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Emelichev Vladimir | Nikulin Yury

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Emelichev Vladimir
Belarusian State University, Department of Mechanics and Mathematics,
Nezavisimisti 4, BLR-20030 Minsk, Belarus
vemelichev@gmail.com
Nikulin Yury
University of Turku, Department of Mathematics and Statistics,
Vesilinnantie 5, FIN-20100 Turku, Finland
yury.nikulin@utu.fi

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Abstract

We consider a multicriteria Boolean programming problem of finding the Pareto set. Partial criteria are given as quadratic functions, and they are exposed to independent perturbations. We study quantitative characteristic of stability (stability radius) of the problem. The lower and upper bounds on the stability radius are obtained in the situation where solution space and problem parameter space are endowed with various Hölder's norms.

Keywords: Boolean programming, quadratic problem, multicriteria optimization, Pareto set, stability radius, Hölder's norms

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1 Introduction

One of the most well-known approaches to multicriteria discrete otimization problem stability investigation is that focusing on finding quantitative bounds that characterize the level of stability. The so-called stability radius is a key concept that holds information about an extreme level of problem parameter perturbations leading to a situation where no new Pareto optima (efficient solutions) appear. Most of the results obtained in this direction specify stability radius analytical formulae or bounds for multicriteria problems of Boolean and integer programming with linear [1–7] and minmax (or maxmin) [8–14] criteria.

In this paper, we analyze stability of multicriteria variant of the well-known quadratic optimization problem with Boolean variables (see e.g. [15]). We obtain the lower and upper bounds on stability radius of the problem considered.

2 Problem formulation and basic definitions

Let $A = [a_{ijk}] \in \mathbf{R}^{n\times n\times m}$ be a matrix with corresponding cuts $A_k \in \mathbf{R}^{n\times n},\ k \in N_m = \{1, 2, \dots, m\}$. Let also $X \subseteq \mathbf{E}^n = \{0, 1\}^n,\ |X| > 1$, be a set of feasible solutions (Boolean vectors) $x = (x_1, x_2, \dots, x_n)^T$. We define a vector criterion

$$f(x, A) = (f_1(x, A_1), f_2(x, A_2), \dots, f_m(x, A_m)) \to \min_{x\in X},$$

with partial criteria being quadratic functions

$$f_k(x, A_k) = x^T A_k x,\ k \in N_m.$$

Denote

$$X(x, A) = \{x' \in X :\ g(x, x', A) \geq 0_{(m)}\ \&\ g(x, x', A) \neq 0_{(m)}\},$$

$$g(x, x', A) = (g_1(x, x', A_1), g_2(x, x', A_2), \dots, g_m(x, x', A_m)),$$

$$g_k(x, x', A_k) = f_k(x, A_k) - f_k(x', A_k) = (x - x')^T A_k(x - x'),$$

$$0_{(m)} = (0, 0, \dots, 0) \in \mathbf{R}^m.$$

Under m-criteria *quadratic problem* $Z_m(A)$ we understand the problem of finding the *Pareto set* (the set of *efficient* solutions)

$$P_m(A) = \{x \in X :\ X(x, A) = \emptyset\}.$$

The solutions which are not efficient are generally termed *inefficient*.

If $m = 1$, the multicriteria problem is transformed into scalar quadratic programming problem with Boolean variables which has lots of applications. The quadratic assignment problem and different optimization problems on graphs are represented in the scheme of the problem [15]. It has many applications in electronics design, partitioning problem, covering problem, packing problem etc. It also has application to statistical physics [16]. In [17], it was discussed how a molecular conformation problem can be formulated as the Boolean quadratic programming problem. In [18], an application of the problem to cellular radio channel assignment was mentioned.

It has been known for a long time that the Boolean quadratic problem is equivalent to the problem of finding a maximum cut in a graph. In [19] and [20], it was also shown that a number of graph problems (maximum clique, maximum vertex packing, minimum vertex cover (maximum independent set) maximum weight independent set can all be formulated as scalar Boolean quadratic problem. It has numerous applications in computer-aided design [21], capital budgeting and financial analysis [22, 23], traffic message management [24], and machine scheduling [25].

