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Abstract

A subset A of $\mathbb{N}$ is called an IP-set if A contains all finite sums of distinct terms of some infinite sequence $(x_n)_{n\in\mathbb{N}}$ of natural numbers. Central sets, first introduced by Furstenberg using notions from topological dynamics, constitute a special class of IP-sets possessing additional nice combinatorial properties: Each central set contains arbitrarily long arithmetic progressions, and solutions to all partition regular systems of homogeneous linear equations. In this paper we show how certain families of aperiodic words of low factor complexity may be used to generate a wide assortment of central sets having additional nice properties inherited from the rich combinatorial structure of the underlying word. We consider Sturmian words and their extensions to higher alphabets (so-called Arnoux-Rauzy words), as well as words generated by substitution rules including the famous Thue-Morse word. We also describe a connection between central sets and the strong coincidence condition for fixed points of primitive substitutions which represents a new approach to the strong coincidence conjecture for irreducible Pisot substitutions. Our methods simultaneously exploit the general theory of combinatorics on words, the arithmetic properties of abstract numeration systems defined by substitution rules, notions from topological dynamics including proximality and equicontinuity, the spectral theory of symbolic dynamical systems, and the beautiful and elegant theory, developed by N. Hindman, D. Strauss and others, linking IP-sets to the algebraic/topological properties of the Stone-Čech compactification of $\mathbb{N}$. Using the key notion of p-$\lim_n$, regarded as a mapping from words to words, we apply ideas from combinatorics on words in the framework of ultrafilters.

Keywords: Sturmian words, numeration systems, IP-sets, central sets and the Stone-Čech compactification.

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Contents

1 Introduction

Let $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ denote the set of natural numbers, and $\mathrm{Fin}(\mathbb{N})$ the set of all non-empty finite subsets of $\mathbb{N}$.

Definition 1.1. *A subset A of $\mathbb{N}$ is called an IP-set if A contains $\{\sum_{n\in F} x_n \mid F \in$ Fin$(\mathbb{N})\}$ for some infinite sequence of natural numbers $x_0 < x_1 < x_2 \cdots$. A subset $A \subseteq \mathbb{N}$ is called an IP*-set if $A \cap B \neq \emptyset$ for every IP-set $B \subseteq \mathbb{N}$.*

By a celebrated result of N. Hindman [23], given any finite partition of $\mathbb{N}$, at least one element of the partition is an IP-set. It follows from Hindman's theorem that every IP*-set is an IP-set, but the converse is in general not true. In fact, more generally Hindman shows that given any finite partition of an IP-set, at least one element of the partition is again an IP-set. In other words the property of being an IP-set is *partition regular*, i.e., cannot be destroyed via a finite partitioning. Other examples of partition regularity are given by the pigeonhole principle, sets having positive upper density, and sets having arbitrarily long arithmetic progressions (Van der Waerden's theorem). In [22], Furstenberg introduced a special class of IP-sets, called central sets, having a substantial combinatorial structure. The property of being central is also partition regular. Central sets were originally defined in terms of topological dynamics:

Definition 1.2. *A subset $A \subset \mathbb{N}$ is called central if there exists a compact metric space (X, d) and a continuous map $T : X \to X$, points $x, y \in X$ and a neighborhood U of y such that*

- *y is a uniformly recurrent point in X,*
- *x and y are proximal,*
- *$A = \{n \in \mathbb{N} \,|\, T^n(x) \in U\}$.*

We say $A \subset \mathbb{N}$ is central if $A \cap B \neq \emptyset$ for every central set $B \subseteq \mathbb{N}$.*

Recall that x is said to be *uniformly recurrent* if each factor of x occurs in x with bounded gap. Two points $x, y \in X$ are said to be *proximal* if for every $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $d(T^n(x), T^n(y)) < \epsilon$. We remark that from the above definition, it is not at all evident that central sets are IP-sets. We later give an alternative definition (see Definition 3.5) which makes this point clear. The equivalence between the two definitions is due to Bergelson and Hindman [7].

The question of determining whether a given subset $A \subseteq \mathbb{N}$ is an IP-set or a central set is typically quite difficult, even if for every A, either A or its complement is an IP-set (resp. central set). It turns out that in each case this question may be reformulated in terms of whether or not the set A belongs to a certain class of ultrafilters on $\mathbb{N}$ (see Theorem 5.12 in [26] in the case of IP-sets and [7] in the case of central sets). But the question of belonging or not to a given (non-principal) ultrafilter is generally equally mysterious. An equivalent word combinatorial reformulation of this question is as follows: Given a binary word $\omega = \omega_0\omega_1\omega_2 \ldots \in \{0, 1\}^\infty$, put $\omega\big|_0 = \{n \in \mathbb{N} \,|\, \omega_n = 0\}$ and $\omega\big|_1 = \{n \in \mathbb{N} \,|\, \omega_n = 1\}$. The question is then to determine whether the set $\omega\big|_0$ or $\omega\big|_1$ is an IP-set or central set. Of course in general, this reformulation is as difficult as the original question. However, should the word ω be characterized by some rich combinatorial properties, or be generated by some "simple" combinatorial or geometric algorithm (such as a substitution rule, a finite state automaton, a Toeplitz rule...) or arise as a natural coding of a reasonably simple symbolic dynamical system, then the underlying rigid combinatorial structure of the word may provide insight to our previous question. Furthermore, such families of words may be used to obtain simple constructions of central sets having additional nice properties inherited from the rich underlying combinatorial structure. One of our objectives here is to illustrate this latter point.

Let $\mathcal{A}$ denote a finite non-empty set (called the alphabet) and $\omega = \omega_0\omega_1\omega_2\ldots \in \mathcal{A}^{\mathbb{N}}$. For each finite word u on the alphabet $\mathcal{A}$ we set

$$\omega\big|_u = \{n \in \mathbb{N} \,|\, \omega_n\omega_{n+1}\ldots\omega_{n+|u|-1} = u\}.$$

In other words, $\omega\big|_u$ denotes the set of all occurrences of u in ω.
In this paper we investigate partitions of $\mathbb{N}$ by sets of the form $\omega\big|_u$ defined by words ω of low factor complexity. Our goal is to study these partitions in the framework of IP-sets and central sets. All infinite words $\omega \in \mathcal{A}^{\mathbb{N}}$ considered in this paper are uniformly recurrent. As we shall see, in our framework IP-sets and central sets are one and the same:

Theorem 1. *Let $\omega \in \mathcal{A}^{\mathbb{N}}$ be uniformly recurrent. Then the set $\omega\big|_u$ is an IP-set if and only if it is a central set.*

The above theorem allows us to simultaneously state our results in terms of IP-sets and central sets.

We begin by considering the simplest aperiodic infinite words, namely Sturmian words. Sturmian words are infinite words over a binary alphabet having exactly $n+1$ factors of length n for each $n \geq 0$. Their origin can be traced back to the astronomer J. Bernoulli III in 1772. A fundamental result due to Morse and Hedlund [31] states that each aperiodic (meaning non-ultimately periodic) infinite word must contain at least $n+1$ factors of each length $n \geq 0$. Thus Sturmian words are those aperiodic words of lowest factor complexity. They arise naturally in many different areas of mathematics including combinatorics, algebra, number theory, ergodic theory, dynamical systems and differential equations. Sturmian words are also of great importance in theoretical physics and in theoretical computer science and are used in computer graphics as digital approximation of straight lines.

Let $\omega \in \{0,1\}^{\mathbb{N}}$ be a Sturmian word, and let Ω denote the shift orbit closure of ω. Then Ω contains a unique word $\tilde{\omega}$ (called the *characteristic word*) having the property that both $0\tilde{\omega}, 1\tilde{\omega} \in \Omega$. In order to state our results, we must distinguish between two cases:

Definition 1.3. *A Sturmian word ω is called nonsingular if it does not contain the characteristic word $\tilde{\omega}$ as a proper tail. Otherwise it is said to be singular.*

Theorem 2. *Let $\omega \in \Omega$ be a nonsingular Sturmian word, and u a factor of ω. Then $\omega\big|_u$ is an IP-set (resp. central set) if and only if u is a prefix of ω. In other words, for every prefix u of ω, the set $\omega\big|_u$ is an IP*- set (resp. central*-set).*

As a corollary we deduce that

Corollary 1. *Let $\omega \in \Omega$ be a nonsingular Sturmian word. For every factor v of ω and $n \in \omega\big|_v$ the set $\omega\big|_v - n$ is a central* set.*

We note that in general the property of being an IP*-set is not translation invariant. See also Theorem 1.1 in [8]. As an immediate consequence to the previous corollary, we have

Corollary 2. *For each $r \geq 1$ there exists a partition of $\mathbb{N}$ into sets $A_0, A_1, \ldots, A_r$ such that for each $0 \leq i \leq r$ and $n \in \mathbb{N}$, exactly one of the sets $\{A_0 - n, A_1 - n, \ldots, A_r - n\}$ is an IP*-set (resp. central* set).*

In fact, given $r \geq 1$, let ω be any nonsingular Sturmian word (for instance the Fibonacci word) and let $\mathcal{F}_\omega(r)$ denote the set of all factors of ω of length r. Then the $r+1$ sets $\omega\big|_u$ with $u \in \mathcal{F}_\omega(r)$ define a partition of $\mathbb{N}$ with the required property.

For singular Sturmian words ω we have

Theorem 3. *Let $\omega \in \Omega$ be a Sturmian word such that $T^{n_0}(\omega) = \tilde{\omega}$ with $n_0 \geq 1$. Then $\omega\big|_u$ is an IP-set (resp. central set) if and only if either u is a prefix of ω or a prefix of ω' where ω' is the unique other element of Ω with $T^{n_0}(\omega') = \tilde{\omega}$.*

Some (but not all) of the results on Sturmian partitions extend to so-called Arnoux-Rauzy words, which may be regarded as natural combinatorial extensions of Sturmian words to larger alphabets [1].

We also consider partitions defined by words generated by substitution rules. For instance, by considering partitions of $\mathbb{N}$ defined by words generated by the generalized Thue-Morse substitution to an alphabet of size $r \geq 2$, we show that

Theorem 4. *For each pair of positive integers r and N there exists a partition of*

$$\mathbb{N} = A_1 \cup A_2 \cup \cdots \cup A_r$$

such that

- *$A_i - n$ is a central set for each $1 \leq i \leq r$ and $1 \leq n \leq N$.*
- *For each $n > N$, exactly one of the sets $\{A_1 - n, A_2 - n, \ldots, A_r - n\}$ is a central set.*

The second assertion of Theorem 4 relies on the fact that each fixed point of the generalized Thue-Morse substitution is distal. At least in the case of the Thue-Morse substitution itself this may already be known, but the authors have been unable to locate this result anywhere in the literature. Our proof of this fact uses a result of V. Baker, M. Barge and J. Kwapisz which states that for subshifts (X, T) generated by primitive substitutions of Pisot type, the maximal equicontinuous factor $\pi : X \to X_{\text{eq}}$ is finite to one [3].

By considering partitions defined by words generating minimal subshifts which are topologically weak mixing (for example the subshift generated by the substitution $0 \mapsto 001$ and $1 \mapsto 11001)$ we prove that

Theorem 5. *For each positive integer r there exists a partition of $\mathbb{N} = A_1 \cup A_2 \cup \cdots \cup A_r$ such that for each $1 \leq i \leq r$ and $n \geq 0$, the set $A_i - n$ is a central set.*

We also consider words on infinite alphabets. Via iterated palindromic closures (see Definition 7.1), we construct a uniformly recurrent infinite word ω on an infinite alphabet $\mathcal{A}$ which gives rise to an infinite partition of $\mathbb{N}$ into central sets:

Theorem 6. *Let Δ be a right infinite word on a finite or infinite alphabet $\mathcal{A}$ with the property that each letter $a \in \mathcal{A}$ occurs in Δ an infinite number of times. Let ψ denote the iterated palindromic operator and set $\omega = \psi(\Delta)$. Then*

1. *ω is uniformly recurrent and closed under reversal, i.e., if $v = v_1 v_2 \ldots v_k$ is a factor of ω, then so is its mirror image $v_k \ldots v_2 v_1$.*

2. *The set $\omega\big|_a + 1$ is a central set for each letter $a \in A$.*

In particular if we take the word Δ to be on an infinite alphabet, the sets $\{\omega\big|_a + 1\}_{a \in \mathcal{A}}$ form a countably infinite collection of pairwise disjoint central subsets of $\mathbb{N}$.[1]

An important open problem in the theory of substitutions is the so-called *strong coincidence conjecture* which states that each pair of fixed points x and y of an irreducible primitive substitution of Pisot type satisfy the following condition called the *strong coincidence condition*: There exist a letter a and a pair of Abelian equivalent words $s, t,$ such that sa is a prefix of x and ta is a prefix of $y.$ This combinatorial condition, originally due to P. Arnoux and S. Ito, is an extension of a similar condition considered by F.M. Dekking in [14] in the case of uniform substitutions. In this case Dekking proves that the condition is satisfied if and only if the associated substitutive subshift has *pure discrete spectrum*, i.e., is metrically isomorphic with translation on a compact Abelian group. The strong coincidence conjecture has been verified for irreducible primitive substitutions of Pisot type on a binary alphabet by M. Barge and B. Diamond [4]. The following establishes a link between the strong coincidence conjecture and central sets:

Theorem 7. *Let τ be a primitive substitution verifying the strong coincidence condition. Then for any pair of fixed points x and y, and any prefix u of y, we have that $x\big|_u$ is a central set.*

[1] This is a special case of a prior result of Hindman, Leader and Strauss [25] in which they show that every central set in $\mathbb{N}$ is a countable union of pairwise disjoint central sets.

