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Postoptimal analysis for one vector venturesome investment problem

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Abstract

Lower and upper bounds are obtained for the stability radius of a vector venturesome investment problem with the Hölder norms in the parameter spaces. As corollaries several results are obtained for special cases.

Keywords: vector venturesome investment problem, Pareto optimal portfolio, stability radius, perturbing matrix, Hölder's metric

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1 Introduction

While solving applied optimization problems, we have to take into account different factors of uncertainty and randomness such as inadequacy of mathematical models to real processes, round-off errors, measurement errors etc. In all these cases, a mathematical problem can not be correctly posed and solved without stability theory results. In multicriteria discrete optimization problems stability research is usually tied with studying a discrete analogue of the Hausdorff continuity (semicontinuity) for dot-valued mappings i.e. mappings that assign each gang of possible data to a given set of optimal solutions [1,2].

Despite the abundance of approaches to the stability analysis in discrete optimization problems (the comprehensive idea of multiple studies on stability is provided in [3, 4]), the two mainstream approaches can be defined: qualitative and quantitative.

As a part of the qualitative approach, research is concentrated on identifying the different types of problem stability [5 – 12] or establishing interconnections between different types of stability [13, 14]. In addition, searching and describing the stability region for optimal solutions is also done (see e.g. [15]).

Big part of quantitative approach is particularised in [16], and it is tied with obtaining valid assessments of changes in the initial data, retaining some predetermined property of optimal solutions [17 – 21], and developing algorithms for calculating these bounds [22 – 26]. The key point here is stability radius, which is defined as the radius of the largest uncertain data neighbourhood preserving some optimal property in the space of perturbed problem parameters. Any perturbed problem with parameter point within the neighbourhood is «close» to the original problem

This article belongs to the second approach, and it is a continuation of publication series [27 – 34]. In that previous research, the author estimate bounds for the stability radii of solutions in vector investment Boolean problems with different criteria of optimism and pessimism (Wald, Savage criterion etc.) under a variety of combinations of linear l_1, Chebyshev l_∞ and Hölder l_p metrics in three-dimensional parameter space. In this study, we consider the most general case where different Hölder norms are used in the parameter space mentioned above. Here we obtain lower and upper bounds for the stability radius of the vector investment problem with well-known in the theory of decision-making criteria of extreme optimism regarding portfolio returns.

The results presented here were also announced in [35].

2 Problem formulation and the basic definitions

We consider a multicriteria discrete version of optimal investment problem. Let introduce some notation:

$N_n = \{1, 2, ..., n\}$ be various alternative investment projects (assets);

N_m be a set of possible states for the financial market (scenarios, the situations in the market);

N_s be many types of (indicators) economic efficiency of investment projects;

e_{ijk} be expected economic evaluation (profitability) of the form $k \in N_s$ for investment project $j \in N_n$ in the case, when the market is in the state $i \in N_m$;

$E = [e_{ijk}] \in \mathbf{R}^{m \times n \times s}$;

$x = (x_1, x_2, ..., x_n)^T \in X \subseteq \mathbf{E}^n$ is the investment portfolio, where $\mathbf{E} = \{0, 1\}$,

$$x_j = \begin{cases} 1, & \text{if you select a portfolio } j \in N_n, \\ 0 & \text{otherwise;} \end{cases}$$

So here $X \subset \mathbf{E}^n$ is the set of all possible investment portfolios, i.e. which realization does not exceed the investor's initial capital;

$\mathbf{R}^m$ is the state space of the financial market;

$\mathbf{R}^n$ is the space of the investment projects;

$\mathbf{R}^s$ is the space criterion of the cost efficiency.

For a given set of investment portfolios (Boolean vectors) X, $|X| \geq 2$, the vector criterion is defined

$$f(x, E) = (f_1(x, E_1), f_2(x, E_2), ..., f_s(x, E_s)),$$

whose components are optimistic criteria (MAX-MAX) of investment portfolio returns:

$$f_k(x, E_k) = \max_{i \in N_m} e_{ik}x = \max_{i \in N_m} \sum_{j \in N_n} e_{ijk}x_j \to \max_{x \in X}, \ k \in N_s,$$

where $E_k \in \mathbf{R}^{m \times n}$ is the k-th cut of the matrix $E = [e_{ijk}] \in \mathbf{R}^{m \times n \times s}$ with rows $e_{ik} = (e_{i1k}, e_{i2k}, ..., e_{ink}) \in \mathbf{R}^n$, $i \in N_m$.