For example, in [23] a classical model of investment portfolio risk evaluation is formulated where one of the objective represents the risk measured by variance that lead us to quadratic programming. Contrary to classical Markowitz's model, our model operates with binary alternatives only, i.e. instead of investment proportions we are dealing with Boolean decision alternatives either to invesst a given asset or not.

In the solution space $\mathbf{R}^n$, we define an arbitrary *Hölder's norm* $l_p,\ p \in [1, \infty]$, i.e. under norm of vector $a = (a_1, a_2, \dots, a_n)^T \in \mathbf{R}^n$ we understand the number

$$\|a\|_p = \begin{cases} \Big(\sum\limits_{j \in N_n} |a_j|^p\Big)^{1/p} & \text{if } 1 \le p < \infty, \\ \\ \max\{|a_j| : \ j \in N_n\} & \text{if } p = \infty. \end{cases}$$

Thus, for any matrix $A_k \in \mathbf{R}^{n \times n}$, the norm of the matrix is defined as a norm of vector composed of all the matrix elements.

In the criterion space $\mathbf{R}^m$, we define another *Hölder's norm* $l_q,\ q \in [1, \infty]$, i.e. under norm of matrix $A \in \mathbf{R}^{n \times n \times m}$ we understand the number

$$\|A\|_{pq} = \|(\|A_1\|_p, \|A_2\|_p, \dots, \|A_m\|_p)\|_q,$$

It is easy to see that

$$\|A_k\|_p \le \|A\|_{pq},\ k \in N_m. \tag{1}$$

Let ζ be either p or q. It is well-known that l_ζ norm, defined in $\mathbf{R}^n$, induces conjugated l_{ζ^*} norm in $(\mathbf{R}^n)^*$. For ζ and ζ^*, the following relations hold

$$\frac{1}{\zeta} + \frac{1}{\zeta^*} = 1, \quad 1 < \zeta < \infty.$$

In addition, if $\zeta = 1$ then $\zeta^* = \infty$. Obviously, if $\zeta^* = 1$ then $\zeta = \infty$. Also notice that ζ and ζ^* belong to the same range $[1, \infty]$. We also set $\frac{1}{\zeta} = 0$ if $\zeta = \infty$.

For any two vectors a and b of the same dimension, the following *Hölder's inequalities* are well-known (see e.g. [26])

$$|a^T b| \le \|a\|_\zeta \|b\|_{\zeta^*}. \tag{2}$$

To any vector $x = (x_1, x_2, \dots, x_n)^T \in \mathbf{E}^n$ we assign a vector $\tilde{x}$ composed of all the possible products $x_i x_j$, i.e.

$$\tilde{x} = (x_{11}, x_{12}, \dots, x_{n-1n}, x_{nn})^T \in \mathbf{E}^{n^2},$$

where

$$x_{ij} = \begin{cases} 1 & \text{if } \ x_i x_j = 1, \\ 0 & \text{if } \ x_i x_j = 0. \end{cases}$$

Taking into account Hölder's inequalities (2), we can see that for any $x, x' \in \mathbf{E}^n$ and $k \in N_m$ the following inequalities hold

$$|f_k(x, A_k)| = |x^T A_k x| \leq \|A_k\|_p \|\tilde{x}\|_{p^*}, \tag{3}$$

$$|g_k(x, x', A_k)| \leq \|A_k\|_p \|\tilde{x} - \tilde{x}'\|_{p^*}. \tag{4}$$

Using the well-known condition (see [26]) that transforms (3) and (4) into equalities, the validity of the following statements becomes transparent

$$\forall\, x, x' \in \mathbf{R}^n \quad \forall\, \delta > 0 \quad \exists\, B \in \mathbf{R}^{n\times n}$$

$$\Big(\|B\|_p = \delta \ \ \& \ \ |(x - x')^T B (x - x')| = \delta \|\tilde{x} - \tilde{x}'\|_{p^*}\Big). \tag{5}$$

