. Horn and C. R. Johnson, *Matrix Analysis.* Cambridge U, 1985.

[62] M. Katz and H. Rosen, *Microeconomics.* McGraw Hill, 1997.

[63] R. L. Keeney and H. Raiffa, *Decisions with multiple objectives: Preferences and value tradeoffs.* Wiley, New York, 1976.

[64] P. Keys, *Understanding the process of operational research*, P. Keys, Ed. Wiley, 1994.

[65] G. J. Klir, *Uncertainty and Information: Foundations of Generalized Information Theory.* Wiley, 2005.

[66] G. J. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic: Theory and Applications.* Pretience Hall, 1995.

[67] W. W. Koczkodaj, “New definition of consistency of pairwise comparisons,” *Mathematical and Computer Modelling*, vol. 18, no. 7, pp. 79–84, 1993.

[68] H.-S. Lee, “A fuzzy method for evaluating suppliers,” in *KES*, 2006, pp. 1035–1043.

[69] H.-S. Lee, P.-D. Shen, and W.-L. Chyr, “Prioritization of incomplete fuzzy preference relation,” in *KES*, vol. 2, 2008, pp. 974–979.

[70] H.-S. Lee and W.-K. Tseng, “Goal programming methods for constructing additive consistency fuzzy preference relations,” in *KES*, 2006, pp. 910–916.

[71] H.-S. Lee and C.-H. Yeh, “Fuzzy multi-criteria decision making based on fuzzy preference relation,” in *KES*, vol. 2, 2008, pp. 980–985.

[72] C.-C. Lin, “A revised framework for deriving preference values from pairwise comparison matrices,” *European Journal of Operational Research*, vol. 176, no. 2, pp. 1145–1150, 2007.

[73] P. Linares, “Are inconsistent decisions better? an experiment with pairwise comparisons,” *European Journal of Operational Research*, vol. 193, no. 2, pp. 492–498, 2009.

[74] S. Lipovetsky and M. W. Conklin, "Robust estimation of priorities in the AHP," *European Journal of Operational Research*, vol. 137, no. 1, pp. 110–122, 2002.

[75] R. Luce and A. Perry, "A method of matrix analysis of group structure," *Psychometrika*, vol. 14, no. 2, pp. 95–116, 1949.

[76] J. Ma, Z.-P. Fan, Y.-P. Jiang, J.-Y. Mao, and L. Ma, "A method for repairing the inconsistency of fuzzy preference relations," *Fuzzy Sets and Systems*, vol. 157, no. 1, pp. 20–33, 2006.

[77] G. A. Miller, "The magical number seven, plus or minus two: some limits on our capacity for processing information," *Psychological Review*, vol. 63, no. 5, pp. 81–97, 1956.

[78] J. Mingers, "Soft OR comes of age—but not everywhere!" *Omega*, vol. 39, no. 6, pp. 729–741, 2011.

[79] J. N. Mordeson and P. S. Nair, *Fuzzy Graphs and Fuzzy Hypergraphs*, ser. Studies in Fuzziness and Soft Computing. Phisica-Verlag, 2000, vol. 46.

[80] M. Newman, A.-L. Barabási, and D. J. Watts, *The structure and dynamics of networks*. Princeton University Press, 2006.

[81] H. Nurmi, *Comparing voting systems*, ser. Theory and decision library. Series A, Philosophy and methodology of the social sciences. Dortrecht, 1987.

[82] M. O'Hagan, "Aggregating template or rule antecedents in real-time expert systems with fuzzy set logic," in *Twenty-Second Asilomar Conference on Signals, Systems and Computers*, 1988.

[83] S. A. Orlovsky, "Decision making with a fuzzy preference relation," *Fuzzy Sets and Systems*, vol. 1, pp. 155–167, 1978.

[84] E. R. Peay, "Hierarchical clique structures," *Sociometry*, vol. 37, no. 1, pp. 54–65, 1974.

[85] J. I. Peláez and M. T. Lamata, "A new measure of inconsistency for positive reciprocal matrices," *Computers and Mathematics with Applications*, vol. 46, no. 12, pp. 1839–1845, 2003.

[86] G. Qian and X.-Q. Feng, "Intuitionistic weight generation approach from intuitionistic preference relations," in *Proceedings of the Seventh International Conference on Machine Learning and Cybernetics*, vol. 1. IEEE Computer Society, 2008, pp. 536–541.

[87] ——, "The satisfactory consistency of fuzzy preference relations," in *Proceedings of the Seventh International Conference on Machine Learning and Cybernetics*, vol. 1. IEEE Computer Society, 2008, pp. 600–605.

[88] M. Rademaker and B. De Baets, "Aggregation of monotone reciprocal relations with application to group decision making," *Fuzzy Sets and Systems*, inpress.

[89] R. Ramanathan and L. S. Ganesh, "Group preference aggregation methods employed in AHP: An evaluation and an intrinsic process for deriving members' weightages," *European Journal of Operational Research*, vol. 79, no. 2, pp. 249–265, 1994.

[90] J. Ramík and P. Korviny, "Inconsistency of pair-wise comparison matrix with fuzzy elements based on geometric mean," *Fuzzy Sets and Systems*, vol. 161, no. 11, pp. 1604–1613, 2010.

[91] J. Ramík and R. Perzina, "A method for solving fuzzy multicriteria decision problems with dependent criteria," *Fuzzy Optimization and Decision Making*, vol. 9, no. 2, pp. 123–141, 2010.

[92] B. Roy, "Classement et choix en présence de points de vue multiples (la méthode ELECTRE)"." *la Revue d'Informatique et de Recherche Opérationelle*, vol. 8, pp. 57–75, 1968.

[93] T. L. Saaty, *Mathematical Methods of Operational Research.* McGraw Hill, 1959.

[94] ——, "Measuring the fuzziness of sets," *Journal of Cybernetics*, vol. 4, no. 4, pp. 53–61, 1974.

[95] ——, "A scaling method for priorities in hierarchical structures," *Journal of Mathematical Psychology*, vol. 15, no. 3, pp. 234–281, 1977.

[96] ——, *The Analythical Hierarchy Process.* New York: McGraw-Hill, 1980.

[97] ——, *The logic of priorities: Applications in business, energhy, health and transportation.* Boston: Kluwer-Nijhoff, 1982.

[98] ——, "Axiomatic foundation of the analythic hierarchy process," *Management Science*, vol. 32, no. 7, pp. 841–855, 1986.