Our proof of Theorem 7 makes use of the so-called Dumont-Thomas numeration systems defined by substitutions, and constitutes a new approach to the strong coincidence conjecture.

The main results in this paper rely on various interactions between different areas of mathematics, some of which had not previously been directly linked: They include the general theory of combinatorics on words, the arithmetic properties of abstract numeration systems defined by substitutions, topological dynamics, the spectral theory of symbolic dynamical systems, and the beautiful theory, developed by Hindman, Strauss and others, linking IP-sets and central sets to the algebraic/topological properties of the Stone-Čech compactification $\beta\mathbb{N}$. We regard $\beta\mathbb{N}$ as the collection of all ultrafilters on $\mathbb{N}$. An ultrafilter may be thought of as a $\{0,1\}$-valued finitely additive probability measure defined on all subsets of $\mathbb{N}$. This notion of measure induces a notion of convergence (p-$\lim_n$) for sequences indexed by $\mathbb{N}$, which we regard as a mapping from words to words. This key notion of convergence allows us to apply ideas from combinatorics on words in the framework of ultrafilters.

The paper is organized as follows: In §2 we present some of the basic ideas and tools from combinatorics on words which will be used throughout the paper. In §3 we outline the key features of the algebraic and topological properties of the Stone-Čech compactification $\beta\mathbb{N}$ in connection with IP-sets and central sets. Since the material in §2 may be unfamiliar to specialists in topological semigroups and vice-versa, we take some care to explain both topics in an attempt to make the paper more accessible. In §4 we analyze some concrete examples which illustrate some of the results mentioned above in Theorems 2 and 3. We use nothing more than the combinatorial properties of the words considered (all generated by substitutions) and the arithmetic properties of the underlying Dumont-Thomas numeration system. In §5 we extend the results in §4 to all Sturmian words, in particular those not generated by substitutions. Here we make use of the algebraic properties of the semigroup $\beta\mathbb{N}$. In §6 we consider partitions defined by the generalized Thue-Morse substitution and prove Theorem 4. Also in §6 we prove Theorem 5 by considering subshifts which are topologically weak mixing. In §7 we consider some infinite words on an infinite alphabet generated by iteration of the palindromic closure operator. Using these words we construct infinite partitions of $\mathbb{N}$ and prove Theorem 6. Finally in §8, after a brief review of the Dumont-Thomas numeration systems defined by substitutions, we discuss a connection between central sets and the strong coincidence condition for substitutions.

2 Words and substitutions

In this section we give a brief summary of some of the basic background in combinatorics on words.

2.1 Words & subshifts

Given a finite non-empty set $\mathcal{A}$ (called the *alphabet*), we denote by $\mathcal{A}^*$, $\mathcal{A}^{\mathbb{N}}$ and $\mathcal{A}^{\mathbb{Z}}$ respectively the set of finite words, the set of (right) infinite words, and the set of bi-infinite words over the alphabet $\mathcal{A}$. Given a finite word $u = a_1a_2\ldots a_n$ with $n \geq 1$ and $a_i \in \mathcal{A}$, we denote the length n of u by $|u|$. The *empty word* will be denoted by ε and we set $|\varepsilon| = 0$. We put $\mathcal{A}^+ = \mathcal{A}^* - \{\varepsilon\}$. For each $a \in \mathcal{A}$, we let $|u|_a$ denote the number of occurrences of the letter a in u. Two words u and v in A^* are said to be *Abelian equivalent*, denoted $u \sim_{\text{ab}} v$, if and only if $|u|_a = |v|_a$ for all $a \in \mathcal{A}$. It is readily verified that $\sim_{\text{ab}}$ defines an equivalence relation on $\mathcal{A}^*$.

Given an infinite word $\omega \in \mathcal{A}^{\mathbb{N}}$, a word $u \in \mathcal{A}^+$ is called a *factor* of ω if $u = \omega_i\omega_{i+1}\cdots\omega_{i+n}$ for some natural numbers i and n. We denote by $\mathcal{F}_\omega(n)$ the set of all factors of ω of length n, and set

$$\mathcal{F}_\omega = \bigcup_{n\in\mathbb{N}} \mathcal{F}_\omega(n).$$

A factor u of ω is called *right special* if both ua and ub are factors of ω for some pair of distinct letters $a, b \in \mathcal{A}$. Similarly u is called *left special* if both au and bu are factors of ω for some pair of distinct letters $a, b \in \mathcal{A}$. The factor u is called $bispecial$ if it is both right special and left special. For each factor $u \in \mathcal{F}_\omega$ set

$$\omega\big|_u = \{n \in \mathbb{N} \,|\, \omega_n\omega_{n+1}\ldots\omega_{n+|u|-1} = u\}.$$

We say ω is *recurrent* if for every $u \in \mathcal{F}_\omega$ the set $\omega\big|_u$ is infinite. We say ω is *uniformly recurrent* if for every $u \in \mathcal{F}_\omega$ the set $\omega\big|_u$ is syndedic, i.e., of bounded gap.

We endow $\mathcal{A}^{\mathbb{N}}$ with the topology generated by the metric

$$d(x, y) = \frac{1}{2^n} \text{ where } n = \inf\{k : x_k \neq y_k\}$$

whenever $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}}$ are two elements of $\mathcal{A}^{\mathbb{N}}$. Let $T : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ denote the *shift* transformation defined by $T : (x_n)_{n\in\mathbb{N}} \mapsto (x_{n+1})_{n\in\mathbb{N}}$. By a *subshift* on $\mathcal{A}$ we mean a pair (X, T) where X is a closed and T-invariant subset of $\mathcal{A}^{\mathbb{N}}$. A subshift (X, T) is said to be *minimal* whenever X and the empty set are the only T-invariant closed subsets of X. To each $\omega \in \mathcal{A}^{\mathbb{N}}$ is associated the subshift (X, T) where X is the shift orbit closure of ω. If ω is uniformly recurrent, then the associated subshift (X, T) is minimal. Thus any two words x and y in X have exactly the same set of factors, i.e., $\mathcal{F}_x = \mathcal{F}_y$. In this case we denote by $\mathcal{F}_X$ the set of factors of any word $x \in X$.

Two points x, y in X are said to be *proximal* if and only if for each $N > 0$ there exists $n \in \mathbb{N}$ such that

$$x_n x_{n+1} \dots x_{n+N} = y_n y_{n+1} \dots y_{n+N}.$$

Two points $x, y \in X$ are said to be *regionally proximal* if for every prefix u of x and v of y, there exist points $x', y' \in X$ with x' beginning in u and y' beginning in v and with x' proximal to y'. Clearly if two points in X are proximal, then they are regionally proximal. A point $x \in X$ is called *distal* if the only point in X proximal to x is x itself. A minimal subshift (X, T) is said to be *topologically mixing* if for every any pair of factors $u, v \in \mathcal{F}_X$ there exists a positive integer N such that for each $n \geq N$, there exists a block of the form $uWv \in \mathcal{F}_X$ with $|W| = n$. A minimal subshift (X, T) is said to be *topologically weak mixing* if for every any pair of factors $u, v \in \mathcal{F}_X$ the set

$$\{n \in \mathbb{N} \,|\, u\mathcal{A}^n v \cap \mathcal{F}_X \neq \emptyset\}$$

is thick, i.e., for every positive integer N, the set contains N consecutive positive integers.

2.2 Substitutions

Many of the words and subshifts considered in this paper are generated by substitutions. A *substitution* τ on an alphabet $\mathcal{A}$ is a mapping $\tau : \mathcal{A} \to \mathcal{A}^+$. The mapping τ extends by concatenation to maps (also denoted τ) $\mathcal{A}^* \to \mathcal{A}^*$ and $\mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$. The *Abelianization* of τ is the square matrix M_τ whose ij-th entry is equal to $|\tau(j)|_i$, i.e., the number of occurrences of i in $\tau(j)$. A substitution τ is said to be *primitive* if there is a positive integer n such that for each pair $(i, j) \in \mathcal{A} \times \mathcal{A}$, the letter i occurs in $\tau^n(j)$. Equivalently if all the entries of M_τ^n are strictly positive. In this case it is well known that the matrix M_τ has a simple positive Perron-Frobenius eigenvalue called the *dilation* of τ. A substitution τ is said to be *irreducible* if the minimal polynomial of its dilation is equal to the characteristic polynomial of its Abelianization M_τ. A substitution τ is said to be of *Pisot type* if its dilation is a Pisot number. Recall that a Pisot number is an algebraic integer greater than 1 all of whose algebraic conjugates lie strictly inside the unit circle.

Let τ be a primitive substitution on $\mathcal{A}$. A word $\omega \in \mathcal{A}^{\mathbb{N}}$ is called a *fixed point* of τ if $\tau(\omega) = \omega$, and is called a *periodic point* if $\tau^m(\omega) = \omega$ for some $m > 0$. Although τ may fail to have a fixed point, it has at least one periodic point. Associated to τ is the topological dynamical system (X, T), where X is the shift orbit closure of a periodic point ω of τ. The primitivity of τ implies that (X, T) is independent of the choice of periodic point and is minimal.

An important example of a primitive substitution is the *Thue-Morse substitution* defined by the morphism $0 \mapsto 01$ and $1 \mapsto 10$. It has two fixed points

$$\mathbf{u} = 0110100110010110100101100111010\dots$$

and

$$\mathbf{v} = 1001011001101001011010011001 01\ldots$$

where $u_n = 1 - v_n$ for every $n \geq 0$. Alternatively, it can be shown that u_n is equal to 0 if and only if the binary expansion of n contains an even number of 1s. For example, $u_5 = u_6 = 0$, and in fact $5 = 101$ and $6 = 110$ expressed in base 2. Two other primitive substitutions we will make reference to, first introduced some thirty years ago by F.M. Dekking and M. Keane, are the substitutions $0 \mapsto 001$, $1 \mapsto 11100$ and $0 \mapsto 001$, $1 \mapsto 11001$. Both have two fixed points, and have the same Abelianization. It is shown in [15] that the subshift generated by the first substitution is topologically mixing, but not the second. But both are topologically weak mixing.

2.3 Sturmian words & generalizations

Let $\omega \in \mathcal{A}^{\mathbb{N}}$ and set

$$\rho_\omega(n) = \text{Card}(\mathcal{F}_\omega(n)).$$

The function $\rho_\omega : \mathbb{N} \to \mathbb{N}$ is called the *factor complexity function* of ω. Given a minimal subshift (X,T) on A, we have $\mathcal{F}_\omega(n) = \mathcal{F}_{\omega'}(n)$ for all $\omega, \omega' \in X$ and $n \in \mathbb{N}$. Thus we can define the factor complexity $\rho_{(X,T)}(n)$ of a minimal subshift (X,T) by

$$\rho_{(X,T)}(n) = \rho_\omega(n)$$

for any $\omega \in X$.

A word $\omega \in \mathcal{A}^{\mathbb{N}}$ is *periodic* if there exists a positive integer p such that $\omega_{i+p} = \omega_i$ for all indices i, and it is *ultimately periodic* if $\omega_{i+p} = \omega_i$ for all sufficiently large i. An infinite word is *aperiodic* if it is not ultimately periodic. By a celebrated result due to Hedlund and Morse [31], a word is ultimately periodic if and only if its factor complexity is uniformly bounded. In particular, $p_\omega(n) < n$ for all n sufficiently large. Words whose factor complexity $\rho_\omega(n) = n+1$ for all $n \geq 0$ are called *Sturmian words*. Thus, Sturmian words are those aperiodic words having the lowest complexity. Since $\rho_\omega(1) = 2$, it follows that Sturmian words are binary words. The most extensively studied Sturmian word is the so-called Fibonacci word

$$\mathbf{f} = 0100101001001010010100100101001001010010100100101001 0\cdots$$

fixed by the morphism $0 \mapsto 01$ and $1 \mapsto 0$. Let $\omega \in \{0,1\}^{\mathbb{N}}$ be a Sturmian word, and let Ω denote the shift orbit closure of ω. The condition $\rho_\omega(n) = n+1$ implies the existence of exactly one right special and one left special factor of each length. Clearly, given any two left special factors, one is necessarily a prefix of the other. It follows that Ω contains a unique word all of whose prefixes are left special factors of ω. Such a word is called the *characteristic word* and denoted $\tilde{\omega}$. It follows that both $0\tilde{\omega}, 1\tilde{\omega} \in \Omega$. It is readily verified that the Fibonacci word above is a characteristic Sturmian word. A Sturmian word ω is called *singular* if $T^n(\omega) = \tilde{\omega}$ for some $n \geq 1$. Otherwise it is said to be *nonsingular.*

Sturmian words admit various types of characterizations of geometric and combinatorial nature. We give two such characterizations which will be used in the paper: as irrational rotations on the unit circle and as mechanical words. In [31] Hedlund and Morse showed that each Sturmian word may be realized measure-theoretically by an irrational rotation on the circle. That is, every Sturmian word is obtained by coding the symbolic orbit of a point x on the circle (of circumference one) under a rotation R_α by an irrational angle α, $0 < \alpha < 1$, where the circle is partitioned into two complementary intervals, one of length α and the other of length $1-\alpha$. And conversely each such coding gives rise to a Sturmian word. The quantity α is called the *slope*. Namely, the *rotation* by angle α is the mapping R_α from $[0, 1)$ (identified with the unit circle) to itself defined by $R_\alpha(x) = \{x+\alpha\}$, where $\{x\} = x-[x]$ is the fractional part of x. Considering a partition of $[0, 1)$ into $I_0 = [0, 1-\alpha)$, $I_1 = [1-\alpha, 1)$, define a word

$$s_{\alpha,\rho}(n) = \begin{cases} 0, & \text{if } R^n_\alpha(\rho) = \{\rho + n\alpha\} \in I_0, \\ 1, & \text{if } R^n_\alpha(\rho) = \{\rho + n\alpha\} \in I_1 \end{cases}$$

One can also define $I'_0 = (0, 1-\alpha]$, $I'_1 = (1-\alpha, 1]$, the corresponding word is denoted by $s'_{\alpha,\rho}$. For a Sturmian word w of slope α its subshift Ω is given by $\Omega = \{s_{\alpha,\rho}, s'_{\alpha,\rho} | \rho \in [0, 1)\}$.