In addition it is easy to see that for any vector $a = (a_1, a_2, \ldots, a_n)^T \in \mathbf{R}^n$ with condition $|a_j| = \alpha$, $j \in N_n$ and any matrix $A_k = [a_{ijk}] \in \mathbf{R}^{n\times n}$ with condition $|a_{ijk}| = \alpha$, $(i, j) \in N_n \times N_n$, the following inequalities are valid

$$\|a\|_p = \alpha n^{\frac{1}{p}}, \tag{6}$$

$$\|A_k\|_p = \alpha n^{\frac{2}{p}}. \tag{7}$$

Given $\varepsilon > 0$, let

$$\Omega_{pq}(\varepsilon) = \Big\{A' \in \mathbf{R}^{n\times n\times m} : \ \ \|A'\|_{pq} < \varepsilon\Big\}$$

be the set of *perturbing matrices* A' with cuts $A'_k \in \mathbf{R}^{n\times n},\ k \in N_m$, and $\|A'\|_{pq}$ is the norm of $A' = [a'_{ijk}] \in \mathbf{R}^{n\times n\times m}$. Denote

$$\Xi_{pq} = \Big\{\varepsilon > 0 : \ \forall\, A' \in \Omega_{pq}(\varepsilon) \ \ (P_m(A + A') \subseteq P_m(A))\Big\}.$$

Following [2, 7, 13, 14], the number

$$\rho = \rho_m(p, q) = \begin{cases} \sup\ \Xi_{pq} & \text{if } \ \Xi_{pq} \neq \emptyset, \\ \\ 0 & \text{if } \ \Xi_{pq} = \emptyset. \end{cases}$$

is called the *stability radius* of problem $Z_m(A)$, $m \in \mathbf{N}$, with Hölder's norms l_p and l_q in the spaces $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively. Thus, the stability radius of problem $Z_m(A)$ defines the extreme level of independent perturbations of the elements of matrix A in the space $\mathbf{R}^{n\times n\times m}$ that do not lead to the situation where new Pareto optimal solutions appear.

It is evident that if $P_m(A) = X$, the inclusion

$$P_m(A + A') \subseteq P_m(A)$$

holds for any perturbing matrix $A' \in \Omega_{pq}(\varepsilon)$ with $\varepsilon > 0$. So, the stability radius is infinite when $P_m(A) = X$. The problem $Z_m(A)$ that satisfies $P_m(A) \neq X$ is called *non-trivial*.

3 Bounds on stability radius

Given $p, q \in [1, \infty]$, for non-trivial problem $Z_m(A)$, $m \in \mathbf{N}$, we set

$$\phi = \phi_m(p) = \min_{x \notin P_m(A)} \max_{x' \in P_m(x,A)} \min_{k \in N_m} \frac{g_k(x, x', A_k)}{\|\tilde{x} - \tilde{x}'\|_{p^*}},$$

$$\psi = \psi_m(p, q) = \min \left\{ n^{\frac{2}{p}} m^{\frac{1}{q}} \phi_m(\infty), \sigma_m(p) \right\},$$

where

$$P_m(x, A) = P_m(A) \cap X(x, A).$$

$$\sigma_m(p) = \min\{\|A_k\|_p : \ k \in N_m\}.$$

Theorem 1 *Given $p, q \in [1, \infty]$ and $m \in \mathbf{N}$, for the stability radius $\rho_m(p, q)$ of non-trivial problem $Z_m(A)$, the following lower and upper bounds are valid*

$$\phi_m(p) \leq \rho_m(p, q) \leq \psi_m(p, q).$$

Proof. First, we prove that $\rho \geq \phi$. If $\phi = 0$, then it is self-evident. Let $\phi > 0$, and let the perturbing matrix $A' \in \mathbf{R}^{n \times n \times m}$ with cuts A'_k, $k \in N_m$, belong to the set $\Omega_{pq}(\phi)$, i.e. $\|A'\|_{pq} < \phi$. According to (1) and the definition of the number ϕ, for any solution $x \notin P_m(A)$, there exists $x^0 \in P_m(x, A)$ such that

$$\frac{g_k(x, x^0, A_k)}{\|\tilde{x} - \tilde{x}^0\|_{p^*}} \geq \phi > \|A'\|_{pq} \geq \|A'_k\|_p \ , \ \ k \in N_m.$$