[99] ——, *Decision making for leaders. The analytic hierarchy process for decisions in a complex world.* University of Pittsburgh Press, 1988.

[100] T. L. Saaty and L. G. Vargas, *The Logic of Priorities: Applications in Business, Energy, Health, Transportation*, ser. The International Series in Management Science & Operations Research. Kluwer-Nijhoff Publishing, Boston, 1982.

[101] ——, "Modeling behavior in competition: The analytic hierarchy process," *Applied Mathematics and Computation*, vol. 16, no. 1, pp. 49–92, 1989.

[102] ——, *Models, methods, concepts & applications of the analytic hierarchy process.* Kluwer Academic, 2001.

[103] J. Scott, *Social Network Analysis: A Handbook.* Sage, 2000.

[104] M. Shimura, "Fuzzy sets concept in rank-ordering objects," *Journal of Mathematical analysis and applications*, vol. 43, no. 3, pp. 717–733, 1973.

[105] S. Shiraishi and T. Obata, "On a maximization problem arising from a positive reciprocal matrix in the AHP," *Bulletin of Informatics and Cybernetics*, vol. 34, pp. 91–96, 2002.

[106] S. Shiraishi, T. Obata, and M. Daigo, "Properties of a positive reciprocal matrix and their application to AHP," *Journal of the Operations Research Society of Japan*, vol. 41, no. 3, pp. 404–414, 1998.

[107] S. Shiraishi, T. Obata, M. Daigo, and N. Nakajima, "Assessment for an incomplete comparison matrix and improvement of an inconsistent comparison: computational experiments," in *ISAHP*, 1999.

[108] C. Spearman, "The proof and measurement of association between two things," *The American Journal of Psychology*, vol. 15, no. 1, pp. 72–101, 1904.

[109] M. E. Squillante, "Decision making in social networks," *International Journal of Intelligent Systems*, vol. 25, no. 3, [special issue] 2010.

[110] W. E. Stein and P. J. Mizzi, "The harmonic consistency index for the analytic hierarchy process," *European Journal of Operational Research*, vol. 177, no. 1, pp. 488–497, 2007.

[111] S. S. Stevens, "On the scales of measurement," *Science*, vol. 103, no. 2684, pp. 677–680, 1946.

[112] T. Tanino, "Fuzzy preference orderings in group decision making," *Fuzzy Sets and Systems*, vol. 12, no. 2, pp. 117–131, 1984.

[113] J. Temesi, "Pairwise comparison matrices and the error-free property of the decision maker," *Central European Journal of Operations Research*, vol. 19, no. 2, pp. 239–249, 2011.

[114] L. L. Thurstone, "A law of comparative judgments," *Psychological Review*, vol. 34, pp. 273–286, 1927.

[115] E. Triantaphyllou, *Multi-Criteria Decision Making Methods: A Comparative Study.* Springer, 2000.

[116] R. C. Van Der Honert and F. A. Lootsma, "Group preference aggregation in the multiplicative AHP the model of the group decision process and pareto optimality," *European Journal of Operational Research*, vol. 96, no. 2, pp. 363–370, 1997.

[117] J. von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior.* Princeton University Press, 1944.

[118] J. Wallenius, J. S. Dyer, P. C. Fishburn, R. E. Steuer, S. Zionts, and K. Deb, "Multiple criteria decision making, multiattribute utility theory: Recent accomplishments and what lies ahead," *Management Science*, vol. 54, no. 7, pp. 1336–1349, 2008.

[119] Y.-M. Wang and Z.-P. Fan, "Fuzzy preference relations: Aggregation and weight determination," *Computers & Industrial Engineering*, vol. 53, no. 1, pp. 163–172, 2007.

[120] Y.-M. Wang, Z.-P. Fan, and H. Zhongsheng, "A chi-square method for obtaining a priority vector from multiplicative and fuzzy preference relations," *European Journal of Operational Research*, vol. 182, no. 1, pp. 356–366, 2007.

[121] Y.-M. Wang and C. Parkan, “Multiple attribute decision making based on fuzzy preference information on alternatives: Ranking and weighting,” *Fuzzy Sets and Systems*, vol. 153, pp. 331–346, 2005.

[122] S. Wasserman and K. Faust, *Social Network Analysis: Methods and Applications.* Cambridge University Press, 1994.

[123] Z. Xu, “Goal programming models for obtaining the priority vector of incomplete fuzzy preference relations,” *International Journal of Appoximate Reasoning*, vol. 36, no. 3, pp. 261–270, 2004.

[124] ——, “A procedure for decision making based on incomplete fuzzy preference relation,” *Fuzzy Optimization and Decision Making*, vol. 4, no. 3, pp. 175–189, 2005.

[125] ——, “Approaches to multiple attribute decision making with intuitionistic fuzzy preference information,” *Systems Engineering - Theory & Practice*, vol. 27, no. 11, pp. 62–71, 2007.

[126] ——, “Dynamic fuzzy preference relations,” in *Fifth International Conference on Fuzzy Systems and Knowledge Discovery*, vol. 1, 2007, pp. 125–129.

[127] ——, “A method for estimating criteria weights from intuitionistic preference relations,” in *Proceedings of the International Conference of Fuzzy Information and Engineering (ICFIE)*, 2007, pp. 503–512.

[128] ——, “Multiple-attribute group decision making with different formats of preference information on attributes,” *IEEE Transactions on Systems, Man, and Cybernetics, Part B*, vol. 37, no. 6, pp. 1500–1511, 2007.

[129] ——, “A survey of preference relations,” *International Journal of General Systems*, vol. 36, no. 2, pp. 179–203, 2007.

[130] ——, “A method for estimating criteria weights from intuitionistic preference relations,” *Fuzzy Information and Engineering*, vol. 1, no. 1, pp. 79–89, 2009.

[131] ——, “An integrated model-based interactive approach to FMAGDM with incomplete preference information,” *Fuzzy Optimization and Decision Making*, vol. 9, no. 3, pp. 333–357, 2010.

[132] Z. Xu and J. Chen, “A subjective and objective integrated method for MAGDM problems with multiple types of exact preference formats,” in *IDEAL*, 2007, pp. 145–154.

[133] ——, “Group decision-making procedure based on incomplete reciprocal relations,” *Soft Computing*, vol. 12, no. 6, pp. 515–521, 2008.