With a help of such criterion a venturesome investor optimizes the efficiency of the portfolio under the assumption that the market is in the most profitable condition, namely when portfolio is in the marginal revenue position. Obviously, this approach is based on the stereotype of reckless optimism behaviour «or sink or swim», «who do not risk, nothing gained» etc.

We mean by vector s-criterion problem $Z^s_m(E)$, $s \in \mathbf{N}$, problem of finding the Pareto set $P^s(E)$, that is set of the Pareto-optimal portfolios

$$P^s(E) = \{x \in X : X(x, E) = \emptyset\},$$

where

$$X(x,E) = \{x' \in X : f(x,E) \leq f(x',E) \ \& \ f(x,E) \neq f(x',E)\}.$$

It's clear that $P^s(E) \neq \emptyset$ for any matrix $E \in \mathbf{R}^{m\times n\times s}$. Note also that the problem $Z^s_m(E)$ can be interpreted as «the best optimisation case».

Let different Hölder's norms l_p, l_q and l_r be given in the spaces $\mathbf{R}^n$, $\mathbf{R}^m$, $\mathbf{R}^s$ respectively, where $p,\ q,\ r \in [1,\infty]$. We understand by norm of matrix $E \in \mathbf{R}^{m\times n\times s}$ the number

$$||E||_{pqr} = ||(||E_1||_{pq}, ||E_2||_{pq}, ..., ||E_s||_{pq})||_r,$$

where

$$||E_k||_{pq} = ||(||e_{1k}||_p, ||e_{2k}||_p, ..., ||e_{mk}||_p)||_q, \ \ k \in N_s.$$

Recall that the Hölder norm l_p in space $\mathbf{R}^n$ is defined by the formula

$$||a||_p = \begin{cases} (\sum\limits_{j\in N_n} |a_j|^p)^{1/p}, & \text{if } 1 \leq p < \infty, \\ \max\{|a_j| : j \in N_n\}, & \text{if } p = \infty, \end{cases}$$

where $a = (a_1, a_2, ..., a_n)^T \in \mathbf{R}^n$.
It is easy to see that the following inequalities hold

$$||e_{ik}||_p \leq ||E_k||_{pq} \leq ||E||_{pqr}, \ \ i \in N_m, \ \ k \in N_s \tag{1}$$

for any $p,\ q,\ r \in [1,\infty]$. Following [21, 28, 30, 32], the number

$$\rho = \rho^s_m(p,q,r) = \begin{cases} \sup \Xi, & \text{if } \Xi \neq \emptyset, \\ 0, & \text{if } \Xi = \emptyset, \end{cases}$$

is called the stability radius of the problem $Z^s(E)$, where

$$\Xi = \{\varepsilon > 0 \ : \ \forall E' \in \Omega(\varepsilon) \ (P^s(E+E') \subseteq P^s(E))\},$$

$$\Omega(\varepsilon) = \{E' \in \mathbf{R}^{m\times n\times s} \quad : \quad ||E'||_{pqr} < \varepsilon\}.$$

Here $\Omega(\varepsilon)$ is a set of perturbing matrices, $P^s(E+E')$ is the Pareto set of the perturbed problem $Z^s(E+E')$, $||E'||_{pqr}$ is the norm of matrix $E' = ||e'_{ijk}||$. Thus, the stability radius $\rho^s_m(p,q,r)$ for a problem $Z^s_m(E)$ is the limiting level of elements perturbation for matrix E in the spase $\mathbf{R}^{m\times n\times s}$ that does not lead to the emergence of new Pareto optimal portfolios. Obviously, when $P^s(E) = X$, the stability radius of a problem should be considered infinitely large. If $P^s(E) \neq X$, we call the problem nontrivial.