A straightforward computation shows that

$$s_{\alpha,\rho}(n) = \lfloor \alpha(n+1) + \rho \rfloor - \lfloor \alpha n + \rho \rfloor,$$

$$s'_{\alpha,\rho}(n) = \lceil \alpha(n+1) + \rho \rceil - \lceil \alpha n + \rho \rceil;$$

$s_{\alpha,\rho}$ and $s'_{\alpha,\rho}$ are called the *upper* and *lower mechanical words* (of slope α) based at ρ.

In [1] Arnoux and Rauzy introduced a class of uniformly recurrent (minimal) sequences ω on a m-letter alphabet of complexity $\rho_\omega(n) = (m-1)n + 1$ characterized by the following combinatorial criterion known as the $\star$ condition: ω admits exactly one right special and one left special factor of each length. We call them *Arnoux-Rauzy sequences*. This condition distinguishes them from other sequences of complexity $(m-1)n+1$ such as those obtained by coding trajectories of m-interval exchange transformations. These words are generally regarded as natural combinatorial generalizations of Sturmian words to higher alphabets. In particular, the Fibonacci word generalizes to the m-bonacci word fixed by the substitution

$$\sigma_m : \{0, 1, \ldots, m-1\} \to \{0, 1, \ldots, m-1\}^*$$

given by

$$\sigma_m(i) = \begin{cases} 0(i+1) & \text{for } 0 \leq i < m-1 \\ 0 & \text{for } i = m-1 \end{cases}$$

However, many of the dynamical and geometrical interpretations of Sturmian words do not extend to this new class of words (see [12] for example).

In the subsequent sections we will consider partitions of $\mathbb{N}$ defined by words. Let $\omega \in \mathcal{A}^{\mathbb{N}}$, and let $\mathcal{F}$ denote the set of factors of ω. A finite subset X is called a *$\mathcal{F}$-prefix code* if $X \subset \mathcal{F}$ and given any two distinct elements of X, neither one is a prefix of the other. A $\mathcal{F}$-prefix code is *$\mathcal{F}$-maximal* if it is not properly contained in any other $\mathcal{F}$-prefix code. The simplest example of a $\mathcal{F}$-maximal prefix code is the set of all elements of $\mathcal{F}$ of some fixed length d. Each $\mathcal{F}$-maximal prefix code X defines a partition

$$\mathbb{N} = \bigcup_{u \in X} \omega\big|_u$$

If ω is a Sturmian word, then the corresponding partition is called a *Sturmian partition*.

3 Ultrafilters, IP-sets and central sets

3.1 Stone-Čech compactification

Many of our results rely on the algebraic/topological properties of the Stone-Čech compactification of $\mathbb{N}$. The Stone-Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$ is one of many compactifications of $\mathbb{N}$. It is in fact the largest compact Hausdorff space generated by $\mathbb{N}$. More precisely $\beta\mathbb{N}$ is a compact and Hausdorff space together with a continuous injection $i : \mathbb{N} \hookrightarrow \beta\mathbb{N}$ satisfying the following universal property: any continuous map $f : \mathbb{N} \to X$ into a compact Hausdorff space X lifts uniquely to a continuous map $\beta f : \beta\mathbb{N} \to X$, i.e., $f = \beta f \circ i$. This universal property characterizes $\beta\mathbb{N}$ uniquely up to homeomorphism. While there are different methods for constructing the Stone-Čech compactification of $\mathbb{N}$, we shall regard $\beta\mathbb{N}$ as the set of all ultrafilters on $\mathbb{N}$ with the *Stone topology.*

Recall that a set $\mathcal{U}$ of subsets of $\mathbb{N}$ is called an *ultrafilter* if the following conditions hold:

- $\emptyset \notin \mathcal{U}$.
- If $A \in \mathcal{U}$ and $A \subseteq B$, then $B \in \mathcal{U}$.
- $A \cap B \in \mathcal{U}$ whenever both A and B belong to $\mathcal{U}$.
- For every $A \subseteq \mathbb{N}$ either $A \in \mathcal{U}$ or $A^c \in \mathcal{U}$ where A^c denotes the complement of A.

For every natural number $n \in \mathbb{N}$, the set $\mathcal{U}_n = \{A \subseteq \mathbb{N} \,|\, n \in A\}$ is an example of an ultrafilter. This defines an injection $i : \mathbb{N} \hookrightarrow \beta\mathbb{N}$ by: $n \mapsto \mathcal{U}_n$. An ultrafilter of this form is said to be *principal*. By way of Zorn's lemma, one can show the existence of non-principal (or *free*) ultrafilters.

It is customary to denote elements of $\beta\mathbb{N}$ by letters $p, q, r \dots$. For each set $A \subseteq \mathbb{N}$, we set $A^\circ = \{p \in \beta\mathbb{N} | A \in p\}$. Then the set $\mathcal{B} = \{A^\circ | A \subseteq \mathbb{N}\}$ forms a basis for the open sets (as well as a basis for the closed sets) of $\beta\mathbb{N}$ and defines a topology on $\beta\mathbb{N}$ with respect to which $\beta\mathbb{N}$ is both compact and Hausdorff.[2] It is not difficult to see that the injection $i : \mathbb{N} \hookrightarrow \beta\mathbb{N}$ is continuous and satisfies the required universal property. In fact, given a continuous map $f : \mathbb{N} \to X$ with X compact Hausdorff, for each ultrafilter $p \in \beta\mathbb{N}$, the pushfoward $f(p) = \{f(n) \,|\, n \in p\}$ defines an ultrafilter on X having a unique limit point $\beta f(p)$.

There is a natural extension of the operation of addition $+$ on $\mathbb{N}$ to $\beta\mathbb{N}$ making $\beta\mathbb{N}$ a compact *left-topological semigroup*. More precisely we define addition of two ultrafilters p, q by the following rule:

$$p + q = \{A \subseteq \mathbb{N} \,|\, \{n \in \mathbb{N} | A - n \in p\} \in q\}.$$

It is readily verified that $p + q$ is once again an ultrafilter and that for each fixed $p \in \beta\mathbb{N}$, the mapping $q \mapsto p + q$ defines a continuous map from $\beta\mathbb{N}$ into itself.[3] The operation of addition in $\beta\mathbb{N}$ is associative and for principal ultrafilters we have $\mathcal{U}_m + \mathcal{U}_n = \mathcal{U}_{m+n}$. However in general addition of ultrafilters is highly non-commutative. In fact it can be shown that the center is precisely the set of all principal ultrafilters [26].

3.2 IP-sets and central sets

Let $(\mathcal{S}, +)$ be a semigroup. An element $p \in \mathcal{S}$ is called an *idempotent* if $p+p = p$. We recall the following result of Ellis [20]:

Theorem 3.1 (Ellis [20])**.** *Let $(\mathcal{S}, +)$ be a compact left-topological semigroup (i.e., $\forall x \in \mathcal{S}$ the mapping $y \mapsto x + y$ is continuous). Then $\mathcal{S}$ contains an idempotent.*

It follows that $\beta\mathbb{N}$ contains a non-principal ultrafilter p satisfying $p + p = p$. In fact, we could simply apply Ellis's result to the semigroup $\mathbb{N} - \{0\}$. This would then exclude the only principal idempotent ultrafilter, namely $\mathcal{U}_0$. From here on, by an idempotent ultrafilter in $\beta\mathbb{N}$ we mean a free idempotent ultrafilter.

We will make use of the following striking result due to Hindman linking IP-sets and idempotents in $\beta\mathbb{N}$:

[2] Although the existence of free ultrafilters requires Zorn's lemma, the cardinality of $\beta\mathbb{N}$ is $2^{2^{\mathbb{N}}}$ from which it follows that $\beta\mathbb{N}$ is not metrizable.

[3] Our definition of addition of ultrafilters is the same as that given in [6] but is the reverse of that given in [26] in which $A \in p + q$ if and only if $\{n \in \mathbb{N} | A - n \in q\} \in p\}$. In this case, $\beta\mathbb{N}$ becomes a compact right-topological semigroup.

Theorem 3.2 (Theorem 5.12 in [26]). *A subset $A \subseteq \mathbb{N}$ is an IP-set if and only if $A \in p$ for some idempotent $p \in \beta\mathbb{N}$.*

It follows immediately that A is an IP*-set if and only if $A \in p$ for every idempotent $p \in \beta\mathbb{N}$ (see Theorem 2.15 in [6]). We also note that the property of being an IP-set is partition regular.

To see the connection between idempotent ultrafilters and IP-sets, consider a set $A_0 \subseteq \mathbb{N}$ belonging to some idempotent $p \in \beta\mathbb{N}$. Then as $A_0 \in p + p$ it follows that there exist $x_0 \in A_0$ such that $A_0 \cap A_0 - x_0 \in p$. Set $A_1 = A_0 \cap A_0 - x_0$. Since $A_1 \in p+p$ we can choose $x_1 \in A_1$ $(x_1 \neq x_0)$ such that $A_1 \cap A_1 - x_1 \in p$. Note that thus far we have x_0, x_1 and $x_0 + x_1$ all belong to A_0. Set $A_2 = A_1 \cap A_1 - x_1$. Again since $A_2 \in p + p$ we can choose $x_2 \in A_2$ (distinct from both x_0, x_1) such that $A_2 \cap A_2 - x_2 \in p$. Since $x_2 \in A_2$, it follows that $x_2, x_2 + x_1 \in A_1 \subseteq A_0$. Since $x_2, x_2 + x_1 \in A_1$ it follows that $x_2 + x_0, x_2 + x_1 + x_0 \in A_0$. Thus $\{x_0, x_1, x_2, x_0 + x_1, x_0 + x_2, x_1 + x_2, x_0 + x_1 + x_2\} \subseteq A_0$. Iterating this process we obtain an infinite sequence of distinct points $(x_n)_{n\in\mathbb{N}}$ such that for any finite subset $F \subset \mathbb{N}$ the sum $\sum_{n\in F} x_n$ belongs to A_0. In other words, A_0 is an IP-set.

In [22], Furstenberg introduced a special class of IP-sets, called central sets, having additional rich combinatorial properties. They were originally defined in terms of topological dynamics (see Definition 1.2). As in the case of IP-sets, they may be alternatively defined in terms of belonging to a special class of free ultrafilters, called minimal idempotents[4]. To define a minimal idempotent we must first review some basic properties concerning ideals in $\beta\mathbb{N}$.

Let $(\mathcal{S}, +)$ be any semigroup. Recall that a subset $\mathcal{I} \subseteq \mathcal{S}$ is called a *right (resp. left) ideal* if $\mathcal{I} + \mathcal{S} \subseteq \mathcal{I}$ (resp. $\mathcal{S} + \mathcal{I} \subseteq \mathcal{I}$). It is called a *two sided ideal* if it is both a left and right ideal. A right (resp. left) ideal $\mathcal{I}$ is called *minimal* if every right (resp. left) ideal $\mathcal{J}$ included in $\mathcal{I}$ coincides with $\mathcal{I}$.

We recall some useful facts concerning minimal right ideals of a semigroup (similar considerations apply to minimal left ideals):

Facts:

1. Let $\mathcal{M}$ be a minimal right ideal of $\mathcal{S}$. Then every element x in $\mathcal{M}$ generates $\mathcal{M}$ in the sense that $\mathcal{M} = x + \mathcal{S} = x + \mathcal{M}$.

2. If $\mathcal{R}$ is a right ideal of $\mathcal{S}$ with the property that $\mathcal{R} = x + \mathcal{R}$ for every $x \in \mathcal{R}$, then $\mathcal{R}$ is a minimal right ideal.

3. Let $\mathcal{M}$ be a minimal right ideal of $\mathcal{S}$. Then $\mathcal{M} = x + \mathcal{M}$ for every $x \in \mathcal{S}$.

4. Every minimal right ideal $\mathcal{M}$ is contained in every two sided ideal $\mathcal{I}$.

[4]The equivalence between the two definitions is due to Bergelson and Hindman [7].

Minimal right/left ideals do not necessarily exist e.g. the commutative semigroup $(\mathbb{N}, +)$ has no minimal right/left ideals (the ideals in $\mathbb{N}$ are all of the form $\mathcal{I}_n = [n, +\infty) = \{m \in \mathbb{N} \,|\, m \geq n\}$.) However,

Proposition 3.3. *Every compact Hausdorff left-topological semigroup (e.g., $\beta\mathbb{N}$) admits a minimal right ideal and a minimal left ideal.*

Let $\mathcal{M}$ be a minimal right ideal of a left-topological semigroup. Since $\mathcal{M}$ is of the form $x + \mathcal{S}$ with $x \in \mathcal{M}$, it follows that $\mathcal{M}$ is closed. Thus $\mathcal{M}$ is a compact left-topological semigroup and hence by Ellis [20] contains an idempotent p. It is verified that $S+p$ is then a minimal left ideal, that $p \in S+p$ and that $p+\mathcal{S} \cap \mathcal{S}+p = p + \mathcal{S} + p$ is a group. More generally the intersection of any minimal right ideal with any minimal left ideal is a group and hence contains an idempotent.