[134] ——, “MAGDM linear programming models with distinct uncertain preference structures,” *IEEE Transactions on Systems, Man, and Cybernetics, Part B*, vol. 38, no. 5, pp. 1356–1370, 2008.

[135] ——, “Some models for deriving the priority weights from interval fuzzy preference relations,” *European Journal of Operational Research*, vol. 184, no. 1, pp. 266–280, 2008.

[136] Z. Xu and Q. Da, "A least deviation method to obtain a priority vector of a fuzzy preference relation," *European Journal of Operational Research*, vol. 164, pp. 206–216, 2005.

[137] R. R. Yager, "Ordered weighted averaging operators in multicriteria decision making," *IEEE Transactions on Systems, Man, and Cybernetics*, vol. 18, no. 1, pp. 183–190, 1988.

[138] ——, "Intelligent social network analysis using granular computing," *International Journal of Intelligent Systems*, vol. 23, no. 11, pp. 1197–2020, 2008.

[139] R. R. Yager and J. Kacprzyk, *The ordered weighted averaging operators: Theory and applications.* Kluwer Academic Publishers, 1997.

[140] L. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, pp. 338–353, 1965.

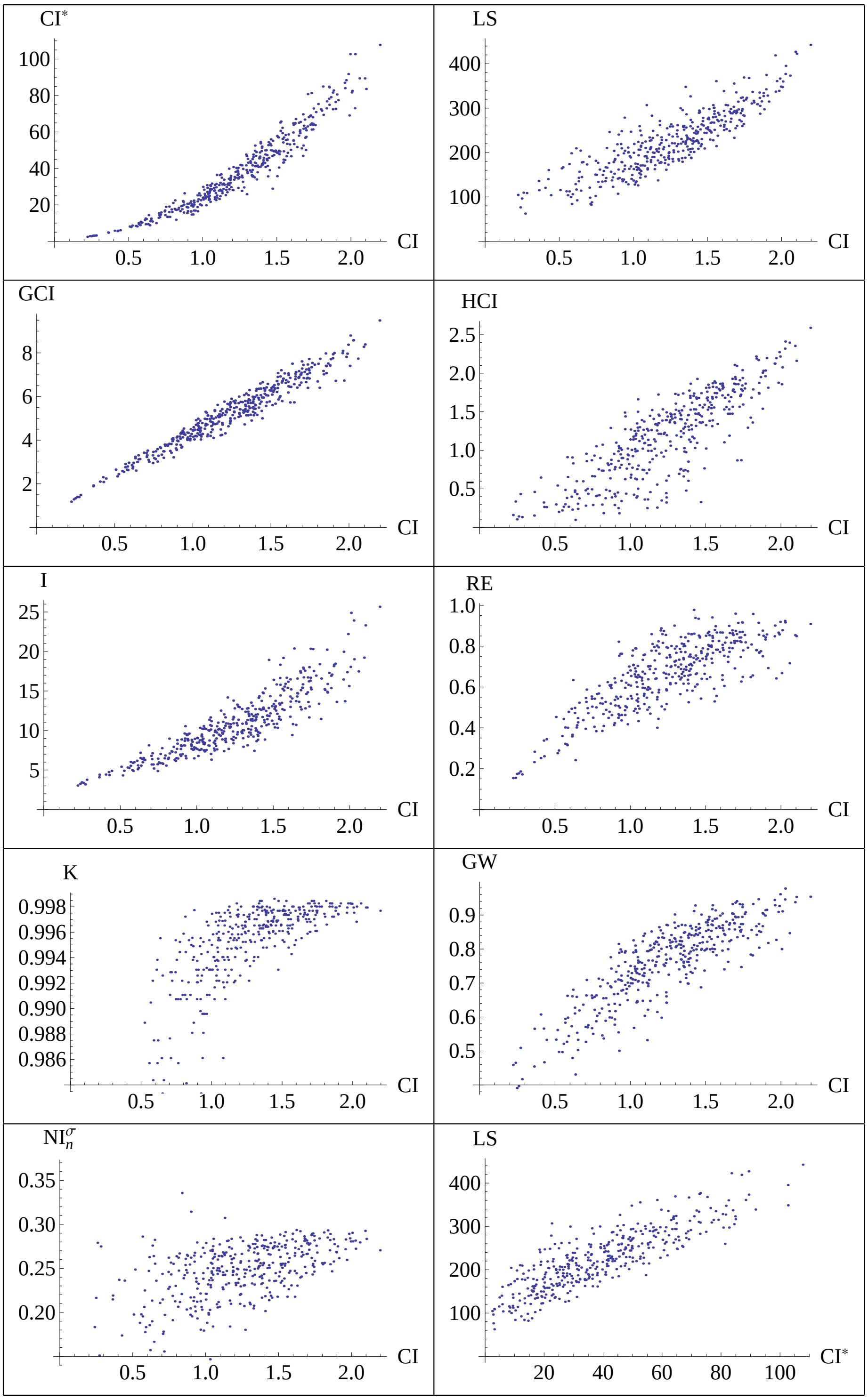
[141] L. A. Zadeh, "Similarity relations and fuzzy orderings," *Information Sciences*, vol. 3, no. 2, pp. 177–200, 1978.