3 Auxiliary assertions

Let u be one of the numbers p, q, r introduced above. With number u we associate a number u' as follows

$$1/u + 1/u' = 1, \quad 1 < u < \infty.$$

Moreover, let assume that $u' = 1$ if $u = \infty$, and $u = \infty$ if $u' = 1$. Therefore in what follows we assume that the range of the numbers u and u' is interval $[1, \infty]$, and the numbers themselves are connected by the conditions above. Additionally let's assume that $1/u = 0$ if $u = \infty$.

Next, we will use the well-known Hölder's inequality

$$|a^T b| \leq ||a||_u ||b||_{u'}, \tag{2}$$

that holds for any vector a and b within the same dimension.

Lemma 1. *For any portfolios $x, x^0 \in X$ and indexes $i, i' \in N_n,\ k \in N_s$, the following inequality holds*

$$e_{i'k}x^0 - e_{ik}x \geq -||E_k||_{pq}||(||x^0||_{p'}, ||x||_{p'})||_v, \tag{3}$$

where

$$v = \min\{p', q'\}.$$

Indeed, if $i \neq i'$, then using Hölder's inequality (2) we have

$$e_{i'k}x^0 - e_{ik}x \geq -(||e_{i'k}||_p||x^0||_{p'} + ||e_{ik}||_p||x||_{p'}) \geq$$

$$\geq -||(||e_{i'k}||_p, ||e_{ik}||_p)||_q \ ||(||x^0||_{p'}, ||x||_{p'})||_{q'} \geq$$

$$\geq -||E_k||_{pq}||(||x^0||_{p'}, ||x||_{p'})||_{q'} \geq -||E_k||_{pq}||(||x^0||_{p'}, ||x||_{p'})||_v.$$

If $i = i'$, then using (1) and Hölder's inequality (2) it comes out that

$$e_{i'k}x^0 - e_{ik}x \geq -||e_{ik}||_p||x^0 - x||_{p'} \geq -||E_k||_{pq}||x^0 - x||_{p'} \geq$$

$$-||E_k||_{pq}||(||x^0||_{p'}, ||x||_{p'})||_{p'} \geq -||E_k||_{pq}||(||x^0||_{p'}, ||x||_{p'})||_v.$$

Also it is easy to see that for the vector $a = (a_1, a_2, ..., a_n)^T \in \mathbf{R}^n$ with conditions $|a_j| = \alpha$, $j \in N_n$ the equality

$$||a||_p = \alpha n^{1/p} \tag{4}$$

holds for any number $p \in [1, \infty]$.

4 Bounds for the stability radius

For the non-trivial problem $Z_m^s(E)$, we assume

$$\varphi = \min_{x \notin P^s(E)} \max_{x' \in P(x,E)} \frac{\gamma(x',x)}{||(||x'||_{p'}, ||x||_{p'})||_v},$$

$$\psi = \min_{x \notin P^s(E)} \max_{x' \in P(x,E)} \frac{\gamma(x',x)}{||x'-x||_1},$$

$$\gamma(x',x) = \min\{f_k(x',E_k) - f_k(x,E_k) : k \in N_s\},$$

$$P(x,E) = X(x,E) \cap P^s(E),$$

$$v = \min\{p', q'\},$$

$$\sigma = \min\{||E_k||_{pq} : k \in N_s\}.$$

It is easy to see that $\varphi, \psi \geq 0$.

Theorem 1. *For the stability radius $\rho_m^s(p,q,r)$ of non-trivial problem $Z_m^s(E)$, $s \geq 1$, the following estimates are true*

$$\varphi \leq \rho_m^s(p,q,r) \leq \min\{n^{1/p} m^{1/q} s^{1/r} \psi,\ \sigma\}, \tag{5}$$

for any $s, m \in \mathbf{N}$ and $p, q, r \in [1, \infty]$

Proof. At first we show that $\rho \geq \varphi$. With $\varphi = 0$ this inequality is obvious. Let $\varphi > 0$. We assume that the perturbing matrix $E' = ||e'_{ijk}|| \in \mathbf{R}^{m \times n \times s}$ with cut E'_k, $k \in N_s$, belongs to $\Omega(\varphi)$. According to the definition φ, for any portfolio $x \notin P^s(E)$ there exists portfolio $x^0 \in P(x,E)$ such that