Let $K(\mathcal{S})$ denote the union of all minimal right ideals of $\mathcal{S}$. Then $K(\mathcal{S})$ is a two sided ideal and is in fact the smallest such ideal. To see this we first note that $K(\mathcal{S})$ is a right ideal (being the union of right ideals). To see that $K(\mathcal{S})$ is also a left ideal, let $x \in K(\mathcal{S})$ and $y \in \mathcal{S}$. Then $x \in \mathcal{M}$ for some minimal right ideal $\mathcal{M}$. Thus $y + x \in y + \mathcal{M}$ which by Fact (3) is a minimal right ideal. Hence $y + x \in K(\mathcal{S})$. This shows that $K(\mathcal{S})$ is a two sided ideal of $\mathcal{S}$. By Fact (4) it follows that $K(\mathcal{S})$ is contained in every two sided ideal $\mathcal{I}$.
We could have defined $K(\mathcal{S})$ to be the union of all minimal left ideals of $\mathcal{S}$ and in an analogous way deduced that $K(\mathcal{S})$ is the smallest two sided ideal of $\mathcal{S}$. Thus

$$\begin{aligned} K(\mathcal{S}) &= \bigcup \{\mathcal{L} \,|\, \mathcal{L} \text{ is a minimal left ideal of } \mathcal{S}\} \\ &= \bigcup \{\mathcal{R} \,|\, \mathcal{R} \text{ is a minimal right ideal of } \mathcal{S}\}. \end{aligned}$$

Definition 3.4. *An idempotent p is called a minimal idempotent if it belongs to $K(\mathcal{S})$.*

Thus as every compact left-topological semigroup (e.g. $\beta\mathbb{N}$) contains a minimal right ideal, and by Ellis every minimal right ideal contains an idempotent, we deduce that every compact left-topological semigroup contains a minimal idempotent. Alternatively, given two idempotents $p, q \in \mathcal{S}$ we write $p \preceq q$ if

$$p + q = q + p = p.$$

It turns out that an idempotent p is minimal if and only if it is minimal with respect to the relation $\preceq$.

Definition 3.5. *A subset $A \subset \mathbb{N}$ is called central if it is a member of some minimal idempotent in $\beta\mathbb{N}$. It is called a central*-set if it belongs to every minimal idempotent in $\beta\mathbb{N}$.*

It follows from the above definition that every central set is an IP-set and that the property of being central is partition regular. Central sets are known to have substantial combinatorial structure. For example, any central set contains arbitrarily long arithmetic progressions, and solutions to all partition regular systems of homogeneous linear equations (see for example [8]). Many of the rich properties of central sets are a consequence of the so-called *Central Sets Theorem* first poved by Furstenberg in Proposition 8.21 in [22] (see also [13, 8, 27]). Furstenberg pointed out that as an immediate consequence of the Central Sets Theorem one has that whenever $\mathbb{N}$ is divided into finitely many classes, and a sequence $(x_n)_{n\in\mathbb{N}}$ is given, one of the classes must contain arbitrarily long arithmetic progressions whose increment belongs to $\{\sum_{n\in F} x_n | F \in \mathrm{Fin}(\mathbb{N})\}$.

3.3 Limits of ultrafilters

It is often convenient to think of an ultrafilter p as a $\{0,1\}$-valued, finitely additive probability measure on the power set of $\mathbb{N}$. More precisely, for any subset $A \subseteq \mathbb{N}$, we say A has p-measure 1, or is p-large if $A \in p$. This notion of measure gives rise to a notion of convergence of sequences indexed by $\mathbb{N}$ which is the key tool in allowing us to apply ideas from combinatorics on words to the framework of ultrafilters. However, from our point of view, it is more natural to define it alternatively as a mapping from words to words (see Remark 3.13). Let $\mathcal{A}$ denote a non-empty finite set. Then each ultrafilter $p \in \beta\mathbb{N}$ naturally defines a mapping

$$p^* : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$$

as follows:

Definition 3.6. *For each $p \in \beta\mathbb{N}$ and $\omega \in \mathcal{A}^{\mathbb{N}}$, we define $p^*(\omega) \in \mathcal{A}^{\mathbb{N}}$ by the condition: $u \in \mathcal{A}^*$ is a prefix of $p^*(\omega) \Longleftrightarrow \omega\big|_u \in p$.*

We note that if $u, v \in \mathcal{A}^*$, $\omega\big|_u, \omega\big|_v \in p$ and $|v| \geq |u|$, then u is a prefix of v. In fact, if v' denotes the prefix of v of length $|u|$ then as $\omega\big|_v \subseteq \omega\big|_{v'}$, it follows that $\omega\big|_{v'} \in p$ and hence $u = v'$. Thus $p^*(\omega)$ is well defined.

We note that if $\omega, \nu \in \mathcal{A}^{\mathbb{N}}$ and if each prefix u of ν is a factor of ω, then there exists an ultrafilter $p \in \beta\mathbb{N}$ such that $p^*(\omega) = \nu$. In fact, the set

$$\mathcal{C} = \{\omega\big|_u \mid u \text{ is a prefix of } \nu\}$$

satisfies the finite intersection property, and hence by a routine argument involving Zorn's lemma it follows that there exists a $p \in \beta\mathbb{N}$ with $\mathcal{C} \subseteq p$.

It follows immediately from the definition of p^*, Definition 3.5 and Theorem 3.2 that

Lemma 3.7. *The set $\omega\big|_u$ is an IP-set (resp. central set) if and only if u is a prefix of $p^*(\omega)$ for some idempotent (resp. minimal idempotent) $p \in \beta\mathbb{N}$.*

Lemma 3.8. *For each $p \in \beta\mathbb{N}$, $\omega \in \mathcal{A}^{\mathbb{N}}$ and $u \in \mathcal{A}^*$ we have*

$$p^*(\omega)\big|_u = \{m \in \mathbb{N} \mid \omega\big|_u - m \in p\}$$

where $\omega\big|_u - m$ is defined as the set of all $n \in \mathbb{N}$ such that $n + m \in \omega\big|_u$.

Proof. Suppose $m \in p^*(\omega)\big|_u$. Then by definition u occurs in position m in $p^*(\omega)$. Let v denote the prefix of $p^*(\omega)$ of length $|v| = m + |u|$. Then, as u is a suffix of v we have $\omega\big|_v + m \subseteq \omega\big|_u$ and hence $\omega\big|_v \subseteq \omega\big|_u - m$. But as v is a prefix of $p^*(\omega)$ we have $\omega\big|_v \in p$ and hence $\omega\big|_u - m \in p$ as required.

Conversely, fix $m \in \mathbb{N}$ such that $\omega\big|_u - m \in p$. Let Z be the set of all factors v of ω of length $|v| = m + |u|$ ending in u. Then

$$\omega\big|_u - m \subseteq \bigcup_{v \in Z} \omega\big|_v.$$

It follows that there exists $v \in Z$ such that $\omega\big|_v \in p$. In other words, there exists $v \in Z$ such that v is a prefix of $p^*(\omega)$. It follows that u occurs in position m in $p^*(\omega)$. $\square$

Lemma 3.9. *For $p, q \in \beta\mathbb{N}$ and $\omega \in \mathcal{A}^{\mathbb{N}}$, we have $(p + q)^*(\omega) = q^*(p^*(\omega))$. In particular, if p is an idempotent, then $p^*(p^*(\omega)) = p^*(\omega)$.*

Proof. For each word $u \in \mathcal{A}^*$ we have that u is a prefix of $(p+q)^*(\omega)$ if and only if

$$\omega\big|_u \in p + q \Longleftrightarrow \{m \in \mathbb{N} \mid \omega\big|_u - m \in p\} \in q.$$

On the other hand, u is a prefix of $q^*(p^*(\omega))$ if and only if $p^*(\omega)\big|_u \in q$. The result now follows immediately from the preceding lemma. $\square$

Lemma 3.10. *For each $p \in \beta\mathbb{N}$ and $\omega \in \mathcal{A}^{\mathbb{N}}$ we have $p^*(T(\omega)) = T(p^*(\omega))$ where $T : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ denotes the shift map.*

Proof. Assume $u \in \mathcal{A}^*$ is a prefix of $p^*(T(\omega))$. Then $T(\omega)\big|_u \in p$. But

$$T(\omega)\big|_u = \bigcup_{a \in \mathcal{A}} \omega\big|_{au}.$$

It follows that there exists $a \in \mathcal{A}$ such that $\omega\big|_{au} \in p$. Thus au is a prefix of $p^*(\omega)$ and hence u is a prefix of $T(p^*(\omega))$. $\square$

In what follows, we will make use of the following key result in [6] (see also Theorem 1 in [10]):

Theorem 3.11 (Theorem 3.4 in [6])**.** *Let (X, T) be a topological dynamical system. Then if two points $x, y \in X$ are proximal with y uniformly recurrent, then there exists a minimal idempotent $p \in \beta\mathbb{N}$ such that $p^*(x) = y$.*

As a consequence we have

Theorem 3.12. *Let $\omega \in \mathcal{A}^{\mathbb{N}}$ be a uniformly recurrent word, and let $u \in \mathcal{A}^{+}$. Then $\omega\big|_u$ is an IP-set if and only if $\omega\big|_u$ is a central set.*

Proof. For any $A \subset \mathbb{N}$ we have that if A is central then A belongs to some minimal idempotent $p \in \beta\mathbb{N}$ and hence in particular A belongs to an idempotent in $\beta\mathbb{N}$. Hence by Theorem 3.2 we have that A is an IP-set. Now suppose that $\omega\big|_u$ is an IP-set. Then $\omega\big|_u$ belongs to some idempotent $p \in \beta\mathbb{N}$. Set $\nu = p^*(\omega)$. Then u is a prefix of ν. Also, since p is idempotent we have $p^*(\nu) = p^*(p^*(\omega)) = p^*(\omega) = \nu$. Hence for every prefix v of ν we have that $\nu\big|_v \in p$ and $\omega\big|_v \in p$ and hence $\nu\big|_v \cap \omega\big|_v \in p$. In particular $\nu\big|_v \cap \omega\big|_v \neq \emptyset$. Hence ω and ν are proximal. Since ω is uniformly recurrent, it follows that ν is also uniformly recurrent. Hence by Theorem 3.11 there exists a minimal idempotent q with $q^*(\omega) = \nu$. Hence $\omega\big|_u \in q$, whence $\omega\big|_u$ is central. $\square$

Remark 3.13. It is readily verified that our definition of p^* coincides with that of $p\text{-}\lim_n$. More precisely, given a sequence $(x_n)_{n\in\mathbb{N}}$ in a topological space and an ultrafilter $p \in \beta\mathbb{N}$, we write $p\text{-}\lim_n x_n = y$ if for every neighborhood U_y of y one has $\{n \,|\, x_n \in U_y\} \in p$. In our case we have $p^*(\omega) = p\text{-}\lim_n(T^n(\omega))$ (see [24]). With this in mind, the preceding two lemmas are well known (see for instance [10, 24]). However, our defining condition of p^* in Definition 3.6 does not directly rely on the topology and so may be applied in other general settings. For instance, let $\Omega \subseteq \mathcal{A}^{\mathbb{N}}$ be a subshift, and $\mathcal{N} = \{n_0 < n_1 < n_2 < \cdots\}$ an infinite sequence of natural numbers. For each $\omega \in \Omega$ we put

$$X_k^{\mathcal{N}} = \{\omega_{n+n_0}\omega_{n+n_1}\ldots\omega_{n+n_{k-1}} \,|\, n \geq 0\} \subseteq \mathcal{A}^k.$$

For each $u \in X_k^{\mathcal{N}}$ we define the set

$$\omega^{\mathcal{N}}\big|_u = \{n \in \mathbb{N} \,|\, \omega_{n+n_0}\omega_{n+n_1}\ldots\omega_{n+n_{k-1}} = u\}.$$

Then the sets $\omega^{\mathcal{N}}\big|_u$ with $u \in X_k^{\mathcal{N}}$ partition $\mathbb{N}$. So, given $p \in \beta\mathbb{N}$, for each $k \geq 1$ there exists a unique $u \in X_k^{\mathcal{N}}$ with $\omega^{\mathcal{N}}\big|_u \in p$. Moreover if $v \in X_{k+1}^{\mathcal{N}}$ and $\omega^{\mathcal{N}}\big|_v \in p$, then u is a prefix of v. So using the condition in Definition 3.6, each infinite sequence $\mathcal{N}$ and ultrafilter $p \in \beta\mathbb{N}$ defines a mapping $\Omega \to \Omega$. Of particular interest is the case in which Ω is a uniform set in the sense of T. Kamae and $\mathcal{N}$ is chosen such that $\omega[\mathcal{N}]$ is a super-stationary set (see [28, 29]).