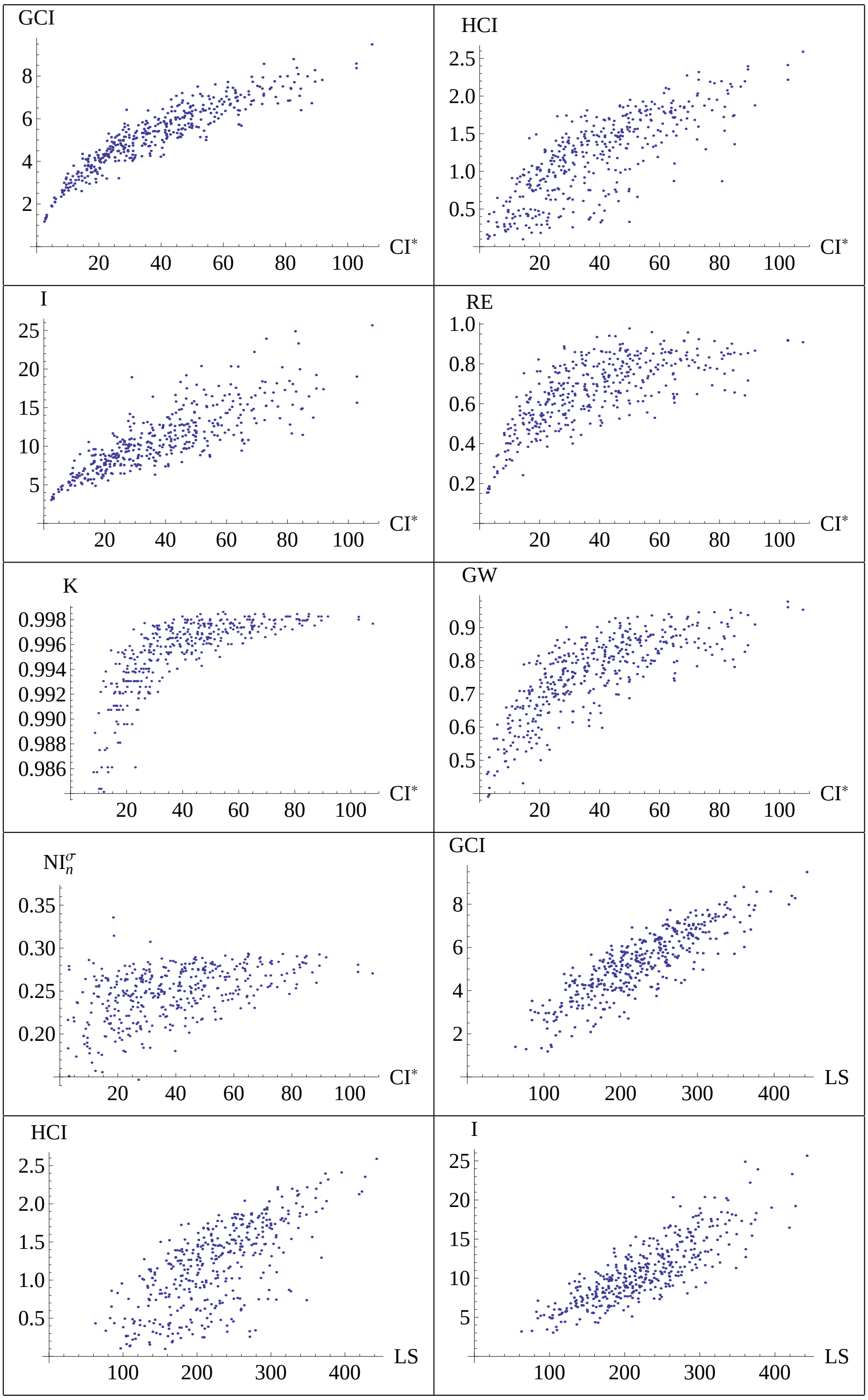
[142] M. Zelený, *Multi Criteria Decision Making.* McGraw Hill, 1982.