Therefore, by (4), we have

$$g_k(x, x^0, A_k + A'_k) = g_k(x, x^0, A_k) + g_k(x, x^0, A'_k) \geq$$

$$g_k(x, x^0, A_k) - \|A'_k\|_p \|\tilde{x} - \tilde{x}^0\|_{p^*} > 0, \ \ k \in N_m.$$

Thus, any solution that is not efficient in the problem $Z_m(A)$ stays inefficient in the problem $Z_m(A + A')$. So, we conclude that for any perturbing matrix $A' \in \Omega_{pq}(\phi)$ the inclusion holds

$$P_m(A + A') \subseteq P_m(A),$$

and hence $\rho \geq \phi$.

Further, we prove that

$$\rho \leq n^{\frac{2}{p}} m^{\frac{1}{q}} \phi_m(\infty). \tag{8}$$

According the definition of number $\phi_m(\infty)$, there exists a solution $x^0 \notin P_m(A)$ such that for any solution $x \in P_m(x^0, A)$ we can point out the index $s = s(x) \in N_m$ such that

$$g_s(x^0, x, A_s) \leq \phi_m(\infty) \|\tilde{x}^0 - \tilde{x}\|_1 > 0. \tag{9}$$

Setting $\varepsilon > n^{\frac{2}{p}} m^{\frac{1}{q}} \phi_m(\infty)$, we define the elements a^0_{ijk} of any cut A^0_k, $k \in N_m$, of the perturbing matrix A^0 according to the formula

$$a^0_{ijk} = \begin{cases} \alpha & \text{if } \ x^0_i x^0_j = 0, \ k \in N_m, \\ -\alpha & \text{if } \ x^0_i x^0_j = 1, \ k \in N_m, \end{cases}$$

where

$$\phi_m(\infty) < \alpha < \frac{\varepsilon}{n^{\frac{2}{p}} m^{\frac{1}{q}}}. \tag{10}$$

Then according to (6) and (7), we get

$$\|A_k^0\|_p = \alpha n^{\frac{2}{p}}, \ k \in N_m,$$

$$\|A^0\|_{pq} = \alpha n^{\frac{2}{p}} m^{\frac{1}{q}},$$

$$A^0 \in \Omega_{pq}(\varepsilon).$$

In addition, due to the construction of matrix A_k^0, for any solution $x \neq x^0$ and any $k \in N_m$ we have

$$g_k(x^0, x, A_k^0) = (x^0 - x)^T A_k^0 (x^0 - x) = \sum_{i \in N_n} \sum_{j \in N_n} a_{ijk}^0 (x_i^0 x_j^0 - x_i x_j) = -\alpha \|\tilde{x}^0 - \tilde{x}\|_1. \tag{11}$$

Using (9) and (10), we continue

$$g_s(x^0, x, A_s + A_s^0) = g_s(x^0, x, A_s) + g_s(x^0, x, A_s^0) \leq (\phi_m(\infty) - \alpha)\|\tilde{x}^0 - \tilde{x}\|_1 < 0.$$

So, we deduce

$$\forall\, x \in P_m(x^0, A) \ \ (x \notin X(x^0, A + A^0)). \tag{12}$$

Obviously, in the case $X(x^0, A + A^0) = \emptyset$, the solution x^0 is efficient in the perturbed problem $Z_m(A + A^0)$, i.e. $x^0 \in P_m(A + A^0)$. Now it is time to recall that $x^0 \notin P_m(A)$.