$$\gamma(x^0, x) \geq \varphi ||(||x^0||_{p'}, ||x||_{p'})||_v,\ k \in N_s.$$

Hence, taking into account inequality (1) and (3), for any index $k \in N_s$ we deduce

$$f_k(x^0, E_k + E'_k) - f_k(x, E_k + E'_k) = \max_{i \in N_m}(e_{ik} + e'_{ik})x^0 - \max_{i \in N_m}(e_{ik} + e'_{ik})x =$$

$$= \min_{i \in N_m} \max_{i' \in N_m} (e_{i'k}x^0 + e'_{i'k}x^0 - e_{ik}x - e'_{ik}x) \geq$$

$$\geq f_k(x^0, E_k) - f_k(x, E_k) - ||E'||_{pqr} ||(||x^0||_{p'}, ||x||_{p'})||_v \geq$$

$$\geq (\varphi - ||E'||_{pqr}) ||(||x^0||_{p'}, ||x||_{p'})||_v > 0,$$

where e'_{ik} is the i-th row of the cut E'_k of matrix E'. Thus, portfolio x doesn't belong to the Pareto set $P^s(E+E')$. Therefore, we conclude that the following inclussion is correct

$$P^s(E+E') \subseteq P^s(E)$$

for any perturbing matrix $E' \in \Omega(\varphi)$.

Therefore, $\rho \geq \varphi$.

Next we prove the inequality $\rho \leq m^{1/p} n^{1/q} s^{1/r} \psi$.

According to the definition of the number ψ, there is a portfolio $x^0 \notin P^s(E)$, such that for any portfolio $x \in P(x^0, E)$ there exists index $l \in N_s$, whereby

$$f_l(x, E_l) - f_l(x^0, E_l) \leq \psi ||x - x^0||_1. \tag{6}$$

Assuming

$$\varepsilon > n^{1/p} m^{1/q} s^{1/r} \psi,$$

we define elements e^0_{ijk} for any k-th cut E^k_0, $k \in N_s$, of the perturbed matrix $E^0 = ||e^0_{ijk}|| \in \mathbf{R}^{m \times n \times s}$ according to the rule

$$e^0_{ijk} = \begin{cases} \delta, & \text{if } i \in N_m,\ x^0_j = 1, \\ -\delta, & \text{if } i \in N_m,\ x^0_j = 0, \end{cases}$$

where

$$\psi < \delta < \varepsilon / n^{1/p} m^{1/q} s^{1/r}. \tag{7}$$

Hence, according to (4), we obtain

$$||e^0_{ik}||_p = \delta n^{1/p},\ i \in N_m,\ k \in N_s,$$

$$||E^0_k||_{pq} = \delta n^{1/p} m^{1/q},\ k \in N_s,$$

$$||E^0||_{pqr} = \delta n^{1/p} m^{1/q} s^{1/r}.$$

It means that $E^0 \in \Omega(\varepsilon)$. In addition, all rows e^0_{ik}, $i \in N_m$, $k \in N_s$, are identical and they consist of component δ and $-\delta$. Therefore, if we set $c = e^0_{ik}$, $i \in N_m$, $k \in N_s$, we deduce that the correlation

$$c(x - x^0) = -\delta ||x - x^0||_1 < 0, \tag{8}$$

is valid for any portfolio $x \neq x^0$.

Hence, taking into account (6) and (7), we conclude that for any portfolio $x \in P(x^0, E)$ there exists an index $l \in N_s$ such that

$$f_l(x, E_l + E^0_l) - f_l(x^0, E_l + E^0_l) = \max_{i \in N_m} (e_{il} + e^0_{il}) x - \max_{i \in N_m} (e_{il} + e^0_{il}) x^0 =$$

$$= \min_{i \in N_m} \max_{i' \in N_m} (e_{i'l} x - e_{il} x^0 + e^0_{i'l} x - e^0_{il} x^0) =$$

$$= f_l(x, E_l) - f_l(x^0, E_l) + c(x - x^0) \leq (\psi - \delta) ||x - x^0||_1 < 0.$$

Thus, the following formula holds

$$\forall x \in P(x^0, E)\ (x \notin X(x^0, E + E^0)). \tag{9}$$

If $X(x^0, E+E^0) = \emptyset$, then it is obvious that portfolio x^0 (it is not Pareto optimal portfolio for problem $Z_m^s(E)$) appears to be optimal in the perturbed problem $Z_m^s(E+E^0)$, i.e. $x^0 \in P^s(E+E^0)$.