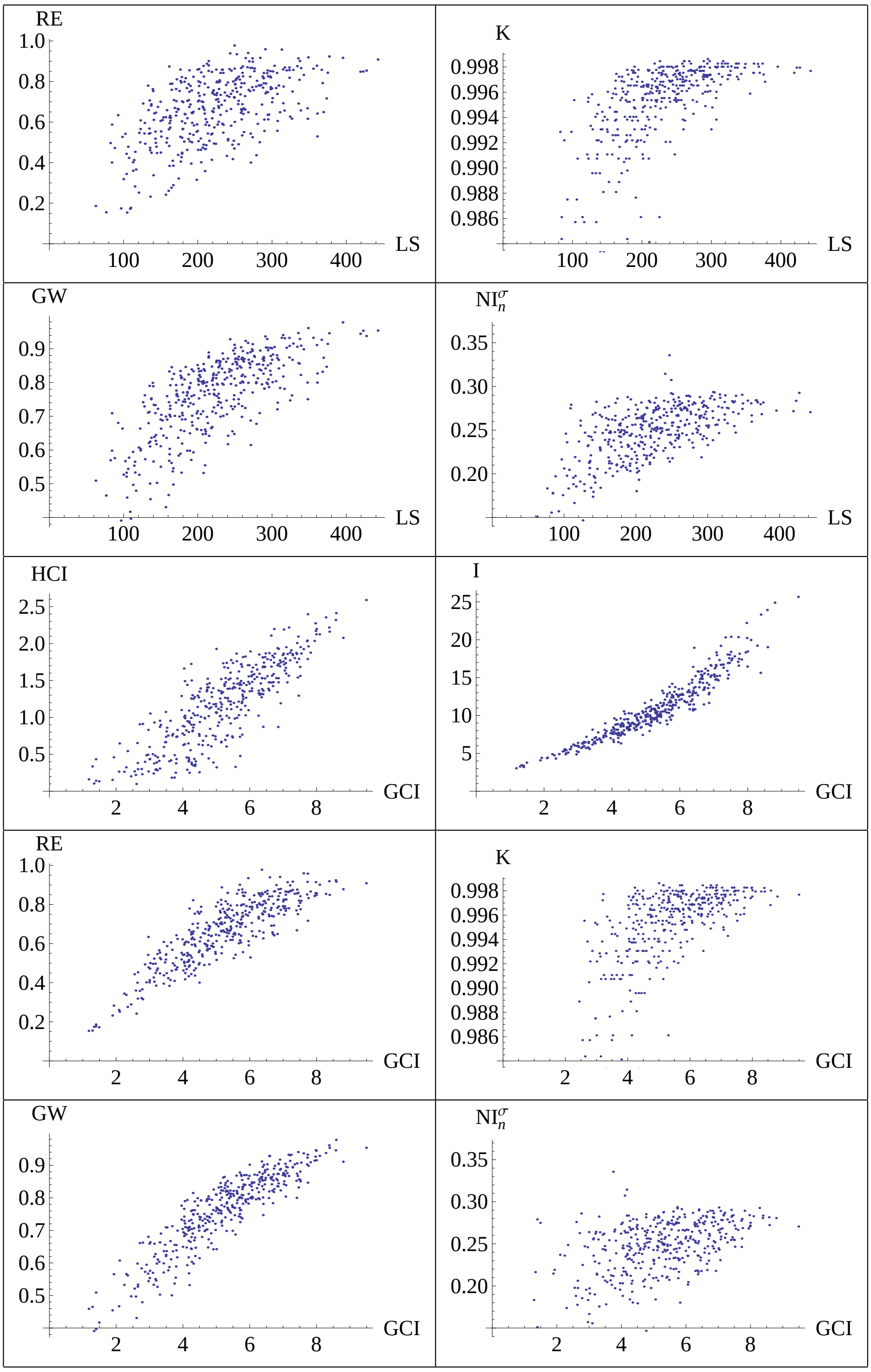
Appendix A

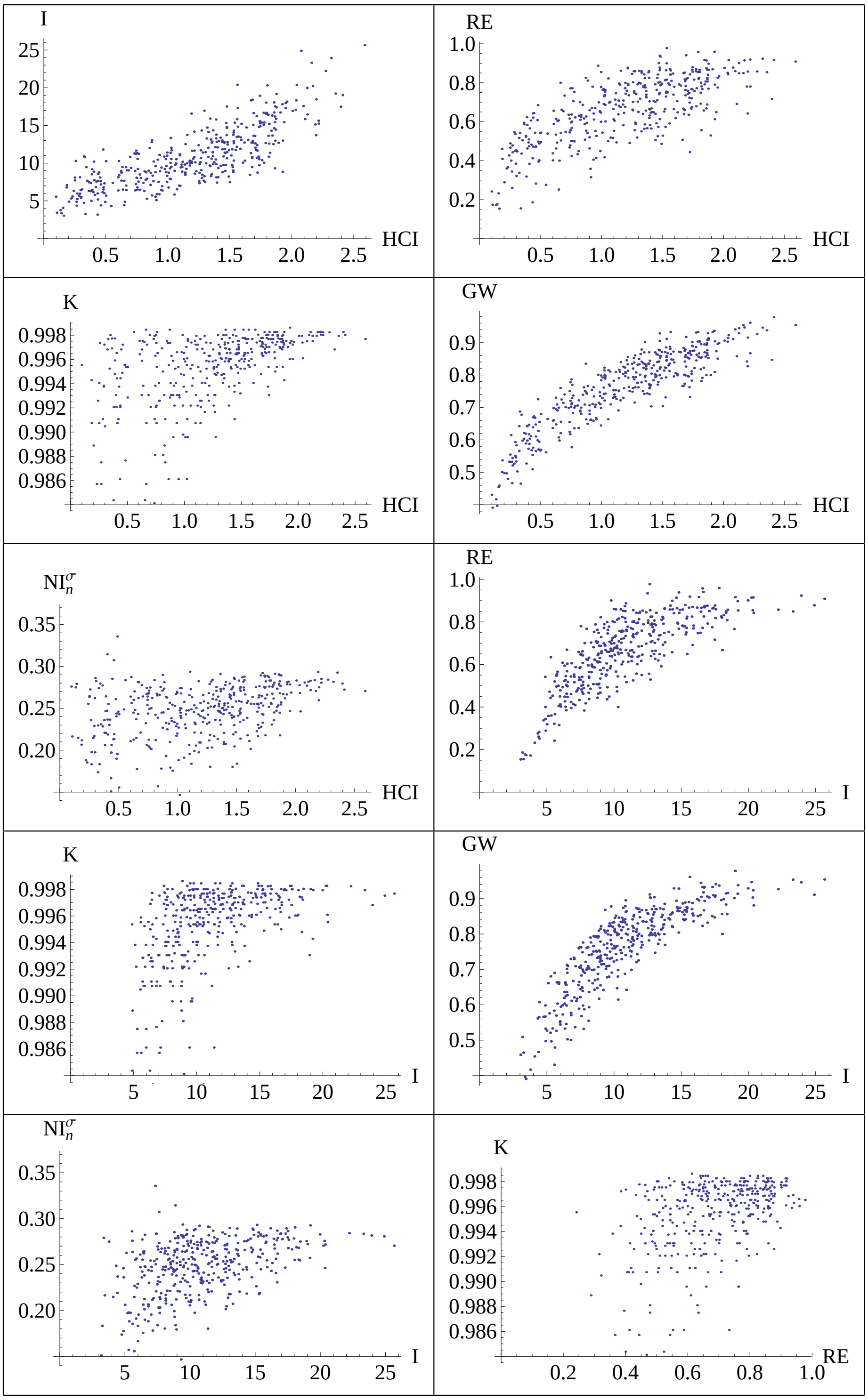
Scatter plots of consistency indices

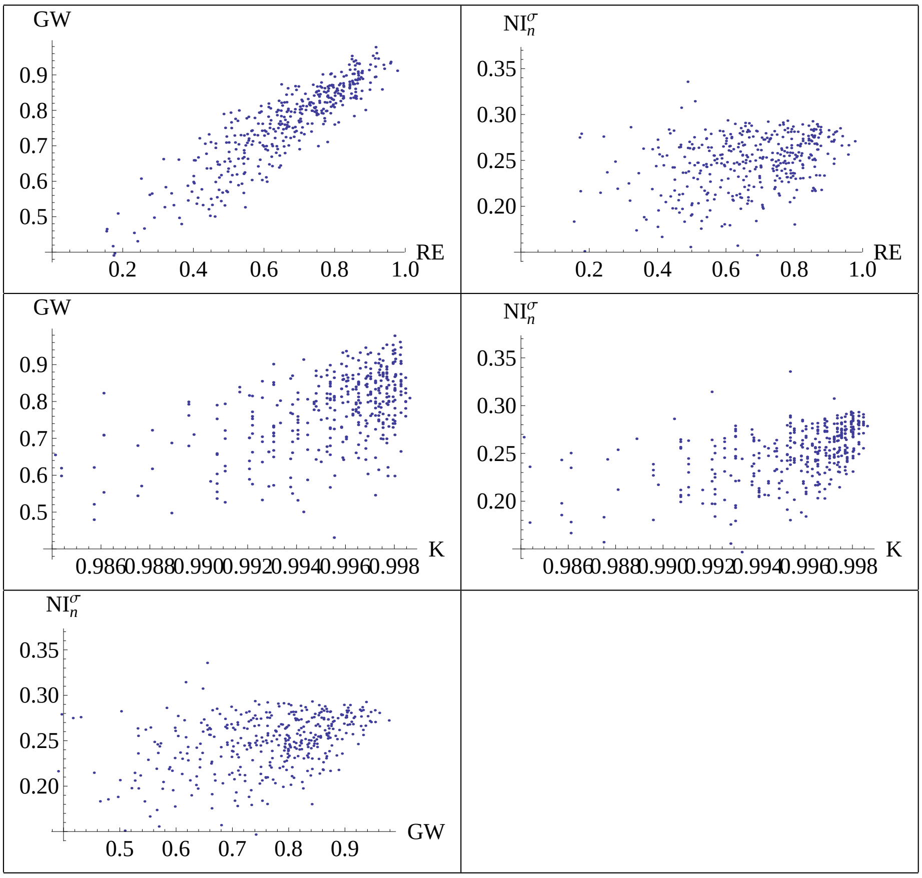
This appendix contains the scatter plots of the consistency indices presented in chapter 5 and previously summarized in figure 5.1. However, in light of the results on the proportionality between some inconsistency indices, only nine of them are here presented. The following plots are simply enlargements of the upper triangle of figure 5.1. However, more information is gathered in the following, as, not only their size is larger but values are reported on the axes thus showing some possible unexpected behavior. Under this point of view, due to their low co-monotonicity with the other indices, K and NI_n^σ are surely interesting quantities.