Further, we should prove $X(x^0, A + A^0) \neq \emptyset$. If so, then due to the outer stability of the Pareto set ([27], p. 34) there exists a solution $x^* \in P_m(x^0, A + A^0)$. Let us show that $x^* \notin P_m(A)$. We prove by contradiction. Suppose that $x^* \in P_m(A)$. Then by (12), we have

$$x^* \in P_m(A) \backslash P_m(x^0, A).$$

Then two cases are possible only.

Case 1. $f(x^*, A) = f(x^0, A)$. Then for any $k \in N_m$ equations (11) imply that

$$g_k(x^0, x^*, A_k + A_k^0) = g_k(x^0, x^*, A_k) + g_k(x^0, x^*, A_k^0) = -\alpha \|\tilde{x}^0 - \tilde{x}^*\|_1 < 0.$$

Case 2. There exists an index s such that $f_s(x^*, A_s) > f_s(x^0, A_s)$. Then using (11), we get

$$g_s(x^0, x^*, A_s + A_s^0) = g_s(x^0, x^*, A_s) - \alpha \|\tilde{x}^0 - \tilde{x}^*\|_1 < 0.$$

As a result, in both cases we get a contradiction with $x^* \in P_m(x^0, A + A^0)$.

Summarizing, we have just shown that for any $\varepsilon > n^{\frac{2}{p}} m^{\frac{1}{q}} \phi_m(\infty)$ we can guarantee the existence of the perturbing matrix $A^0 \in \Omega_{pq}(\varepsilon)$ and existence of the solution (x^0 or x^*) such that the solution is not efficient in the problem $Z_m(A)$ and efficient in the perturbed problem $Z_m(A + A^0)$. Thus the following statement is valid

$$\forall\, \varepsilon > n^{\frac{2}{p}} m^{\frac{1}{q}} \phi_m(\infty) \ \ \exists\, A^0 \in \Omega_{pq}(\varepsilon) \ \ (P_m(A + A^0) \not\subseteq P_m(A)).$$

Hence inequality (8) is true.

We are finally left with a need to demonstrate that $\rho \leq \sigma_m(p)$. To do that it is sufficient to show that for any index $k \in N_m$ we should have

$$\rho \leq \|A_k\|_p.$$

Let $x^0 = (x_1^0, x_2^0, \ldots, x_n^0)^T \notin P_m(A)$, Let fix an arbitrary index $s \in N_m$ and denote

$$\gamma_s = \|A_s\|_p \ . \tag{13}$$

Assuming $\varepsilon > \gamma_s$, we define a number δ such that

$$0 < \delta n^{\frac{2}{p}} < \varepsilon - \gamma_s. \tag{14}$$

Now consider an auxiliary matrix $U = U(x^0) = [u_{ij}] \in \mathbf{R}^{n\times n}$ with elements

$$u_{ij} = \begin{cases} \delta & \text{if } x_i^0 x_j^0 = 0, \\ -\delta & \text{if } x_i^0 x_j^0 = 1. \end{cases}$$

Using (7), we deduce

$$\|U\|_p = \delta n^{\frac{2}{p}}. \tag{15}$$

Besides that, for any solution $x \in X\backslash\{x^0\}$ the following is obvious

$$(x^0 - x)^T U(x^0 - x) = -\delta\|\tilde{x}^0 - \tilde{x}\|_1 < 0. \tag{16}$$

Let $A^0 \in \mathbf{R}^{n\times n\times m}$ be a perturbing matrix with cuts $A_k^0 \in \mathbf{R}^{n\times n}$, $k \in N_m$, defined as follows

$$A_k^0 = \begin{cases} U - A_k & \text{if } k = s, \\ 0^{(n\times n)} & \text{if } k \neq s, \end{cases}$$

where $0^{(n\times n)}$ is $(n\times n)$-matrix with all zero elements. Then according to (13)-(15), we get

$$\|A^0\|_{pq} = \|A_s^0\|_{pq} = \|U - A_s\|_p \leq \|U\|_p + \|A_s\|_p = \delta n^{\frac{2}{p}} + \gamma_s < \varepsilon,$$

i.e. $A^0 \in \Omega_{pq}(\varepsilon)$. In addition, due to (16), we get

$$g_s(x^0, x, A_s^0) = g_s(x^0, x, U - A_s) = -\delta\|\tilde{x}^0 - \tilde{x}\|_1 - g_s(x^0, x, A_s).$$

