Emelichev Vladimir | Nikulin Yury

Stability of Extremum Solutions in Vector Quadratic Discrete Optimization

Turku Centre for Computer Science

TUCS Technical Report
No 1189, September 2017

Stability of Extremum Solutions in Vector Quadratic Discrete Optimization

Emelichev Vladimir
Belarusian State University, Department of Mechanics and Mathematics,
Nezavisimisti 4, BLR-20030 Minsk, Belarus
vemelichev@gmail.com
Nikulin Yury
University of Turku, Department of Mathematics and Statistics,
Vesilinnantie 5, FIN-20100 Turku, Finland
yury.nikulin@utu.fi

TUCS Technical Report
No 1189, September 2017

Abstract

We consider a wide class of quadratic optimization problems with Boolean variables. Such problems can be found in economics, planning, project management, artificial intelligence and computer-aided design. The problems are known to be NP-hard. In this paper, the lower and upper bounds on the stability radius of the set of extremum solutions are obtained in the situation where solution space and criterion space are endowed with various Hölder's norms.

Keywords: Boolean programming, quadratic problem, multicriteria optimization, extremum solutions, stability radius, Hölder's norms

TUCS Laboratory
Turku Optimization Group

Many problems from the area of finance, economy, project management and computer-aided design can be modelled as a quadratic optimization with either discrete or continuous variables, see e.g. [1,2]. We consider a quadratic problem with Boolean variables. It has been known for a long time that the Boolean quadratic problem is equivalent to the problem of finding a maximum cut in a graph. In [3] and [4], it was also shown that a number of graph problems (maximum clique, maximum vertex packing, minimum vertex cover (maximum independent set), maximum weight independent set) can all be formulated as scalar Boolean quadratic problem. Quadratic Boolean programming is also related to some problems where graph theory meets combinatorial optimization indirectly (see e.g. [5], [6])

Unconstrained quadratic Boolean programming problem, as well as their constrained counterparts, are generally belonging to the class of NP-hard problems [7], and considered as classic problems in combinatorial optimization, see, for example [8], for the characterization of the polytope of an unconstrained quadratic Boolean programming problem.

The current work is filled with new results specifying attainable bounds on stability radius for multicriteria quadratic Boolean programming problem of finding the extremum set in the situation where solution space and criterion space are endowed with various Hölder's norms. Notice that similar results were obtained in [9] for the multicriteria linear Boolean programming problem of finding the Pareto set.

1. Problem statement and main definitions

Let $A = [a_{ijk}]$ be a $n \times n \times m$-matrix with corresponding cuts $A_k \in \mathbf{R}^{n\times n},\ k \in N_m = \{1, 2, \ldots, m\},\ m \geq 1$. Let also $X \subseteq \mathbf{E}^n = \{0,1\}^n,\ |X| \geq 2$, be a set of feasible solutions (Boolean vectors) $x = (x_1, x_2, \ldots, x_n)^T$, $n \geq 2$.

We define a vector criterion

$$f(x, A) = (f_1(x, A_1), f_2(x, A_2), \ldots, f_m(x, A_m)) \to \min_{x\in X},$$

with partial criteria being quadratic functions

$$f_k(x, A_k) = x^T A_k x,\ k \in N_m.$$

In decision making theory, along with the well-known Pareto optimality principle (see e.g. [10]), various choice functions are considered [11–14]. In this paper, under m-criteria quadratic problem $Z_m(A)$ we understand the problem of finding the set of extremum solutions defined in traditional way (see e.g. [11–13]):

$$C_m(A) = \{x \in X :\ \exists s \in N_m\ \ \forall x' \in X\ \ (g_s(x, x', A_s) \leq 0)\},$$

where

$$g_s(x, x', A_s) = f_s(x, A_s) - f_s(x', A_s) = (x - x')^T A_s (x - x').$$

Thus, the choice of extremum solutions can be interpreted as finding best solutions for each of m criteria, and then combining them into one set. In other words, the set of extremum solutions contains all the individual minimizers of each objective. Obviously, $C_1(A),\ A \in \mathbf{R}^{n\times n}$ is the set of optimal solutions for scalar problem $Z_1(A)$ with $A \in \mathbf{R}^{n\times n}$.