CI*
100
80
60
40
20
0.5
1.0
1.5
2.0
CI
LS
400
300
200
100
0.5
1.0
1.5
2.0
CI
GCI
8
6
4
2
0.5
1.0
1.5
2.0
CI
HCI
2.5
2.0
1.5
1.0
0.5
0.5
1.0
1.5
2.0
CI
I
25
20
15
10
5
0.5
1.0
1.5
2.0
CI
RE
1.0
0.8
0.6
0.4
0.2
0.5
1.0
1.5
2.0
CI
K
0.998
0.996
0.994
0.992
0.990
0.988
0.986
0.5
1.0
1.5
2.0
CI
GW
0.9
0.8
0.7
0.6
0.5
0.5
1.0
1.5
2.0
CI
NI$_n^\sigma$
0.35
0.30
0.25
0.20
0.5
1.0
1.5
2.0
CI
LS
400
300
200
100
20
40
60
80
100
CI*


GCI
8
6
4
2
20 40 60 80 100
CI*
HCI
2.5
2.0
1.5
1.0
0.5
20 40 60 80 100
CI*
I
25
20
15
10
5
20 40 60 80 100
CI*
RE
1.0
0.8
0.6
0.4
0.2
20 40 60 80 100
CI*
K
0.998
0.996
0.994
0.992
0.990
0.988
0.986
20 40 60 80 100
CI*
GW
0.9
0.8
0.7
0.6
0.5
20 40 60 80 100
CI*
NI_n^σ
0.35
0.30
0.25
0.20
20 40 60 80 100
CI*
GCI
8
6
4
2
100 200 300 400
LS
HCI
2.5
2.0
1.5
1.0
0.5
100 200 300 400
LS
I
25
20
15
10
5
100 200 300 400
LS


RE
1.0
0.8
0.6
0.4
0.2
100 200 300 400
LS
K
0.998
0.996
0.994
0.992
0.990
0.988
0.986
100 200 300 400
LS
GW
0.9
0.8
0.7
0.6
0.5
100 200 300 400
LS
NI$_n^\sigma$
0.35
0.30
0.25
0.20
100 200 300 400
LS
HCI
2.5
2.0
1.5
1.0
0.5
2 4 6 8
GCI
I
25
20
15
10
5
2 4 6 8
GCI
RE
1.0
0.8
0.6
0.4
0.2
2 4 6 8
GCI
K
0.998
0.996
0.994
0.992
0.990
0.988
0.986
2 4 6 8
GCI
GW
0.9
0.8
0.7
0.6
0.5
2 4 6 8
GCI
NI$_n^\sigma$
0.35
0.30
0.25
0.20
2 4 6 8
GCI


I
25
20
15
10
5
0.5 1.0 1.5 2.0 2.5
HCI
RE
1.0
0.8
0.6
0.4
0.2
0.5 1.0 1.5 2.0 2.5
HCI
K
0.998
0.996
0.994
0.992
0.990
0.988
0.986
0.5 1.0 1.5 2.0 2.5
HCI
GW
0.9
0.8
0.7
0.6
0.5
0.5 1.0 1.5 2.0 2.5
HCI
NI_n^σ
0.35
0.30
0.25
0.20
0.5 1.0 1.5 2.0 2.5
HCI
RE
1.0
0.8
0.6
0.4
0.2
5 10 15 20 25
I
K
0.998
0.996
0.994
0.992
0.990
0.988
0.986
5 10 15 20 25
I
GW
0.9
0.8
0.7
0.6
0.5
5 10 15 20 25
I
NI_n^σ
0.35
0.30
0.25
0.20
5 10 15 20 25
I
K
0.998
0.996
0.994
0.992
0.990
0.988
0.986
0.2 0.4 0.6 0.8 1.0
RE



GW
0.9
0.8
0.7
0.6
0.5
0.2
0.4
0.6
0.8
1.0
RE
NI_n^σ
0.35
0.30
0.25
0.20
0.2
0.4
0.6
0.8
1.0
RE
GW
0.9
0.8
0.7
0.6
0.5
0.986
0.988
0.990
0.992
0.994
0.996
0.998
K
NI_n^σ
0.35
0.30
0.25
0.20
0.986
0.988
0.990
0.992
0.994
0.996
0.998
K
NI_n^σ
0.35
0.30
0.25
0.20
0.5
0.6
0.7
0.8
0.9
GW

Appendix B

The Analytic Hierarchy Process

The Analytic Hierarchy Process (AHP) is a multi-criteria decision making method originally introduced by T.L. Saaty in 1974 [94]. Since its official inception [95] it has been widely studied and applied to several real-world decision making problems in business, engineering, political sciences, marketing, forecasting, conflict resolution and so on [96, 97, 99]. It can tackle problems involving several alternatives and criteria and was also axiomatic justified [98]. This appendix aims to present the very fundamentals of AHP in a practical way, thus skipping all the unnecessary details.

As already mentioned, in our framework, the AHP can be applied to most decision making problem involving a plurality of alternatives and criteria. Formally, every decisional process has one *goal* and there is always a finite set of *alternatives* $X = \{x_1, \ldots, x_n\}$ from which the decision maker, is usually asked to pick the best one. Also, in the decision making process the expert has to consider a set of *criteria* $C = \{c_1, \ldots, c_n\}$. Criteria are characteristics which make one alternative preferable to another one. Let us now present one of the most classical among the examples: a family has to decide which city to visit during their holidays. Their goal is the highest overall satisfaction with the trip. Alternatives may be some cities, in our example

$$X = \{\text{Rome}, \text{Barcelona}, \text{Rejkyavik}\}, \tag{B.1}$$

and the set of criteria could be

$$C = \{\text{climate}, \text{sightseeing}, \text{environment}\}, \tag{B.2}$$

Later, we will return to this example in order to illustrate some theoretical matters.