If $X(x^0, E+E^0) \neq \emptyset$, then due to the external stability of the Pareto set $P^s(E+E^0)$ (look, for example, [37] p. 34) we can find portfolio $x^* \in P^s(x^0, E+E^0)$. Let us show that $x^* \notin P^s(E)$.

Let assume, on the contrary, that $x^* \in P^s(E)$. Then according to (9), $x^* \notin P^s(x^0, E)$. This leads to two cases.

At first, let have a look at the case $f(x^*, E) = f(x^0, E)$. Then for any index $k \in N_s$ according to (8), we have

$$f_k(x^*, E_k + E_k^0) - f_k(x^0, E_k + E_k^0) = f_k(x^*, E_k) - f_k(x^0, E_k) + c(x^* - x^0) =$$

$$= -\delta||x^* - x^0||_1 < 0.$$

Next, let's examine the case where exists an index $h \in N_s$, that

$$f_h(x^*, E_h) < f_h(x^0, E_h).$$

Then, again using (8), we arrive at correlation

$$f_h(x^*, E_h + E_h^0) - f_h(x^0, E_h + E_h^0) = f_h(x^*, E^h) - f_h(x^0, E^h) + c(x^* - x^0) < 0.$$

As a result both cases contradict to the inclusion $x^* \in P^s(x^0, E+E^0)$. This proves that $x^* \notin P^s(E)$. Let us remind that $x^* \in P^s(E+E^0)$.

To summarize, we come to a conclusion that for any number $\varepsilon >$ $> n^{1/p} m^{1/q} s^{1/r} \psi$ the existence of the perturbing matrix is guaranteed $E^0 \in \Omega(\varepsilon)$, and there is a portfolio (x^0 or x^*), which is not Pareto optimal portfolio problem $Z_m^s(E)$, which is simultaneously the same in perturbed problem $Z_m^s(E+E^0)$. Thus, the following formula holds

$$\forall \varepsilon > n^{1/p} m^{1/q} s^{1/r} \psi \;\; \exists\, E^0 \in \Omega(\varepsilon) \;\; (P^s(E+E^0)) \nsubseteq P^s(E).$$

Therefore, $\rho \leq n^{1/p} m^{1/q} s^{1/r} \psi$.

Finally, we prove inequality

$$\rho_m^s(p, q, r) \leq \sigma.$$

Let x^0 be any non-Pareto optimal portfolio for problem $Z_m^s(E)$, i.e. $x^0 \notin P^s(C)$. Let index $l \in N_m$ be such that

$$\sigma = ||E_l||_{pq}. \tag{10}$$

Assuming $\varepsilon > \sigma$, let's choose a number δ with condition

$$0 < \delta n^{1/p} m^{1/q} < \varepsilon - \sigma. \tag{11}$$

Next, consider the matrix $V = [v_{ij}] \in \mathbf{R}^{m\times n}$ with elements

$$v_{ij} = \begin{cases} \delta, & \text{if } i \in N_m,\ x_j^0 = 1, \\ -\delta, & \text{if } i \in N_m,\ x_j^0 = 0. \end{cases}$$

Then, using (4) we obtain

$$||V||_{pq} = \delta n^{1/p} m^{1/q}. \tag{12}$$

Moreover, all the rows of matrix V are the same, and they consist of numbers δ and $-\delta$. Having denoted the line as A, we have

$$A(x - x^0) = -\delta||x - x^0||_1 < 0 \tag{13}$$

for any portfolio $x \in X\backslash\{x^0\}$. Cuts $E_k^0,\ k \in N_s$ perturbing matrix $E^0 \in \mathbf{R}^{m\times n\times s}$ let us define according to the rule