Another situation in which the defining condition of Definition 3.6 applies is in the context of infinite permutations [21]. By an infinite permutation π we mean a linear ordering on $\mathbb{N}$. Then for each finite permutation u of $\{1, 2, \ldots, n\}$ we say that u occurs in position m of π if the restriction of π to $\{m, m+1, \ldots, m+n-1\}$ is equal to u. Thus we may define the set $\pi\big|_u$ as the set of all $m \in \mathbb{N}$ such that u occurs in position m in π, and again the sets $\pi\big|_u$ (over all permutations u of $\{1, 2, \ldots, n\}$) determine a partition of $\mathbb{N}$. Hence each $p \in \beta\mathbb{N}$ defines a map from the set of all infinite permutations into itself.

4 A first analysis of some concrete examples

4.1 The Fibonacci word

While most of the proofs of the results announced in the Introduction rely on the algebraic and topological properties of ultrafilters on $\mathbb{N}$ and their links to IP-sets, we begin by analyzing concretely a few examples generated by simple substitution rules. To establish that certain subsets of $\mathbb{N}$ are IP-sets, we will use nothing more than the definition of IP-sets and the abstract numeration systems defined by substitutions first introduced by J.-M. Dumont and A. Thomas [17, 18].

Let us begin with the *Fibonacci* infinite word $\mathbf{f} = f_0 f_1 f_2 \ldots \in \{0,1\}^{\mathbb{N}}$.

We set

$$\mathbf{f}\big|_0 = \{n \in \mathbb{N} | f_n = 0\}$$

and

$$\mathbf{f}\big|_1 = \{n \in \mathbb{N} | f_n = 1\}.$$

So $\mathbf{f}\big|_0 = \{0, 2, 3, 5, 7, 8, 10, 11, 13, 15, 16, \ldots\}$ and $\mathbf{f}\big|_1 = \{1, 4, 6, 9, 12, 14, 17, \ldots\}$. This defines the Sturmian partition $\mathbb{N} = \mathbf{f}\big|_0 \cup \mathbf{f}\big|_1$. Let us denote by F_n the nth Fibonacci number so that $F_0 = 1, F_1 = 2, F_2 = 3, \ldots$. It is well known that each positive integer n has one or more representations when expressed as a sum of distinct Fibonacci numbers. One way of obtaining such a representation is by applying the *greedy algorithm*. This gives rise to a representation of n of the form $n = \sum_{i=0}^{k} t_i F_i$ with $t_i \in \{0,1\}$ and with $t_{i+1}t_i \neq 11$ for each $0 \leq i \leq k-1$. Such a representation of n is necessarily unique and is called the *Zeckendorff representation* [32] (a special case of the Dumont-Thomas numeration system [17, 18]). We shall write $\mathcal{Z}(n) = t_k t_{k-1} \ldots t_0$. For example, applying the greedy algorithm to $n = 50$ we obtain $50 = 34 + 13 + 3 = F_7 + F_5 + F_2$ which gives rise to the representation $\mathcal{Z}(50) = 10100100$. It follows immediately that $\mathcal{Z}(F_n) = 10^n$. The connection between $\mathcal{Z}(n)$ and the entry f_n of the Fibonacci word $\mathbf{f}$ is given by the following well known fact: $f_n = 0$ whenever $\mathcal{Z}(n)$ ends in 0 and $f_n = 1$ whenever $\mathcal{Z}(n)$ ends in 1. Thus

$$\mathbf{f}\big|_0 = \{n \in \mathbb{N} \,|\, \mathcal{Z}(n) \text{ ends in } 0\}$$

and

$$\mathbf{f}\big|_1 = \{n \in \mathbb{N} \,|\, \mathcal{Z}(n) \text{ ends in } 1\}.$$

We now consider the sequence $(x_n)_{n\in\mathbb{N}}$ given by $x_n = F_{2n+1.}$ It is readily verified that for each $A \in \text{Fin}(\mathbb{N})$, the Zeckendorff representation of $\sum_{n\in A} x_n$ ends in 10^{2m+1} where $m = \min(A)$. In fact, the symbolic sum of the individual Zeckendorff representations of each x_n occurring in $\sum_{n\in A} x_n$ does not involve any carry overs. Moreover the resulting expression does not contain any occurrences of 11 and hence is equal to the Zeckendorff representation of $\sum_{n\in A} x_n$. Thus every finite sum of the form $\sum_{n\in A} x_n$ with $A \in \text{Fin}(\mathbb{N})$ belongs to $\mathbf{f}\big|_0$. Thus we have shown that $\mathbf{f}\big|_0$ is an IP-set.

We next verify that $\mathbf{f}\big|_1$ is not an IP-set, and hence $\mathbf{f}\big|_0$ is an IP*-set. We will use the following general observation. Consider a subset $A \subset \mathbb{N}$ partitioned into $k > 0$ non-intersecting sets: $A = A_1 \cup A_2 \cup \cdots \cup A_k$. Suppose that for each $1 \leq j \leq k$ there exists a positive integer N (which may depend on j) such that whenever $m_1, m_2, \ldots, m_N$ are distinct elements of A_j, we have $\sum_{i=1}^{N} m_i \notin A$. Then A is not an IP-set. In fact, if A were an IP-set, then for some $1 \leq j \leq k$, there would exist a sequence $x_1 < x_2 < x_3 < \cdots$ contained in A_j such that $\{\sum_{n\in F} x_n | F \in \text{Fin}(\mathbb{N})\} \subset A$.

Let $\alpha = \frac{3-\sqrt{5}}{2}$. Then the Fibonacci word $\mathbf{f}$ is the orbit of the point α under irrational rotation R_α on the unit circle by α. Let I be the interval $[1-\alpha, 1)$ (the interval coded by 1). So $n \in \mathbf{f}\big|_1$ if and only if $R_\alpha^n(\alpha) = \{\alpha + n\alpha\} = \{(n+1)\alpha\} \in I$.

Fix $(1-\alpha)/3 \leq \alpha' \leq (1-\alpha)/2$ and put $I_1 = [1-\alpha, 1-\alpha')$ and $I_2 = [1-\alpha', 1)$. Since $\alpha' \leq (1-\alpha)/2$ it follows that $\alpha' < \alpha$. Also for $j = 1, 2$ set $A_j = \{n \in \mathbb{N} | R^n(\alpha) \in I_j\}$. Thus A_1, A_2 partitions the set $\mathbf{f}\big|_1$. We now show that $\mathbf{f}\big|_1$ is not an IP-set by showing that the sum of any three elements of A_1 belongs to $\mathbf{f}\big|_0$ and that the sum of any two elements of A_2 belongs to $\mathbf{f}\big|_0$.

Now take any $n_1, n_2, n_3 \in A_1$ and set $x_1 = \{(n_1+1)\alpha\}, x_2 = \{(n_2+1)\alpha\}, x_3 = \{(n_3+1)\alpha\} \in [1-\alpha, 1-\alpha')$. Then $n_1+n_2+n_3$ corresponds to the point $\{(n_1+n_2+n_3+1)\alpha\} = \{x_1+x_2+x_3-2\alpha\}$. Since $x_1, x_2, x_3 \in [1-\alpha, 1-\alpha')$, we have $\{x_1+x_2+x_3-2\alpha\} \in [\{3-5\alpha\}, \{3-3\alpha'-2\alpha\})$. Since $\alpha' \geq \frac{1-\alpha}{3}$ it follows that $2-3\alpha'-2\alpha \leq 1-\alpha$, and hence $\{2-3\alpha'-2\alpha\} \leq 1-\alpha$, which gives $\{3-3\alpha'-2\alpha\} \leq 1-\alpha$ as required.

Similarly take any $n_1, n_2 \in A_2$. Set $x_1 = \{(n_1+1)\alpha\}, x_2 = \{(n_2+1)\alpha\} \in [1-\alpha', 1)$. Then n_1+n_2 corresponds to the point $\{(n_1+n_2+1)\alpha\} = \{x_1+x_2-\alpha\}$. Since $x_1, x_2 \in [1-\alpha', 1)$, we have $\{x_1+x_2-\alpha\} \in [\{2-2\alpha'-\alpha\}, 1-\alpha)$. Since $\alpha' \leq \frac{1-\alpha}{2}$ it follows that $\{1-2\alpha'-\alpha\} \geq 0$, and hence $\{2-2\alpha'-\alpha\} \geq 0$.

The above arguments may be generalized to show that $\mathbf{f}\big|_u$ is an IP*-set for every prefix u of $\mathbf{f}$.

In contrast, let us consider the sets $\mathbf{g}\big|_0$ and $\mathbf{g}\big|_1$ where $\mathbf{g} = 0\mathbf{f} = 001001010010010\ldots$. Thus,

$$\mathbf{g}\big|_0 = \{n \in \mathbb{N} \,|\, g_n = 0\} = \{0\} \cup \{n \geq 1 \,|\, f_{n-1} = 0\}.$$

Consider the sequence $(y_n)_{n\in\mathbb{N}}$ defined by $y_n = F_{2n+2}$. It is readily verified that $\mathcal{Z}(y_n - 1) = (10)^{n+1}$ and hence each y_n belongs to $\mathbf{g}\big|_0$. Now fix $A \in \text{Fin}(\mathbb{N})$. Since the Zeckendorff representation of $\sum_{n\in A} y_n$ ends in 10^{2m+2} where $m = \min(A)$, it follows that $\mathcal{Z}(\sum_{n\in A} y_n - 1)$ ends in $(10)^{m+1}$, and hence $\sum_{n\in A} y_n \in \mathbf{g}\big|_0$. Thus, $\mathbf{g}\big|_0$ is an IP-set. Similarly, it is readily verified that for each $A \in \text{Fin}(\mathbb{N})$, we have that $\sum_{n\in A} x_n \in \mathbf{g}\big|_1$ where $x_n = F_{2n+1}$. Thus this time we obtain the Sturmian decomposition $\mathbb{N} = \mathbf{g}\big|_0 \cup \mathbf{g}\big|_1$ in which both sets $\mathbf{g}\big|_0$ and

$\mathbf{g}\big|_1$ are IP-sets. In this case, neither $\mathbf{g}\big|_0$ nor $\mathbf{g}\big|_1$ is an IP*-set. Once again, these arguments may be extended to show that both $\mathbf{g}\big|_{0u}$ and $\mathbf{g}\big|_{1u}$ are IP-sets for any prefix u of $\mathbf{f}$ and hence neither set is an IP*-set.

In summary, by Theorem 3.12 we have:

Proposition 4.1. *Let* $\mathbf{f}$ *denote the Fibonacci word. Then for every prefix* u *of* $\mathbf{f}$ *the set* $\mathbf{f}\big|_u$ *is an IP*-set (and hence a central* set). Setting* $\mathbf{g} = 0\mathbf{f}$ *we have that for every prefix* u *of* $\mathbf{f}$ *the sets* $\mathbf{g}\big|_{0u}$ *and* $\mathbf{g}\big|_{1u}$ *are both IP-sets (resp. central sets).*

4.2 The m-bonacci word

The above analysis extends more generally to the so-called m-bonacci word. Fix a positive integer $m \geq 2$, and let $\mathbf{t} = t_0t_1t_2\ldots \in \{0, 1, \ldots, m-1\}^{\mathbb{N}}$ denote the m*-bonacci infinite word* fixed by the substitution

$$\sigma_m : \{0, 1, \ldots, m-1\} \to \{0, 1, \ldots, m-1\}^*$$

given by

$$\sigma_m(i) = \begin{cases} 0(i+1) & \text{for } 0 \leq i < m-1 \\ 0 & \text{for } i = m-1 \end{cases}$$

Using the associated Dumont-Thomas numeration system, we will show:

Proposition 4.2. *Let* $m \geq 2$*, and consider the partition of* $\mathbb{N}$ *given by*

$$\mathbb{N} = \bigcup_{0\leq k\leq m-1} \mathbf{g}\big|_k$$

where $\mathbf{g} = 0\mathbf{t} \in \{0, 1, \ldots, m-1\}^{\mathbb{N}}$*. Then for each* $0 \leq k \leq m-1$ *the set* $\mathbf{g}\big|_k$ *is an IP-set (resp. central set).*

The proof is a simple extension of the ideas outlined above in the case of the Fibonacci word. For each $m \geq 2$, we define the m-bonacci numbers by $T_k = 2^k$ for $0 \leq k \leq m-1$ and $T_k = T_{k-1} + T_{k-2} + \cdots + T_{k-m}$ for $k \geq m$. When $m = 2$, these are the usual Fibonacci numbers. Each positive integer n may be written in one or more ways in the form $n = \sum_{i=1}^{k} t_iT_{k-i}$ where $t_i \in \{0, 1\}$ and $t_1 = 1$. By applying the greedy algorithm, one obtains a representation of n of the form $w = t_1t_2\cdots t_k$ with the property that w does not contain m consecutive 1's. Such a representation of n is necessarily unique and is called the m*-Zeckendorff representation* of n, denoted $\mathcal{Z}_m(n)$ (see [19]). Thus $\mathcal{Z}_m(T_n) = 10^n$ for $n \geq 0$.

Proof. Fix $0 \leq k \leq m-1$. We will show that the set $\mathbf{g}\big|_k$ is an IP-set. It is well known that $t_n = k$ if and only if $\mathcal{Z}_m(n)$ ends in 01^k. Hence

$$\mathbf{g}\big|_k = \{n \in \mathbb{N} \mid g_n = k\} = \{n \in \mathbb{N} \mid t_{n-1} = k\} = \{n \in \mathbb{N} \mid \mathcal{Z}_m(n-1) \text{ ends in } 01^k\}.$$

Consider the sequence $(x_n)_{n\in\mathbb{N}}$ given by $x_n = T_{mn+k}$. It is readily verified for any finite subset $A \subset \mathbb{N}$, the m-Zeckendorff representation of the finite sum $s = \sum_{n\in A} x_n$ ends in 10^{mr+k} where $r = \min(A)$ and hence the m-Zeckendorff representation of $s-1$ ends in $(1^{m-1}0)^r 1^k$ and hence $s \in \mathbf{g}\big|_k$ as required.
Having established that each of the sets $\mathbf{g}\big|_k$ is an IP-set (for $0 \le k \le m-1$), it follows that no $\mathbf{g}\big|_k$ is an IP*-set.