Therefore, for any solution $x \in X\backslash\{x^0\}$ we get

$$g_s(x^0, x, A_s + A_s^0) = -\delta\|\tilde{x}^0 - \tilde{x}\|_1 < 0.$$

This implies that for any solution $x \in X\backslash\{x^0\}$ we have $x \notin X(x^0, A + A^0)$. Since $x^0 \notin X(x^0, A + A^0)$, we have $X(x^0, A + A^0) = \emptyset$, i.e.

$$x^0 \in P_m(A + A^0).$$

Summarizing, for any $\varepsilon > \gamma_s$ we can guarantee the existence of the perturbing matrix $A^0 \in \Omega_{pq}(\varepsilon)$ such that the inefficient solution x^0 of $Z_m(A)$ ($x^0 \notin P_m(A)$)

becomes efficient in the perturbed problem $Z_m(A + A^0)$ ($x^0 \in P_m(A + A^0)$). Therefore, the following formula is valid

$$\forall\, \varepsilon > \gamma_s \;\; \exists\, A^0 \in \Omega_{pq}(\varepsilon) \;\; (P_m(A + A^0) \not\subseteq P_m(A)).$$

Hence $\rho \leq \gamma_s = \|A_s\|_p$ for any $s \in N_m$ (recall that s has been chosen arbitrary), i.e. $\rho \leq \sigma_m(p)$.

Thus, we have shown both $\rho \leq \sigma_m(p)$ and (8), so collecting all together we get the valid upper bound specified in the theorem

$$\rho_m(p, q) \leq \psi_m(p, q).$$

Finally, we have just shown the correctness of both the lower bound $\phi_m(p) \leq \rho_m(p, q)$ and the upper bound $\rho_m(p, q) \leq \psi_m(p, q)$. specified in the theorem for non-trivial problem $Z_m(A)$, $m \in \mathbf{N}$, $p, q \in [1, \infty]$. Thus, the theorem has been proven.

Since the equalities are evident

$$\|\tilde{x} - \tilde{x}'\|_1 = \|\tilde{x}\|_1 + \|\tilde{x}'\|_1 - 2(\tilde{x})^T\tilde{x}' = \|x\|_1^2 + \|x'\|_1^2 - 2(x^Tx')^2,$$

the following corollary is concluded directly from the theorem, and it illustrates attainability of the lower and upper bounds for $p = q = \infty$.

Corollary 1 *The stability radius $\rho_m(\infty, \infty)$ of non-trivial problem $Z_m(A)$, $m \in \mathbf{N}$, is expreseed by the following formula*

$$\rho_m(\infty, \infty) = \min_{x \notin P_m(A)} \max_{x' \in P_m(x,A)} \min_{k \in N_m} \frac{(x - x')^T A_k (x - x')}{\|x\|_1^2 + \|x'\|_1^2 - 2(x^Tx')^2}.$$

The next corollary implies that the lower bound for the stability radius specified in the theorem is also attainable in the case $|P_m(A)| = 1$.

Corollary 2 *Let problem $Z_m(A)$, $m \in \mathbf{N}$, have a unigue efficient solution x^0. Then for any $p, q \in [1, \infty]$ we have*

$$\rho_m(p, q) = \min_{x \in X \backslash \{x^0\}} \min_{k \in N_m} \frac{g_k(x, x^0, A_k)}{\|\tilde{x} - \tilde{x}^0\|_{p^*}}. \tag{17}$$