Taking into account that X is finite, the following formulae below are true:

$$C_m(A) = S_m(A)\backslash(P_m(A)\backslash L_m(A)) = L_m(A) \cup (S_m(A)\backslash P_m(A)),$$

$$C_m(A) \cap P_m(A) = L_m(A),$$
$$L_m(A) \subseteq P_m(A) \subseteq S_m(A),$$
$$L_m(A) \subseteq C_m(A) \subseteq S_m(A),$$

where $P_m(A)$ denotes the Pareto set [15], $S_m(A)$ denotes the Slater set [16], and $L_m(A)$ denotes the lexicographic set (see e.g. [10, 17]). Below we define all the three sets in a traditional way see e.g. [9, 18]:

$$P_m(A) = \{x \in X :\ X(x, A) = \emptyset\},$$

$$S_m(A) = \{x \in X :\ \nexists x^0 \in X\ \ \forall k \in N_m\ \ (g_k(x, x^0, A_k) > 0)\},$$
$$L_m(A) = \bigcup_{\pi\in\Pi_m} L(A, \pi),\ L(A, \pi) = \{x \in X :\ \forall x' \in X\ \ (g(x, x', A) \leq_\pi 0_{(m)})\},$$
$$X(x, A) = \{x' \in X :\ g(x, x', A) \geq 0_{(m)}\ \&\ g(x, x', A) \neq 0_{(m)}\},$$
$$g(x, x', A) = (g_1(x, x', A_1), g_2(x, x', A_2), \ldots, g_m(x, x', A_m)),$$
$$0_{(m)} = (0, 0, \ldots, 0) \in \mathbf{R}^m.$$

Here Π_m is the set of all $m!$ permutations of numbers $1, 2, ..., m$; $\pi = (\pi_1, \pi_2, ..., \pi_m) \in \Pi_m$; and the binary relation of lexicographic order between two vectors $y = (y_1, y_2, \ldots, y_m) \in \mathbf{R}^m$ and $y' = (y'_1, y'_2, \ldots, y'_m) \in \mathbf{R}^m$ is defined as follows

$$g(y, y', A) \leq_\pi 0_{(m)} \Leftrightarrow \Big(y = y'\Big) \vee \Big(\exists k \in N_m\ \forall i \in N_{k-1}\ (y_{\pi_k} < y'_{\pi_k}\ \&\ y_{\pi_i} = y'_{\pi_i})\Big),$$

where $N_0 = \emptyset$. Obviously all the sets, $P_m(A), S_m(A), L_m(A)$ and $C_m(A)$, are non-empty for any matrix $A = [a_{ijk}] \in \mathbf{R}^{n\times n\times m}$ due to the finite number of alternatives in X.

We will perturb the elements of matrix $A \in \mathbf{R}^{n\times n\times m}$ by adding elements of the perturbing matrix $A' \in \mathbf{R}^{n\times n\times m}$. Thus the perturbed problem $Z_m(A + A')$ of finding extremum solutions has the following form

$$f(x, A + A') \to \min_{x\in X}.$$

In the solution space $\mathbf{R}^n$, we define an arbitrary Hölder's norm $l_p,\ p \in [1, \infty]$, i.e. under norm of vector $a = (a_1, a_2, \ldots, a_n)^T \in \mathbf{R}^n$ we understand the number

$$\|a\|_p = \begin{cases} \Big(\sum\limits_{j\in N_n} |a_j|^p\Big)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max\{|a_j| :\ j \in N_n\} & \text{if } p = \infty. \end{cases}$$

Thus, for any matrix $A_k \in \mathbf{R}^{n\times n}$, the norm of the matrix is defined as a norm of vector composed of all the matrix elements.

In the criterion space $\mathbf{R}^m$, we define another Hölder's norm $l_q,\ q \in [1,\infty]$, i.e. under norm of matrix $A \in \mathbf{R}^{n\times n\times m}$ we understand the number

$$\|A\|_{pq} = \|(\|A_1\|_p, \|A_2\|_p, \ldots, \|A_m\|_p)\|_q,$$

It is easy to see that

$$\|A_k\|_p \le \|A\|_{pq},\ k \in N_m. \tag{1}$$

Let ζ be either p or q. It is well-known that l_ζ norm, defined in $\mathbf{R}^n$, induces conjugated l_{ζ^*} norm in $(\mathbf{R}^n)^*$. For ζ and ζ^*, the following relations hold

$$\frac{1}{\zeta} + \frac{1}{\zeta^*} = 1, \quad 1 < \zeta < \infty.$$

In addition, if $\zeta = 1$ then $\zeta^* = \infty$. Obviously, if $\zeta^* = 1$ then $\zeta = \infty$. Also notice that ζ and ζ^* belong to the same range $[1,\infty]$. We also set $\frac{1}{\zeta} = 0$ if $\zeta = \infty$.