In usual decision making processes the decision maker assigns a score to each alternative and then he chooses the alternative with the maximum value. That is, given X there is a weight vector

$$\mathbf{w} = (w_1, \ldots, w_n), \tag{B.3}$$

where w_i is a value which can coherently estimates the utility[1] of alternative x_i, the higher it is, the better the alternative is. Weight vectors are nothing other than rankings and their components are scores. Components w_i of $\mathbf{w}$ are also called priorities or weights of the alternatives x_i. For example

$$\mathbf{w} = (0.4, 0.2, 0.3, 0.1) \tag{B.4}$$

implies $x_1 \succ x_3 \succ x_2 \succ x_4$. Making decisions in this way seems to be rather easy. Unfortunately, it becomes a hard task whenever complexity increases. As we will see, complexity augments as the number of alternatives and criteria increases. In the following, at first, we consider the case of complexity in alternatives, then we discuss the complexity in criteria.

The AHP in practice

Complexity in alternatives - From the priority vector to pairwise comparison matrix

It is clear that the decision maker could run into troubles when he/she is asked to submit a rating for a large number of alternatives. It often happens that we cannot decide among several alternatives or, even worse, we decide and then we realize that it was not the right choice.

An effective way to overcome this problem is using pairwise comparison matrices. According to some studies, the most reasonable point in doing so, is that such matrices allow the decision maker to consider two alternatives at a time. More formally, the aspect of a pairwise comparison matrix $\mathbf{A} = (a_{ij})_{n \times n}$ is

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \tag{B.5}$$

with $a_{ij} \in [\frac{1}{9}, 9]$ that is a value which can express the degree of preference of x_i over x_j. More precisely, according to Saaty's theory, each entry is

[1]Similarly to what happens for utility theory, the main rule is that x_i should be preferred to x_j if and only if $w_i \geq w_j$

supposed to approximate the ratio between two weights

$$a_{ij} \approx \frac{w_i}{w_j}. \tag{B.6}$$

As soon as (B.6) is understood, multiplicative reciprocity $a_{ij}a_{ji} = 1$ holds and, still thanks to (B.6), **A** in (B.5) can be simplified and rewritten

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & \cdots & a_{1n} \\ \frac{1}{a_{12}} & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{a_{1n}} & \frac{1}{a_{2n}} & \cdots & 1 \end{pmatrix}. \tag{B.7}$$

The fact that multiplicative reciprocity holds thanks to (B.6) is easy to be verified.

Let us now proceed with the example and build a pairwise comparison matrix for the set of cities X.

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 6 \\ 1/3 & 1 & 2 \\ 1/6 & 1/2 & 1 \end{pmatrix}$$

From this matrix, in particular from entry a_{12}, one can figure out that x_1 is considered to be three times better than x_2. Once a pairwise comparison matrix is fulfilled there are many methods to derive the priority vector **w** [2]. In the example it is

$$\mathbf{w} = \begin{pmatrix} 6/9 \\ 2/9 \\ 1/9 \end{pmatrix},$$

and thus Rome is the best alternative. Using the example it is possible to verify that (B.6) holds for all $i, j = 1, \ldots, n$. To summarize, we have shown that, whenever the number of alternatives is too large, pairwise comparing alternatives is an effective way for obtaining a ranking. Perhaps we have spent a bit more of our time but **w** is now far more reliable that it would be if it had been estimated without using **A**.
Obviously, the case involving three alternatives is trivial and solely illustrative and pairwise comparison matrices are of great help only if the number of alternatives is quite large.

From the pairwise comparison matrix to the hierarchy

At this point, it is time to wonder why **A** was filled in that particular way and what factors could have guided the decision maker. Of course, such decision

[2]The most famous one is the eigenvector method according to which vector **w** is the principal right eigenvector of **A**

factors are few if the expert is choosing the kind of bread to buy (mainly price and taste) whereas they were many when, at the age of eighteen, he/she was asked to choose his/her perspective degree course (location, reputation, future employment and many others). First, we should start using the word criterion instead of factor.

The main problem is that matrix $\mathbf{A}$ compares alternatives without considering criteria. Simply, when filling it the decision maker was only thinking about his/her overall satisfaction with the alternatives and he/she did not make any separate reasoning about the criteria, e.g. cost, sightseeing and environment in the example.

Once again, complexity rises and the best way to overcome this problem is to decompose it. This is why, at this point, Saaty suggested to build a different matrix for each criterion (we will see later what happens). Hence, in the following, a matrix $\mathbf{A}^{(k)}$ is the matrix of pairwise comparisons between alternatives according to criterion k, for example, entry a_{13} of matrix $\mathbf{A}^{(c)}$ entails that the decision maker prefers Rome over Reykjavik if he/she compares them according to the climatic point of view only.

$$\mathbf{A}^{(c)} = \begin{pmatrix} 1 & 1 & 4 \\ 1 & 1 & 4 \\ 1/4 & 1/4 & 1 \end{pmatrix} \quad \mathbf{A}^{(s)} = \begin{pmatrix} 1 & 2 & 6 \\ 1/2 & 1 & 3 \\ 1/6 & 1/3 & 1 \end{pmatrix} \quad \mathbf{A}^{(e)} = \begin{pmatrix} 1 & 1/2 & 1/8 \\ 2 & 1 & 1/4 \\ 8 & 4 & 1 \end{pmatrix}$$

Then, we estimate their priority vectors

$$\mathbf{w}^{(c)} = \begin{pmatrix} 4/9 \\ 4/9 \\ 1/9 \end{pmatrix} \quad \mathbf{w}^{(s)} = \begin{pmatrix} 6/10 \\ 3/10 \\ 1/10 \end{pmatrix} \quad \mathbf{w}^{(e)} = \begin{pmatrix} 1/11 \\ 2/11 \\ 8/11 \end{pmatrix}$$

Now we have three vectors instead of one. The interpretation of these vectors is at least twofold: (i) as they are m vectors of dimension n, then one can imagine them as m points in the n-dimensional space; (ii) vectors are ratings and they are often contradictory: climate-wise Barcelona is preferred over Reykjavik, but, on the other hand, the opposite is true if the criterion is the environment.