□

As an immediate consequence of Proposition 4.2 we have:

Corollary 4.3. *For each positive integer r there exists a partition $\mathbb{N} = A_1 \cup A_2 \cup \cdots \cup A_r$ in which each A_i is a central set.*

Proof. For each $1 \le k \le r$, it suffices to take $A_k = g\big|_{k-1}$. □

5 Sturmian partitions & central sets

In this section we prove the results announced in section 1 concerning Sturmian partitions of $\mathbb{N}$. Throughout this section $\omega = \omega_0\omega_1\omega_2 \ldots \in \{0,1\}^{\mathbb{N}}$ will denote a Sturmian word, $\mathcal{F}$ the set of all factors of ω, and (Ω, T) the subshift generated by ω, where T denotes the shift map. We denote by $\tilde{\omega} \in \Omega$ the characteristic word.

Lemma 5.1. *If $\omega, \omega', \omega'' \in \Omega$ are such that $T^{n_0}(\omega) = T^{n_0}(\omega') = T^{n_0}(\omega'')$, then Card$\{\omega, \omega', \omega''\} \le 2$.*

Proof. This follows immediately from the fact that Ω contains a unique characteristic word and that this word is aperiodic. □

We will make use of the following key lemma which essentially says that two distinct Sturmian words ω and ω' are proximal if and only if $T^n(\omega) = T^n(\omega') = \tilde{\omega}$ for some $n \ge 1$.

Lemma 5.2. *Let ω and ω' be distinct elements of Ω. Then either $T^n(\omega) = T^n(\omega') = \tilde{\omega}$ for some $n \ge 1$, or there exists $N > 0$ such that $\omega_n\omega_{n+1}\ldots\omega_{n+N} \ne \omega'_n\omega'_{n+1}\ldots\omega'_{n+N}$ for every $n \in \mathbb{N}$.*

Proof. We will use a definition of Sturmian words via rotations, which we recalled in Section 2. Notice that $\tilde{\omega} = s_{\alpha,\alpha} = s'_{\alpha,\alpha}$, and singular words correspond to the case when the orbit of a point under rotation map goes through the point α. If $s_{\alpha,\rho}$ is non-singular, then $s_{\alpha,\rho} = s'_{\alpha,\rho}$. If $w \ne w'$ are singular words defined by rotations of the same point, i. e., $w = s_{\alpha,\rho}$, $w' = s'_{\alpha,\rho}$, then they differ only when they pass through $1-\alpha$ and 0, i. e., in maximum two points, so there exists $n_0 \ge 1$ such that $T^{n_0}(\omega) = T^{n_0}(\omega') = \tilde{\omega}$.

Now consider the case when w, w' are defined by rotations of two different points ρ, ρ', $0 \le \rho < \rho' < 1$. To be definite, let us consider the interval exchange of I_0 and I_1 for both w and w'. We should prove that there there exists $N > 0$ such that $\omega_n\omega_{n+1}\ldots\omega_{n+N} \neq \omega'_n\omega'_{n+1}\ldots\omega'_{n+N}$ for every $n \in \mathbb{N}$. We have $w_i \neq w'_i$ if and only if $w_i \in I_0$, $w'_i \in I_1$ or $w_i \in I_1$, $w'_i \in I_0$. This condition is equivalent to $w_i \in [1-\alpha-(\rho'-\rho), 1-\alpha) \cup [1-(\rho'-\rho), 1)$. The distribution of points from the orbit of any point θ under rotation by α is dense, it means that for every ϵ there exists $N(\epsilon)$, such that after $N(\epsilon)$ iterations points split the interval $[0, 1)$ into intervals of length less than ϵ. Putting $\epsilon = \rho' - \rho$, we get that every $N = N(\epsilon)$ consecutive iterations there will be a point in every interval of length $\rho' - \rho$, so there are points in $[1-\alpha-(\rho'-\rho), 1-\alpha)$ and $[1-(\rho'-\rho), 1)$ every N iterations, and hence for every n there exists $i \in [n, n+N-1]$ with $w_i \neq w'_i$.

□

We first consider the case of nonsingular Sturmian words:

Lemma 5.3. *Let $\omega \in \{0,1\}^{\mathbb{N}}$ be a nonsingular Sturmian word and $p \in \beta\mathbb{N}$ an idempotent ultrafilter. Then $p^*(\omega) = \omega$.*

Proof. Suppose to the contrary that $p^*(\omega) \neq \omega$. Then since ω is nonsingular, Lemma 5.2 implies that for all sufficiently long factors u of ω, we have that $\omega\big|_u \cap p^*(\omega)\big|_u = \emptyset$. But, by Lemma 3.9 we have $p^*(p^*(\omega)) = p^*(\omega)$, that is the image under p^* of ω and $p^*(\omega)$ coincides. It follows by definition of p^* that for every prefix u of $p^*(\omega)$ we have $\omega\big|_u \in p$ and $p^*(\omega)\big|_u \in p$ and hence $\omega\big|_u \cap p^*(\omega)\big|_u \in p$, a contradiction. □

Theorem 5.4. *Let $\omega \in \Omega$ be a nonsingular Sturmian word, and u a factor of ω. Then $\omega\big|_u$ is an IP-set (resp. central set) if and only if u is a prefix of ω. Hence for every prefix v of ω and $n \in \omega\big|_v$ the set $\omega\big|_v - n$ is an IP*-set (resp. central* set).*

Proof. Let ω be a nonsingular Sturmian word, u a prefix of ω, and $p \in \beta\mathbb{N}$ an idempotent ultrafilter. Then by Lemma 5.3 u is a prefix of $p^*(\omega)$ and hence $\omega\big|_u \in p$. Thus for each prefix u of ω the set $\omega\big|_u$ belongs to every idempotent ultrafilter and hence is an IP*-set. It follows that if $v \in F$ is not a prefix of ω, then $\omega\big|_v$ is not an IP-set. Finally, let v be any factor of ω and $n \in \mathbb{N}$. Then $\omega\big|_v - n = T^n(\omega)\big|_v$. If $n \in \omega\big|_v$, then v is a prefix of $T^n(\omega)$ from which it follows that

$$\omega\big|_v - n = T^n(\omega)\big|_v. = T^n(\omega)\big|_v \in p.$$

Hence $\omega\big|_v - n$ is an IP*-set □

As a consequence of the above theorem we have

Corollary 5.5. *Let ω and ω' be two nonsingular Sturmian words, not necessarily of the same slope. Then for every prefix u of ω and every prefix u' of ω' we have that $\omega\big|_u \cap \omega'\big|_{u'}$ is an IP*-set (resp. central* set), in particular the intersection is infinite.*

We note that the assumption that ω and ω' be nonsingular is necessary, as for example if we consider $\omega = 0\mathbf{f}$ and $\omega' = 1\mathbf{f}$ with $\mathbf{f}$ the Fibonacci word, then $\omega\big|_0 \cap \omega'\big|_1 = \{0\}$.

Proof. Let ω and ω' be two nonsingular Sturmian words, u a prefix of ω, u' a prefix of ω', and $p \in \beta\mathbb{N}$ an idempotent ultrafilter. Then by Corollary 1 we have that $\omega\big|_u \in p$ and $\omega\big|_{u'} \in p$ and hence $\omega\big|_u \cap \omega\big|_{u'} \in p$. Thus $\omega\big|_u \cap \omega\big|_{u'}$ belongs to every idempotent and hence is an IP*-set. $\square$

We next consider singular Sturmian words.

Lemma 5.6. *Let $\omega, \omega' \in \Omega$ be distinct Sturmian words such that $T^{n_0}(\omega) = T^{n_0}(\omega') = \tilde{\omega}$ for some $n_0 \geq 1$. Then for every $u \in \mathcal{F}$ and every non-principal ultrafilter $p \in \beta\mathbb{N}$ we have*

$$\omega\big|_u \in p \Longleftrightarrow \omega'\big|_u \in p.$$

In particular, $p^*(\omega) = p^*(\omega')$.

Proof. Since p is a non-principal ultrafilter, we have that $\omega\big|_u \in p \Longleftrightarrow \omega\big|_u \cap [N, +\infty) \in p$ for all $N \geq 1$. Similarly $\omega'\big|_u \in p \Longleftrightarrow \omega'\big|_u \cap [N, +\infty) \in p$ for all $N \geq 1$. But for each $u \in \mathcal{F}$, we have $\omega\big|_u \cap [n_0, +\infty) = \omega'\big|_u \cap [n_0, +\infty)$. The result now follows. $\square$

Lemma 5.7. *Let $\omega, \omega' \in \Omega$ be as in the previous lemma, and let $p \in \beta\mathbb{N}$ be an idempotent ultrafilter. Then $p^*(\omega) = p^*(\omega') \in \{\omega, \omega'\}$.*

Proof. That $p^*(\omega) = p^*(\omega')$ follows from the previous lemma and the fact that idempotent ultrafilters are non-principal (see for instance [6]). By Lemma 3.10, p^* commutes with the shift map T, and hence

$$T^{n_0}p^*(\omega) = p^*(T^{n_0}\omega) = p^*(\tilde{\omega}) = \tilde{\omega}$$

where the last equality follows from Lemma 5.3. By Lemma 5.1 applied to $\omega'' = p^*(\omega)$ it follows that $p^*(\omega) = \omega$ or $p^*(\omega) = \omega'$. $\square$

Theorem 5.8. *Let $\omega \in \Omega$ be a Sturmian word such that $T^{n_0}(\omega) = \tilde{\omega}$ with $n_0 \geq 1$. Then $\omega\big|_u$ is an IP-set (or central set) if and only if either u is a prefix of ω or a prefix of ω' where ω' is the unique other element of Ω with $T^{n_0}(\omega') = \tilde{\omega}$.*

Proof. Let $\omega \in \Omega$ and n_0 be as in the statement of the theorem. Then there exists a unique $\omega' \in \Omega$ with $\omega' \neq \omega$ and with $T^{n_0}(\omega') = \tilde{\omega}$. Suppose that $\omega\big|_u$ is an IP-set for some $u \in \mathcal{F}$. Then by Lemma 3.7 it follows that u is a prefix of $p^*(\omega)$ for some idempotent ultrafilter $p \in \beta\mathbb{N}$. It follows from Lemma 5.7 that u is a prefix of ω or a prefix of ω'. This proves one direction.

To establish the other direction, we must show that $\omega\big|_u$ is a central set for each prefix u of ω or of ω'. By Theorem 3.11, there exist minimal idempotent ultrafilters $p_1, p_2 \in \beta\mathbb{N}$ such that $p_1^*(\omega) = \omega$ and $p_2^*(\omega) = \omega'$. The result now follows. $\square$

Remark 5.9. V. Bergelson [9] suggested to us that the above result may be related to a previously known partition of $\mathbb{N}$ into two central sets $X = \{[mx], m \in \mathbb{N}\}$ and $Y = \{[my], m \in \mathbb{N}\}$, where x and y are two irrational numbers satisfying $1/x + 1/y = 1$. In fact, this partition precisely corresponds to our partition of $\mathbb{N}$ into two IP-sets $\omega\big|_0$ and $\omega\big|_1$ where ω is of the form $0\tilde{\omega}$ and $\tilde{\omega}$ is a characteristic Sturmian.

This could be seen using the definition of Sturmian words via mechanical words (see Section 2 for notation). For a slope α we have $s_{\alpha,0} = 0\tilde{\omega}$. Let $\alpha = 1/x$ and $1/y = 1-\alpha$; then $s_{\alpha,0}(n) = 1$ if and only if there exists an integer k such that $\alpha(n+1) \geq k$ and $\alpha n < k$. It is easy to see that this pair of equations is equivalent to $n < kx \leq n+1$, which implies $n \in X$. We have $s_{\alpha,0}(n) = 0$ if and only if there exists an integer k such that $\alpha(n+1) < k+1$ and $\alpha n \geq k$. It is not difficult to see that this pair of equations is equivalent to $n \leq (n-k)y < n+1$, which implies $n \in Y$.

Remark 5.10. We are unable to extend the results on Sturmian partitions to all Arnoux-Rauzy words. In fact, our proof of Lemma 5.2 relies on the geometric interpretation of Sturmian words as codings of orbits under an irrational rotation on the circle. It was shown in [12] that there exist Arnoux-Rauzy words which are not measure-theoretically conjugate to a rotation on the n-torus. In this case, we do not understand for which pairs of Arnoux-Rauzy words in the subshift are proximal.