$$E_k^0 = \begin{cases} V - E_k, & \text{if } k = l, \\ \mathbf{0}, & \text{if } k \neq l, \end{cases}$$

where $\mathbf{0} \in \mathbf{R}^{m\times n}$. Then referring to (10) – (12) we find

$$||E^0||_{pqr} = ||E_l^0||_{pq} = ||V - E_l||_{pq} \leq ||V||_{pq} + ||E_l||_{pq} \leq \delta n^{1/p} m^{1/q} + \sigma < \varepsilon,$$

and taking into consideration the inequality (13), we obtain

$$f_l(x, E_l + E_l^0) - f_l(x^0, E_l + E_l^0) = f_l(x, V) - f_l(x^0, V) =$$

$$= A(x - x^0) = -\delta||x - x^0||_1 < 0.$$

It means that $x \notin X(x^0, E + E^0)$, where $E^0 \in \Omega(\varepsilon)$. But as $x^0 \notin X(x^0, E + E^0)$, then $X(x^0, E + E^0) = \emptyset$. Therefore

$$x^0 \in P_m^s(E + E^0).$$

Taking into account $x^0 \notin P^s(E)$, we conclude that $\rho_m^s(p, q, r) \leq \varepsilon$ for any number $\varepsilon > \sigma$. Therefore, $\rho_m^s(p, q, r) \leq \sigma$, together with previously proven inequality

$$\rho_m^s(p, q, r) \leq n^{1/p} m^{1/q} s^{1/r} \psi$$

it provides wanted upper estimate of the stability radius. Theorem 1 is proved.

5 Corollaries

Corollary 1. *For $s, m \in \mathbf{N}$ and $p \in [1, \infty]$ the inequalities hold*

$$\min_{x \notin P^s(E)} \max_{x' \in P(x,E)} \frac{\gamma(x, x')}{||x' + x||_1^{1/p'}} \leq \rho_m^s(p, p, p) \leq$$

$$\leq (nms)^{1/p} \min_{x \notin P^s(E)} \max_{x' \in P(x,E)} \frac{\gamma(x, x')}{||x' - x||_1}.$$

Hence, we obtain the following obvious statement indicating the assessment reachability

Corollary 2. *If for any two portfolios $x \notin P^s(E)$ and $x' \in P(x, E)$ of the problem $Z_m^s(E)$ the set*

$$\{j \in N_n : x_j = x'_j = 1\}$$

is empty, then the following formula holds

$$\rho_m^s(\infty, \infty, \infty) = \varphi = \psi.$$

Corollary 3[32]. *For $s, m \in \mathbf{N}$ and $p \in [1, \infty]$, the inequalities hold*

$$\min_{x \notin P^s(E)} \max_{x' \in P(x,E)} \frac{\gamma(x', x)}{||x'||_{p'} + ||x||_{p'}} \leq \rho_m^s(p, \infty, p) \leq$$

$$\leq (ns)^{1/p} \min_{x \notin P^s(E)} \max_{x' \in P(x,E)} \frac{\gamma(x', x)}{||x' - x||_1}.$$

Corollary 4[33]. *For $s, m \in \mathbf{N}$ and $p \in [1, \infty]$, the inequalities hold*

$$\min_{x \notin P^s(E)} \max_{x' \in P(x,E)} \frac{\gamma(x', x)}{||x' + x||_1} \leq \rho_m^s(\infty, p, p) \leq$$

$$\leq (ms)^{1/p} \min_{x \notin P^s(E)} \max_{x' \in P(x,E)} \frac{\gamma(x', x)}{||x' - x||_1}.$$

Corollary 5[34]. *For $s, m \in \mathbf{N}$ and $p \in [1, \infty]$, the inequalities hold*

$$\min_{x \notin P^s(E)} \max_{x' \in P(x,E)} \frac{\gamma(x', x)}{||x'||_{p'} + ||x||_{p'}} \leq \rho_m^s(p, \infty, \infty) \leq$$

$$\leq n^{1/p} \min_{x \notin P^s(E)} \max_{x' \in P(x,E)} \frac{\gamma(x', x)}{||x' - x||_1}.$$