Thus, the solution should be a compromise between ratings of different criteria. The simple arithmetic mean is not the best way because criteria have different weights as they have different degrees of importance. For instance, an old and rich man would not care much about the cost as he may only demand a quiet and peaceful place. We need another type of averaging operator and the compromise that we are looking for is the result of a weighted arithmetic mean operation [3]. The question is how to find the

[3] Mathematically, we speak of a convex (linear) combination of vectors

weights to associate to different vectors. The only thing we know is that the weight associated to a vector should be coherent with the importance of the criteria associated with that rating. The proposed solution is using the same technique used so far. First, we build a matrix which compares the importance of criteria respect to the achievement of the goal, $\hat{\mathbf{A}} = (\hat{a}_{ij})_{n \times n}$. In the example, the matrix could be

$$\hat{\mathbf{A}} = \begin{pmatrix} 1 & 1/2 & 1/4 \\ 2 & 1 & 1/2 \\ 4 & 2 & 1 \end{pmatrix}.$$

Then, we derive a vector $\hat{\mathbf{w}} = (\hat{w}_1, \ldots, \hat{w}_n)$

$$\hat{\mathbf{w}} = \begin{pmatrix} 1/7 \\ 2/7 \\ 4/7 \end{pmatrix}$$

whose components are the weights of criteria. According to this vector the expert is really interested in the third criterion, i.e. the environment. We go further with the linear combination of $\mathbf{w}^{(c)}$, $\mathbf{w}^{(s)}$ and $\mathbf{w}^{(e)}$.

$$\begin{aligned} \mathbf{w}^* &= \hat{w}_1 \mathbf{w}^{(c)} + \hat{w}_2 \mathbf{w}^{(s)} + \hat{w}_1 \mathbf{w}^{(e)} \\ &= \frac{1}{7} \begin{pmatrix} 4/9 \\ 4/9 \\ 1/9 \end{pmatrix} + \frac{2}{7} \begin{pmatrix} 6/10 \\ 3/10 \\ 1/10 \end{pmatrix} + \frac{4}{7} \begin{pmatrix} 1/11 \\ 2/11 \\ 8/11 \end{pmatrix} \approx \begin{pmatrix} 0.287 \\ 0.253 \\ 0.460 \end{pmatrix} \end{aligned}$$

The result of the AHP is $\mathbf{w}^* \approx (0.287, 0.253, 0.46)$. We have a final ranking and we can choose the best alternative, which is the one associated with the 'heaviest' weight, then x_3 which, in our example, is Reykjavik. Formally, the best alternative is any element of the set $\{x_i | w_i^* \geq w_j^*, \ \forall i, j\}$.

Turku Centre for Computer Science
TUCS Dissertations

104. **Denis Shestakov**, Search Interfaces on the Web: Querying and Characterizing
105. **Sampo Pyysalo**, A Dependency Parsing Approach to Biomedical Text Mining
106. **Anna Sell**, Mobile Digital Calendars in Knowledge Work
107. **Dorina Marghescu**, Evaluating Multidimensional Visualization Techniques in Data Mining Tasks
108. **Tero Säntt**i, A Co-Processor Approach for Efficient Java Execution in Embedded Systems
109. **Kari Salonen**, Setup Optimization in High-Mix Surface Mount PCB Assembly
110. **Pontus Boström**, Formal Design and Verification of Systems Using Domain-Specific Languages
111. **Camilla J. Hollanti**, Order-Theoretic Methods for Space-Time Coding: Symmetric and Asymmetric Designs
112. **Heidi Himmanen**, On Transmission System Design for Wireless Broadcasting
113. **Sébastien Lafond**, Simulation of Embedded Systems for Energy Consumption Estimation
114. **Evgeni Tsivtsivadze**, Learning Preferences with Kernel-Based Methods
115. **Petri Salmela**, On Communication and Conjugacy of Rational Languages and the Fixed Point Method
116. **Siamak Taati**, Conservation Laws in Cellular Automata
117. **Vladimir Rogojin**, Gene Assembly in Stichotrichous Ciliates: Elementary Operations, Parallelism and Computation
118. **Alexey Dudkov**, Chip and Signature Interleaving in DS CDMA Systems
119. **Janne Savela**, Role of Selected Spectral Attributes in the Perception of Synthetic Vowels
120. **Kristian Nybom**, Low-Density Parity-Check Codes for Wireless Datacast Networks
121. **Johanna Tuominen**, Formal Power Analysis of Systems-on-Chip
122. **Teijo Lehtonen**, On Fault Tolerance Methods for Networks-on-Chip
123. **Eeva Suvitie**, On Inner Products Involving Holomorphic Cusp Forms and Maass Forms
124. **Linda Mannila**, Teaching Mathematics and Programming – New Approaches with Empirical Evaluation
125. **Hanna Suominen**, Machine Learning and Clinical Text: Supporting Health Information Flow
126. **Tuomo Saarni**, Segmental Durations of Speech
127. **Johannes Eriksson**, Tool-Supported Invariant-Based Programming
128. **Tero Jokela**, Design and Analysis of Forward Error Control Coding and Signaling for Guaranteeing QoS in Wireless Broadcast Systems
129. **Ville Lukkarila**, On Undecidable Dynamical Properties of Reversible One-Dimensional Cellular Automata
130. **Qaisar Ahmad Malik**, Combining Model-Based Testing and Stepwise Formal Development
131. **Mikko-Jussi Laakso**, Promoting Programming Learning: Engagement, Automatic Assessment with Immediate Feedback in Visualizations
132. **Riikka Vuokko**, A Practice Perspective on Organizational Implementation of Information Technology
133. **Jeanette Heidenberg**, Towards Increased Productivity and Quality in Software Development Using Agile, Lean and Collaborative Approaches
134. **Yong Liu**, Solving the Puzzle of Mobile Learning Adoption
135. **Stina Ojala**, Towards an Integrative Information Society: Studies on Individuality in Speech and Sign
136. **Matteo Brunelli**, Some Advances in Mathematical Models for Preference Relations

Turku Centre *for* Computer Science

Joukahaisenkatu 3-5 B, 20520 Turku, Finland | www. tucs.fi

University of Turku
Faculty of Mathematics and Natural Sciences
- Department of Information Technology
- Department of Mathematics

Turku School of Economics
- Institute of Information Systems Science

Åbo Akademi University
Division for Natural Sciences and Technology
- Department of Information Technologies

ISBN 978-952-12-2596-3
ISSN 1239-1883

Matteo Brunelli

Some Advances in Mathematical Models for Preference Relations