6 Other central partitions defined by substitutions

We begin by briefly reviewing some notions from topological dynamics in the framework of minimal subshifts (X,T) which will be used in the proof of Theorem 4. For this we consider two-sided subshifts (X,T) meaning that $X \subset \mathcal{A}^{\mathbb{Z}}$. So points in X are bi-infinite words. A subshift (X,T) is said to be *equicontinuous* if for every $\epsilon > 0$, there exists a $\delta > 0$, such that for all $x, y \in X$, if $d(x,y) < \delta$ then $d(T^n(x), T^n(y)) < \epsilon$ for every $n \in \mathbb{Z}$. A subshift (Y,T) is called a *factor* of (X,T) if there exists a continuous surjection

$$\pi : X \to Y$$

which commutes with the shift map T. It is well known (for instance by way of Zorn's lemma) that every subshift (X,T) has a *maximal equicontinuous factor* (Y,T) i.e., (Y,T) is an equicontinuous factor of (X,T) and any equicontinuous factor (Z,T) of (X,T) is also a factor of (Y,T). It is also well known that if $\pi : X \to Y$ is the maximal equicontinuous factor, then for any two points $x, y \in X$ we have that $\pi(x) = \pi(y)$ if and only if x and y are regionally proximal (see [2]).

Proof of Theorem 4. Let us fix positive integers r and N. Consider the constant length substitution

$$\tau : \{1, 2, \ldots, r\} \to \{1, 2, \ldots, r\}^+$$

given by $1 \mapsto 123\cdots r$, $2 \mapsto 23\cdots r1$, $3 \mapsto 34\cdots r12$, $\ldots$, $r \mapsto r12\cdots r-1$. In case $r = 2$ we have the Thue-Morse substitution on the alphabet $\{1, 2\}$. For $1 \leq i \leq r$, let $x^{(i)}$ denote the ith fixed point of τ beginning in the letter i. As in the case of Thue-Morse, for $i \neq j$ the words $x^{(i)}$ and $x^{(j)}$ never coincide, i.e., $x_n^{(i)} \neq x_n^{(j)}$ for each $n \in \mathbb{N}$. Let (X, T) denote the one-sided minimal subshift generated by the primitive substitution τ. We will now show that each of the fixed points $x^{(i)}$ is distal.

Lemma 6.1. *Let x denote any one of the fixed points $x^{(i)}$ of the substitution τ above. Then x is distal. In particular, the two fixed points of the Thue-Morse substitution are each distal.*

Proof. Let $(\tilde{X}, T)$ denote the two-sided subshift generated by τ, and let $\pi : \tilde{X} \to Y$ denote the maximal equicontinuous factor. The substitution τ above is of Pisot type, in fact, the dilation of τ is r and all other eigenvalues are equal to 0. (Note that τ is not an irreducible substitution). In [3], V. Baker, M. Barge and J. Kwapisz show that for a primitive substitution of Pisot type (irreducible or not), the mapping onto the maximal equicontinuous factor is finite to one. Thus there exists a constant C such that for any $z \in \tilde{X}$, there are at most C points $z' \in \tilde{X}$ which are regionally proximal to z In particular, for any $z \in \tilde{X}$, there are at most C points $z' \in \tilde{X}$ which are proximal to z.

Now suppose $y \in X$ is proximal to x. We will show that $y = x$. It is easy to see that the bi-infinite word $z = x_{\text{rev}} \cdot x \in \tilde{X}$ where x_{rev} denotes the reversal or mirror image of x, and where $\cdot$ denotes the origin. Similarly, let y' denote a left infinite word such that the concatenation $z' = y' \cdot y \in \tilde{X}$. Since x and y are proximal, it follows that z and z' are proximal. Set $\sigma = \tau^r$. Since τ, and hence σ, are of constant length, it follows that $\sigma(z')$ is proximal to $\sigma(z)$. But $\sigma(z) = z$. Hence $(\sigma^n(z'))_{n \geq 0}$ defines an infinite sequence of points in $\tilde{X}$ each of which is proximal to z, and which in the limit tends to $x_{\text{rev}}^{(i)} \cdot x^{(j)}$ where i is the first (meaning rightmost) letter of y' and j is the first letter of y. But since there are only finitely many points in $\tilde{X}$ which are proximal to z it follows that $\sigma^n(z') = x_{\text{rev}}^{(i)} \cdot x^{(j)}$ for some $n \geq 0$. Hence by de-substituting we obtain $z' = x_{\text{rev}}^{(i)} \cdot x^{(j)}$ from which it follows that $y = x^{(j)}$. Thus both x and y are fixed points of τ which are assumed proximal. It follows that $y = x$ and hence x is distal as required. $\square$

Put $x = x^{(1)}$. Since x is distal, so is $T^n(x)$ for each $n \geq 1$. On the other hand, it is easy to see that for each positive integer n we have $u^{(i)}[n]x \in X$, where $u^{(i)}[n]$ denotes the reversal of the prefix of $x^{(i)}$ of length n. Thus the r words $\{u^{(1)}[n]x, u^{(2)}[n]x, \ldots, u^{(r)}[n]x\}$ are pairwise proximal and each begin in distinct letters (this is because the fixed points never coincide). Finally let $\omega = u^{(1)}[N+1]x$, and set $A_i = \omega\big|_i$ for each $1 \leq i \leq r$. Then each A_i is a central set.

For each $1 \leq n \leq N$, we have that $A_i - n = T^n(\omega)\big|_i = u^{(1)}[N+1-n]x\big|_i$ is a central set. But for $k \geq 1$, we have that $A_i - (N+k) = T^{k-1}(x)\big|_i$ which is a central set if and only if $T^{k-1}(x)$ begins in i.

$\square$

Proof of Theorem 5. Fix a positive integer r. Let τ be a primitive substitution whose associated subshift Ω is topologically weak mixing. For instance we may take the substitution $0 \mapsto 001$ and $1 \mapsto 11001$ or $0 \mapsto 001$ and $1 \mapsto 11100$ (see [15]). Let $\omega \in \Omega$. Fix m such that $\rho_\omega(m) \geq r$, and put $s = \rho_\omega(m)$. Let $u_1, u_2, \ldots, u_s$ denote the factors of ω of length m. As pointed out to us by V. Bergelson and Y. Son [9], the weak mixing implies that the set of points in Ω proximal to ω is dense in Ω (see for instance page 184 of [22]). Thus for each factor u_i there exists a word $x_i \in \Omega$ beginning in u_i and which is proximal to ω. Hence by Theorem 3.11 there exists a minimal idempotent ultrafilter $p_i \in \beta\mathbb{N}$ such that $p_i^*(\omega) = x_i$. Hence for each $1 \leq i \leq s$ we have that $\omega\big|_{u_i} \in p_i$ and hence $\omega\big|_{u_i}$ is a central set. Finally, for each positive integer n and for each $1 \leq i \leq s$ we have that

$$\omega\big|_{u_i} - n = T^n(\omega)\big|_{u_i}.$$

Again the weak mixing implies that there exists a word $x \in \Omega$ beginning in u_i and proximal to $T^n(\omega)$. Hence there exists a minimal idempotent $p \in \beta\mathbb{N}$ such that $p^*(T^n(\omega)) = x$ from which it follows that $\omega\big|_{u_i} - n \in p$ and hence $\omega\big|_{u_i} - n$ is a central set. Thus we obtain a partition of $\mathbb{N}$

$$\mathbb{N} = \bigcup_{i=1}^{s} \omega\big|_{u_i}$$

into s-many central sets and for each positive integer n and $1 \leq i \leq s$ we have that $\omega\big|_{u_i} - n$ is again a central set. Thus, setting

$$A_i = \omega\big|_{u_i}$$

for $i = 1, \ldots, r-1$, and

$$A_r = \bigcup_{i=r-1}^{s} \omega\big|_{u_i}$$

we obtain the desired partition of $\mathbb{N}$. $\square$

7 Infinite central partitions of $\mathbb{N}$

In this section we construct infinite partitions of $\mathbb{N}$ into central sets by using words on an infinite alphabet and prove Theorem 6. Our construction makes use of the notion of *iterated palindromic closure operator* (first introduced in [16]):

Definition 7.1. *The iterated palindromic operator ψ is defined inductively as follows:*

- $\psi(\varepsilon) = \varepsilon$,
- *For any word w and any letter a, $\psi(wa) = (\psi(w)a)^{(+)}$.*

We denote with $w^{(+)}$ the right palindromic closure *of the word w, i.e., the shortest palindrome which has w as a prefix.*

For example, $\psi(aaba) = aabaaabaa$. The operator ψ has been extensively studied for its central role in constructing standard Sturmian and episturmian words. It follows immediately from the definition that if u is a prefix of v, then $\psi(u)$ is a prefix of $\psi(v)$. Thus, given an infinite word $\omega = \omega_0\omega_1\omega_2\ldots$ on the alphabet A we can define

$$\psi(\omega) = \lim_{n\to\infty} \psi(\omega_0\omega_1\omega_2\ldots\omega_n).$$

The following lemma summarizes the properties of ψ needed.

Lemma 7.2. *Let Δ be a right infinite word over the (finite or infinite) alphabet A and let $\omega = \psi(\Delta)$. Then the following statements hold:*

1. *The word ω is closed under reversal, i.e., if $v = v_1v_2\ldots v_k$ is a factor of ω, then so is its mirror image $v_k\ldots v_2v_1$.*
2. *The word ω is uniformly recurrent.*
3. *If each letter $a \in A$ appears in Δ an infinite number of times, then for each prefix u of ω and each $a \in A$, we have au is a factor of ω.*

Proof. Since any factor of ω is contained in some $\psi(v)$ for a sufficiently long prefix v of Δ, and $\psi(v)$ is by definition a palindrome (and hence closed under reversal), the first statement is proved. The second statement is easily derived from the fact that for any finite prefix va of Δ (a being a letter), we have that $|\psi(va)| \leq 2|\psi(v)| + 1$ and moreover $\psi(va)$ begins and ends in $\psi(v)$. It follows that any factor of length (for example) $3|\psi(v)|$ contains an occurrence of $\psi(v)$.

Finally suppose each $a \in A$ appears infinitely many times in Δ. Thus for any letter a and any prefix v of Δ there exists a prefix of Δ of the form $vv'a$. From the definition of ψ we then have that $\psi(vv')a$ is a prefix of ω and $\psi(vv')$ ends in $\psi(v)$, so $\psi(v)a$ is a factor of ω. Since $\psi(v)$ is a palindrome and ω is closed under reversal, we obtain that for any prefix v of Δ and for any letter a, the word $a\psi(v)$ is a factor of ω and the third statement easily follows. $\square$

With the preceding Lemma, we are now able to construct infinite partitions of $\mathbb{N}$ such that each element of the partition is an IP-set.

Proposition 7.3. *Let $\omega = \psi(\Delta)$ where Δ is a right infinite word on an infinite alphabet $\mathcal{A}$ with the property that each letter $a \in \mathcal{A}$ occurs in Δ an infinite number of times. Then, for any $a \in \mathcal{A}$, the set $a\omega\big|_a$ is a central set, thus $\{\omega\big|_a + 1\}_{a\in\mathcal{A}}$ is an infinite partition of $\mathbb{N}$ into central sets.*

Proof. From 7.2 we clearly have that ω is uniformly recurrent and closed under reversal. Furthermore, since each $a \in \mathcal{A}$ occurs in Δ an infinite number of times, by (2) of the same lemma we also obtain that condition (3) holds, so that for any letter a, the set of factors of $a\omega$ coincides with that of ω. From this and from the uniform recurrence of ω, we have that $a\omega$ is uniformly recurrent as well. Let us denote by π_a the image of ω under the morphism μ_a defined as follows:

- $\mu_a(a) = 0$,
- $\mu_a(x) = 1$ if $x \neq a$.

Since $a\omega$ is uniformly recurrent for any a, it is clear that also $0\pi_a$ is uniformly recurrent for any a. From Theorem 3.11, we then have that for any a there exists a minimal idempotent ultrafilter p_a such that $p_a^*(0\pi_a) = 0\pi_a$. In particular, this means, by Lemma 3.7, that $0\pi_a\big|_0$ (which clearly coincides with $a\omega\big|_a$ by definition) is a central set for any a. The statement can then be easily derived from the fact that $a\omega\big|_a = \omega\big|_a + 1$. $\square$

8 Strong coincidence condition

Let $r \geq 2$ be a positive integer and set $\mathcal{A} = \{1, 2, \ldots, r\}$. A primitive substitution $\tau : \mathcal{A} \to \mathcal{A}^+$ is said to satisfy the *strong coincidence condition* if and only if for any pair of fixed points x and y, we can write $x = scx'$, and $y = tcy'$ for some $s, t \in \mathcal{A}^+$, $c \in \mathcal{A}$, and $x', y' \in \mathcal{A}^\infty$ with $s \sim_{\text{ab}} t$. This combinatorial condition, originally due to P. Arnoux and S. Ito, is an extension of a similar condition considered by F.M. Dekking in [14] in the case of *constant length substitutions*, i.e., when $|\tau(a)| = |\tau(b)|$ for all $a, b \in \mathcal{A}$. In this case Dekking proves that the condition is satisfied if and only if the associated substitutive subshift has *pure discrete spectrum*, i.e., is metrically isomorphic with translation on a compact Abelian group. Clearly not all primitive substitutions satisfy the strong coincidence condition. For instance, it is not satisfied by the Thue-Morse substitution (in fact the two fixed points disagree in each coordinate). It is conjectured however that if τ is an irreducible primitive substitution of Pisot type, then τ satisfies the strong coincidence condition. M. Barge and B. Diamond established this conjecture for binary primitive substitutions of Pisot type [4]. Otherwise the conjecture remains open for substitutions on alphabets greater that two. Substitutions of Pisot type provide a framework for non-constant length substitutions in which the strong coincidence condition implies pure discrete spectrum.
As a consequence of Theorem 3.11 we have

Corollary 8.1. *Let τ be a primitive substitution verifying the strong coincidence condition. Then*

1. *Any two fixed points of τ are proximal.*
2. *For any pair of fixed points x and y, there exists a minimal idempotent ultrafilter $p \in \beta\mathbb{N}$ with $p^*(x) = y$.*
3. *For any pair of fixed points x and y, and any prefix u of y, we have that $x\big|_u$ is a central set.*

Remark 8.2. For irreducible primitive substitutions of Pisot type, it turns out that each of the above conditions (1), (2), and (3) are equivalent and each implies the strong coincidence condition. A proof of this fact will be given in [11]. However, for a general primitive substitution we always have that $(1) \iff (2) \implies (3)$. The two fixed points of the uniform substitution $a \mapsto aaab$, $b \mapsto bbab$ are proximal but do not satisfy the strong coincidence condition. V. Bergelson and Y. Son [9] showed that the fixed points of $a \mapsto aab$, $b \mapsto bbaab$ satisfy (3) but not (1) and (2).