6 The case of linear criteria

With $m = 1$ the investment problem is transformed into vector problem of linear Boolean programming. Let it be written in a convenient form

$$Z_1^s(E) : e_k x \to \max_{x \in X}, \ k \in N_s.$$

where $X \subset \mathbf{E}^n, \ e_k \in \mathbf{R}^n$ is the k-th row of the matrix $E = [e_{kj}] \in \mathbf{R}^{s \times n}$. The very case can be interpreted as a situation in which the state of the financial market doesn't raise doubts. As before, let us consider that in the space of projects $\mathbf{R}^n$ and a criterion space $\mathbf{R}^s$ set respectively the Hölder norm l_p и l_r, where $p, r \in [1, \infty]$. For the problem $Z_1^s(E)$, the same notation $P^s(E), P(x, E)$ and etc will be used.

In this linear case, a lower bound for the stability radius ρ_1^s of a problem $Z_1^s(E)$ can be improved. Indeed, the following theorem holds.

Theorem 2. *For the stability radius $\rho_1^s(p, r)$ of non-trivial problem $Z_1^s(E)$, the following bounds are valid*

$$\xi(p) \leq \rho_1^s(p, r) \leq n^{1/p} s^{1/r} \xi(\infty), \tag{14}$$

where

$$\xi(p) = \min_{x \notin P^s(E)} \ \max_{x' \in P(x,E)} \ \min_{k \in N_s} \frac{e_k(x' - x)}{||x' - x||_{p'}} \tag{15}$$

for any $s \in \mathbf{N}$ and $p, r \in [1, \infty]$.

Proof. The validity of the upper bound, i.e. the inequality

$$\rho_1^s(p, r) \leq n^{1/p} s^{1/r} \xi(\infty),$$

follows from Theorem 1. Next we prove the inequality

$$\rho_1^s(p, r) \geq \xi(p). \tag{16}$$

With $\xi(p) = 0$, it is obvious. Let $\xi(p) > 0$ and $E' \in \mathbf{R}^{s \times n}$ be the perturbing matrix with the rows $e'_k \in \mathbf{R}^n, \ k \in N_s$, and norm

$$||E'||_{pr} = ||(||e'_1||_p, ||e'_2||_p, ..., ||e'_s||_p)||_r < \xi(p).$$

According to (15) for every portfolio $x \notin P^s(E)$ there is a Pareto-optimal portfolio $x^0 \in P^s(E)$ such that

$$||e'_k||_p \leq ||E'||_{pr} < \xi(p) \leq \frac{e_k(x' - x)}{||x' - x||_{p'}}, \ k \in N_s.$$

Therefore, by Hölder's inequality (2) for any index $k \in N_s$ we have

$$(e_k + e'_k)(x^0 - x) = e_k(x^0 - x) + e'_k(x^0 - x) \geq e_k(x^0 - x) - ||e'_k||_p ||x^0 - x||_{p'} > 0,$$

i.e. $x \notin P^s(E+E')$. Hence, we come to a conclusion that $P^s(E+E') \subseteq P^s(E)$ under any perturbing matrix $E' \in \mathbf{R}^{s\times n}$ with the norm $||E'||_{pr} < \xi(p)$. Consequently, inequality (16) holds. Theorem 2 is proved.

Due to Theorem 2, we arrive at the known result, indicating the reachability of lower and upper bounds (14) for the stability radius in the problem $Z_1^s(E)$ with $p = r = \infty$.

Corollary 6[38]. *Given $s \in \mathbf{N}$, for the stability radius of non-trivial problem $Z_1^s(E)$, the next equality holds*

$$\rho_1^s(\infty,\infty) = \min_{x\notin P^s(E)} \max_{x'\in P(x,E)} \min_{k\in N_s} \frac{e_k(x'-x)}{||x'-x||_1}.$$

7 Conclusion

We considered a multicriteria discrete version of optimal investment problem. Here venturesome investor optimizes the efficiency of the portfolio under the assumption that the market is in revenue position. We obtained the lower and upper attainable bounds for the stability radius of a vector Boolean investment problem of portfolio optimization with the Hölder norms in the parameter spaces. Conducting similar kind of research for vector investment problems with non-Boolean decision variables could be an interesting direction for further investigations.

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