Proof. For the sake of brevity, we denote ξ the right-hand side of (17). Let $P_m(A) = \{x^0\}$. Then according to the definition of ξ there exists a solution $x^* \notin P_m(A)$ and an index $s \in N_m$ such that the following equality holds

$$\xi\|\tilde{x}^* - \tilde{x}^0\|_{p^*} = g_s(x^*, x^0, A_s), \tag{18}$$

with $\xi > 0$. Setting $\varepsilon > \xi$, we fix a the number δ that satisfies the condition

$$\xi < \delta < \varepsilon. \tag{19}$$

Due to (5), there exists a matrix $B \in \mathbf{R}^{n \times n}$ such that

$$\|B\|_p = \delta,$$

$$(x^* - x^0)^T B(x^* - x^0) = -\delta\|\tilde{x}^* - \tilde{x}^0\|_{p^*}.$$

Now we define the cuts A_k^0, $k \in N_m$, of the perturbing matrix $A^0 \in \mathbf{R}^{n\times n\times m}$ as follows

$$A_k^0 = \begin{cases} B & \text{if } k = s, \\ 0^{(n\times n)} & \text{if } k \neq s, \end{cases}$$

where $0^{(n\times n)}$ is $(n \times n)$-matrix with all zero elements. Then we get

$$\|A^0\|_{pq} = \|A_s^0\|_p = \|B\|_p = \delta,$$

$$g_s(x^*, x^0, A_s^0) = -\delta\|\tilde{x}^* - \tilde{x}^0\|_{p^*}.$$

Using (18) and (19), we deduce

$$g_s(x^*, x^0, A_s + A_s^0) = g_s(x^*, x^0, A_s) - \delta\|\tilde{x}^* - \tilde{x}^0\|_{p^*} = (\xi - \delta)\|\tilde{x}^* - \tilde{x}^0\|_{p^*} < 0.$$

This implies $x^0 \notin X(x^*, A+A^0)$. If $X(x^*, A+A^0) = \emptyset$, then $x^* \in P_m(A+A^0)$. Otherwise, due to the property of outer stability of the Pareto set (see again [27]), we can point out a solution $\hat{x} \in P_m(x^*, A + A^0)$ such that $\hat{x} \in P_m(A + A^0)$.

Summarizing, for any $\varepsilon > \xi$ we can guarantee the existence of the perturbing matrix $A^0 \in \Omega_{pq}(\varepsilon)$ such that there exists a solution $x' \in X\backslash\{x^0\}$ with the condition $x' \in P_m(A + A^0)$, i.e. $P_m(A + A^0) \not\subseteq P_m(A)$. This confirms that $\rho \leq \xi$. Since the problem $Z_m(A)$ is non-trivial ($|P_m(A)| = 1$), then due to the theorem, we get $\rho \geq \xi$. Since at the same time we have both $\rho \leq \xi$ and $\rho \geq \xi$, the formula (17) holds.

References

[1] Sergienko I.V., Shilo V.P. *Discrete optimization problems. Problems, methods of solutions, investigations*. Naukova dumka, Kiev, 1979.

[2] Emelichev V.A., Girlich E., Nikulin Y.V., Podkopaev D.P. *Stability and regularization of vector problems of integer linear programming*, Optimization, 2002, **51**, No.4, 645–676.

[3] Emelichev V.A., Kuzmin K.G. *On a type of stability of a multicriteria integer linear programming problem in the case of a monotone norm*, Journal of Computer and Systems Sciences International, 2007, **46**, No.5, 714–720.

[4] Libura M., Nikulin Y.V. *Stability and accuracy functions in multicriteria linear combinatorial optimization problems*, Annals of Operations Research, 2007, **147**, 255–267.

[5] Emelichev V.A., Podkopaev D.P. *Quantitative stability analysis for vector problems of 0-1 programming*, Discrete Optimization, 2010, **7**, No.1–2, 48–63.

[6] Emelichev V.A., Kuzmin K.G. *Stability radius of a vector integer linear programming problem: case of a regular norm in the space of criteria*, Cybernetics and Systems Analysis, 2010, **46**, No.1, 72–79.