Proof. Condition (1) is immediate from the definition of strong coincidence. By Theorem3.11 we have that (1) implies (2) and hence (3). □

We present now an alternative and constructive proof of (3) using the so-called Dumont-Thomas numeration systems defined by substitutions [17, 18]. Since in the irreducible Pisot case, condition (3) alone implies the strong coincidence condition, this method of proof constitutes a new approach to the strong coincidence conjecture. We begin with a brief review of these numerations systems.

8.1 Abstract numeration systems defined by substitutions

Let τ denote a substitution on a finite alphabet $\mathcal{A}$. For simplicity we assume that τ has at least one fixed point $x = x_0x_1x_2\dots$ beginning in some letter $a \in \mathcal{A}$. The idea behind the numeration system is quite natural: every coordinate x_n of the fixed point x is in the image of τ of some coordinate x_m with $m \leq n$. More precisely, consider the least positive integer m such that $x_0x_1\dots x_n$ is a prefix of $\tau(x_0x_1\dots x_m)$. In this case we can write $x_0x_1\dots x_n = \tau(x_0x_1\dots x_{m-1})u_nx_n$ where u_nx_n is a prefix of $\tau(x_m)$. We now imagine a directed arc from x_m to x_n labeled u_n. In this way every coordinate x_n is the target of exactly one arc, and the source of $|\tau(x_n)|$-many arcs. It follows that for each n there is a unique path s from x_0 to x_n. Thus every natural number n may be represented by a finite sequence of labels u_i obtained by reading the labels along the path s in the direction from x_0 to x_n.

More formally, associated to τ is a directed graph $\mathcal{G}(\tau)$ defined as follows: the vertex set of $\mathcal{G}(\tau)$ is the set $\mathcal{A}$. Given any pair of vertices a, b we draw a directed edge from a to b labeled $u \in \mathcal{A}^*$ if ub is a prefix of $\tau(a)$. In other words, for every occurrence of b in $\tau(a)$ there is a directed edge from a to b labeled by the prefix (possibly empty) of $\tau(a)$ preceding the given occurrence of b. Figure 1 depicts the graph $\mathcal{G}(\tau)$ for the Fibonacci substitution $a \mapsto ab$, $b \mapsto a$.

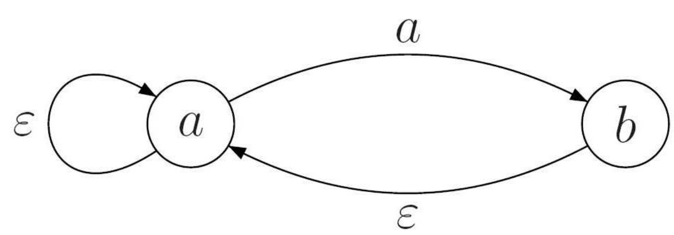

Figure 1: The Fibonacci automaton

For simplicity, in case some letter b occurs multiple times in $\tau(a)$, we draw just one directed edge from a to b having multiple labels as described above. This is shown in Figure 2 in the case of the substitution $a \mapsto aab$, $b \mapsto bbaab$.

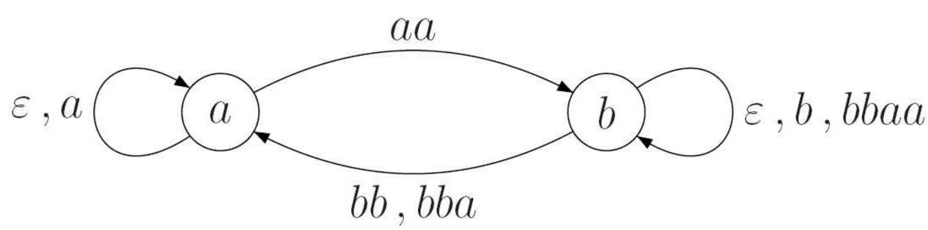

Figure 2: The automaton of $a \mapsto aab$, $b \mapsto bbaab$.

Let $x = x_0x_1x_2\ldots$ denote the fixed point of τ beginning in a. Then the graph $\mathcal{G}(\tau)$ has a singleton loop based at a labeled with the empty word ε. We consider this to be the empty or 0th path at a. More generally by a path at $a \in \mathcal{A}$ we mean a finite sequence of edge labels $u_0u_1u_2\cdots u_n$ corresponding to a path in $\mathcal{G}(\tau)$ originating at vertex a with the condition that $u_0 \neq \varepsilon$ whenever the length of the path $n > 0$. For example in the case of the Fibonacci substitution, except for the path $s = \varepsilon$, each path is given by a word in $\{a, \varepsilon\}$ beginning in a and not containing the factor aa. For each path $s = u_0u_1u_2\cdots u_n$ set

$$\rho(s) = \tau^n(u_0)\tau^{n-1}(u_1)\tau^{n-2}(u_2)\cdots\tau(u_{n-1})u_n$$

and $\lambda(s) = |\rho(s)|$. In [17, 18] it is shown that for each path s at a, the word $\rho(s)$ is a prefix of the fixed point x at a and conversely for each prefix u of x there is a unique path s at a with $\rho(s) = u$. This correspondence defines a numeration system in which every natural number l is represented by the path $s = u_0 u_1 u_2 \cdots u_n$ in $\mathcal{G}(\tau)$ from vertex a to vertex x_l corresponding to the prefix of length l of x, so that

$$(*) \quad l = \lambda(s) = |\tau^n(u_0)| + |\tau^{n-1}(u_1)| + |\tau^{n-2}(u_2)| + \cdots + |\tau(u_{n-1})| + |u_n|.$$

Generally by the numeration system one means the quantities $|\tau^n(u)|$ for all $n \geq 0$ and all proper prefixes u of the images under τ of the letters of $\mathcal{A}$. Then a *proper* representation of l in this numeration is an expression of the form (*) corresponding to a path $s = u_0 u_1 u_2 \cdots u_n$ in $\mathcal{G}(\tau)$.

In the case of a uniform substitution of length k this corresponds to the usual base k-expansion of l. In the case of the Fibonacci substitution, each $u_n \in \{\varepsilon, a\}$ and $u_i u_{i+1} \neq aa$ for each $0 \leq i \leq n-1$. Thus this representation of l is precisely the Zeckendorff representation of l discussed in §4 in which l is expressed as a sum of distinct Fibonacci numbers via the greedy algorithm.

In general, this numeration system not only depends on the substitution τ but also on the choice of fixed point. For example for the substitution in Figure 2 the number 5 is represented by the path a, aa from vertex a or by the path b, ε from vertex b. In fact, $\tau(a)aa = aabaa$ is the prefix of length 5 of $\tau^\infty(a)$ while $\tau(b)\varepsilon = bbaab$ is the prefix of length 5 of $\tau^\infty(b)$.

An alternative reformulation is as follows: Given two distinct paths $s = u_0 u_1 u_2 \cdots u_n$ and $t = v_0 v_1 v_2 \cdots v_m$ both starting from the same vertex a, we write $s < t$ if either $n < m$ or if $n = m$ there exists $i \in \{0, 2, \ldots, n\}$ such that $u_j = v_j$ for $j < i$, and $|u|_i < |v|_i$. This defines a total order on the set of all paths starting from vertex a. In the case of the Fibonacci substitution, we list the paths at a in increasing order

$$\varepsilon, a, a\varepsilon, a\varepsilon\varepsilon, a\varepsilon a, a\varepsilon\varepsilon\varepsilon, a\varepsilon\varepsilon a, a\varepsilon a\varepsilon, a\varepsilon\varepsilon\varepsilon\varepsilon, \ldots$$

Thus there is an order preserving correspondence between $0, 1, 2, 3, \ldots$ and the set of all paths at a ordered in increasing order.

While these numeration systems are very natural and simple to define, they are typically extremely difficult to work with in terms of addition and multiplication.

Let a and b be distinct vertices in $\mathcal{G}(\tau)$. We say a path s originating at a is *synchronizing* relative to b if there exists a path s' originating at b having the same terminal vertex as s and with $\lambda(s) = \lambda(s')$. From this point of view the strong coincidence conjecture implies that

$$\{\lambda(s) \mid s = \text{a synchronizing path relative to } b\}$$

is a thick set.

8.2 Proof of (3) in Corollary 8.1

Let τ be a primitive substitution satisfying the strong coincidence condition. Suppose x and y are fixed points of τ beginning in a and b respectively. Then we can write $x = scx'$, and $y = tcy'$ for some $s, t \in \mathcal{A}^+$, $c \in \mathcal{A}$, and $x', y' \in \mathcal{A}^\infty$ with $s \sim_{\text{ab}} t$. By replacing τ by a sufficiently large power of τ, we can assume that

- sc is a prefix of $\tau(a)$,
- tc is a prefix of $\tau(b)$,
- b occurs in $\tau(c)$.

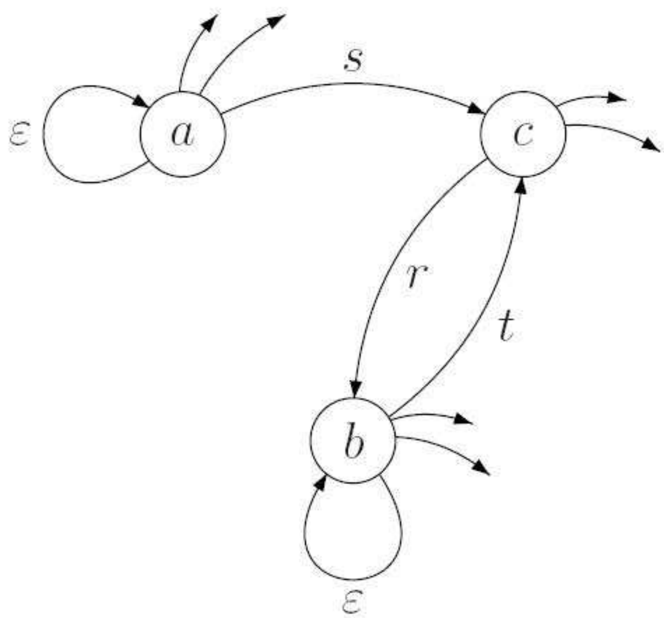

Figure 3: Vertices a, b, c of $\mathcal{G}(\tau)$

Thus in $\mathcal{G}(\tau)$ there is a directed edge from a to c labeled s, a directed edge from b to c labeled t, and a directed edge from c to b labeled r for some prefix r of $\tau(c)$. See Figure 3.

We now define a sequence of paths $(p_i)_{i\geq 0}$ from a to b by

$$p_i = s, r, \underbrace{\varepsilon, \varepsilon, \ldots, \varepsilon}_{2i}.$$

Put $n_i = \lambda(p_i)$. Then clearly $\{n_i \mid i \geq 0\} \subseteq x\big|_b$. We now show that any finite sum of distinct elements from the set $\{n_i \mid i \geq 0\}$ is contained in $x\big|_b$. Set

$$q_i = t, r, \underbrace{\varepsilon, \varepsilon, \ldots, \varepsilon}_{2i}.$$

Then each q_i is a path from b to b and since s and t are Abelian equivalent it follows that $\lambda(p_i) = \lambda(q_i)$. Fix $k \geq 1$ and choose $i_1 < i_2 < \cdots < i_k$. Then

$$\sum_{j=1}^{k} \lambda(p_{i_j}) = \lambda(p_{i_k}) + \sum_{j=1}^{k-1} \lambda(p_{i_j})$$
$$= \lambda(p_{i_k}) + \sum_{j=1}^{k-1} \lambda(q_{i_j})$$
$$= |\tau^{2i_k+1}(s)| + |\tau^{2i_k}(r)| + \sum_{j=1}^{k-1}(|\tau^{2i_j+1}(t)| + |\tau^{2i_j}(r)|)$$
$$= |\tau^{2i_k+1}(s)\tau^{2i_k}(r)\tau^{2i_{k-1}+1}(t)\tau^{2i_{k-1}}(r)\tau^{2i_{k-2}+1}(t)\tau^{2i_{k-2}}(r)\cdots\tau^{2i_1+1}(t)\tau^{2i_1}(r)|$$

which is represented by a path in $\mathcal{G}(\tau)$ from a to b and hence corresponds to an occurrence of b in x. This shows that $x\big|_b$ is an IP-set, and hence by Theorem 3.12 $x\big|_b$ is a central set. A similar argument applies for any prefix u of y by defining the paths p_i by

$$p_i = s, r, \underbrace{\varepsilon, \varepsilon, \ldots, \varepsilon}_{N_i}$$

with N_i sufficiently large.

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