[7] Emelichev V.A., Kuzmin K.G., Mychkov V.I. *Estimates of stability radius of multicriteria Boolean problem with H”lder metrics in parameter spaces*, Bulletin of the Academy of Sciences of Moldova. Mathematics, 2015, **78**, No.2, 74–81.

[8] Libura M., Nikulin Y.V. *Stability and accuracy functions in multicriteria combinatorial optimization problem with* Σ*-MINMAX and* Σ*-MINMIN partial criteria*, Control and Cybernetics, 2004, **33**, No.3, 511–524.

[9] Emelichev V.A., Korotkov V.V. *On stability radius of the multicriteria variant of Markowitz's investment portfolio problem*, Bulletin of the Academy of Sciences of Moldova. Mathematics, 2011, **65**, No.1, 83–94.

[10] Emelichev V.A., Korotkov V.V. *Stability radius of a vector investment problem with Savage's minimax risk criteria*, Cybernetics and Systems Analysis, 2012, **48**, No.3, 378–386.

[11] Emelichev V.A., Korotkov V.V. *On stability of a vector Boolean investment problem with Wald's criteria*, Discrete Math. Appl., 2012, **22**, No.4, 367–381.

[12] Emelichev V.A., Korotkov V.V. *On stability of multicriteria investment Boolean problem with Wald's efficiency criteria*, Bulletin of the Academy of Sciences of Moldova. Mathematics., 2014, **74**, No.1, 3–13.

[13] KOROTKOV V.V., NIKULIN Y.V., EMELICHEV V.A. *Stability of the bicriteria Boolean investment problem subject to extreme optimism and pessimism criteria*, Croatian Operation Research Review, 2015, **6**, No. 1, 195–205.

[14] BUKHTOYAROV S.E., EMELICHEV V.A. *On the stability measure of solutions to a vector version of an investment problem*, Journal of Applied and Industrial Mathematics, 2015, **9**, No. 3, 328–334.

[15] SHOR N.Z., STETSENKO S.I. *Quadratic extreme problems and nondifferential optimization*. Naukova dumka, Kiev, 1989.

[16] BARAHONA FR., GRÖTSCHEL M., JÜNGER M., REINELT G. *An application of combinatorial optimization to statistical physics and circuit layout design*, Operation Research, 1988, **36**, 491–513.

[17] PHILLIPS A.T., ROSEN J.S. *A quadratic assignment formulation of the molecular conformation problem*, Journal of Global Optimization, 1994, **4**, 229–241.

[18] CHARDAIRE P., SUTTER A. *A decomposition method for quadratic zero-one programming*, Management Science, 1995, **41**, 704 – 712.

[19] PARDALOS P.M., RODGERS G.P. *A branch and bound algorithm for the maximum clique problem*, Computers and Operation Research, 1992, **19**, 363–375.

[20] PARDALOS P.M., XUE J. *The maximum clique problem*, Journal of Global Optimization, 1994, **4**, 301–328.

[21] KRARUP J. PRUZAN P.M. *Computer-aided Layot Design*, Mathematical Programming Study, 1978, **9**, 75–94.

[22] LAUGHUNN D.J. *Quadratic Binary Programming*, Operation Research, 1970, **14**, 454–461.

[23] MARKOWITZ H.M. *Portfolio Selection: Efficient Diversification of Investments*. John Wiley & Sons, New York, 1959 (reprinted by Yale University Press, 1970).

[24] GALLO G., HAMMER P.L., SIMEONE B. *Quadratic Knapsack Problems*, Mathematical Programming, 1980, **12**, 132-149.

[25] AHMADIAN A., KOCHENBERGER B.G., ALIDAEE B. *Quadratic Programming approach for the optimal solution of two Scheduling problems*, International Journal of Systems Science, 1994, **25**, 401–408.

[26] HARDY G H., LITTLEWOOD J.E., POLYA G. *Inequalities (2nd ed.)*. Cambridge University Press, Cambridge, 1952.

[27] PODINOVSKIY V.V., NOGIN V.D. *Pareto optimal solutions of multicriteria problems*. Nauka, Moscow, 1982.