For any two vectors a and b of the same dimension, the following Hölder's inequalities are well-known (see e.g. [19])

$$|a^T b| \le \|a\|_\zeta \|b\|_{\zeta^*}. \tag{2}$$

To any vector $x = (x_1, x_2, \ldots, x_n)^T \in \mathbf{E}^n$, we assign a vector $\tilde{x}$ composed of all the possible products $x_i x_j$, i.e.

$$\tilde{x} = (x_{11}, x_{12}, \ldots, x_{nn-1}, x_{nn})^T \in \mathbf{E}^{n^2},$$

where

$$x_{ij} = \begin{cases} 1 & \text{if } x_i x_j = 1, \\ 0 & \text{if } x_i x_j = 0. \end{cases}$$

Taking into account Hölder's inequalities (2), we can see that for any $x, x' \in \mathbf{E}^n$ and $k \in N_m$ the following inequalities hold

$$|f_k(x, A_k)| = |x^T A_k x| = |A_k x x^T| \le \|A_k\|_p \|\tilde{x}\|_{p^*}, \tag{3}$$

$$|g_k(x, x', A_k)| \le \|A_k\|_p \|\tilde{x} - \tilde{x'}\|_{p^*}. \tag{4}$$

Using the well-known condition (see [19]) that transforms (3) and (4) into equalities, the validity of the following statements becomes transparent

$$\forall\, x, x' \in \mathbf{E}^n \quad \forall\, \delta > 0 \quad \exists\, B \in \mathbf{R}^{n\times n}$$

$$\Big(\|B\|_p = \delta \ \ \&\ \ |(x - x')^T B (x - x')| = \delta \|\tilde{x} - \tilde{x'}\|_{p^*}\Big). \tag{5}$$

In addition it is easy to see that for any vector $a = (a_1, a_2, \ldots, a_n)^T \in \mathbf{R}^n$ with condition $|a_j| = \alpha,\ j \in N_n$, and any matrix $A_k = [a_{ijk}] \in \mathbf{R}^{n\times n}$ with condition $|a_{ijk}| = \alpha$, $(i,j) \in N_n \times N_n$, the following inequalities are valid

$$\|a\|_p = \alpha n^{\frac{1}{p}}, \tag{6}$$

$$\|A_k\|_p = \alpha n^{\frac{2}{p}}. \tag{7}$$

Given $\varepsilon > 0$, let

$$\Omega_{pq}(\varepsilon) = \Big\{A' \in \mathbf{R}^{n\times n\times m} : \ \|A'\|_{pq} < \varepsilon\Big\}$$

be the set of perturbing matrices A' with cuts $A'_k \in \mathbf{R}^{n\times n},\ k \in N_m$, and $\|A'\|_{pq}$ is the norm of $A' = [a'_{ijk}] \in \mathbf{R}^{n\times n\times m}$. Denote

$$\Xi_{pq} = \Big\{\varepsilon > 0 : \ \forall\, A' \in \Omega_{pq}(\varepsilon)\ \ (C_m(A + A') \subseteq C_m(A))\Big\}.$$

Following [9, 20–22], the number

$$\rho_m(p, q) = \begin{cases} \sup\ \Xi_{pq} & \text{if } \ \Xi_{pq} \neq \emptyset, \\ 0 & \text{if } \ \Xi_{pq} = \emptyset \end{cases}$$

is called the stability radius of problem $Z_m(A)$, $m \in \mathbf{N}$, with Hölder's norms l_p and l_q in the spaces $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively. Thus, the stability radius of problem $Z_m(A)$ defines the extreme level of independent perturbations of the elements of matrix A in the space $\mathbf{R}^{n\times n\times m}$ that do not lead to the situation where new extremum solutions appear.

It is evident that if $C_m(A) = X$, the inclusion

$$C_m(A + A') \subseteq C_m(A)$$

holds for any perturbing matrix $A' \in \Omega_{pq}(\varepsilon)$ with $\varepsilon > 0$. So, the stability radius is infinite when $C_m(A) = X$. The problem $Z_m(A)$ that satisfies $C_m(A) \neq X$ is called non-trivial.

2. Main result

Given $p, q \in [1, \infty]$, for non-trivial problem $Z_m(A)$, $m \in \mathbf{N}$, we set

$$\phi_m(p) = \min_{k\in N_m}\ \min_{x\notin C_m(A)}\ \max_{x'\in X\setminus\{x\}} \frac{g_k(x, x', A_k)}{\|\tilde{x} - \tilde{x'}\|_{p^*}},$$

$$\psi_m(p, q) = \min\Big\{n^{\frac{2}{p}} m^{\frac{1}{q}} \phi_m(\infty), \gamma_m(p)\Big\},$$

where

$$\gamma_m(p) = \min\{\|A_k\|_p : \ k \in N_m\}.$$

Theorem. *Given $p, q \in [1, \infty]$ and $m \in \mathbf{N}$, for the stability radius $\rho_m(p, q)$ of non-trivial problem $Z_m(A)$, the following lower and upper bounds are valid*

$$0 < \phi_m(p) \leq \rho_m(p, q) \leq \psi_m(p, q).$$

Proof. Since the formula

$$\forall k \in N_m\ \ \forall x \notin C_m(A)\ \ \exists x^0 \in X\ \ (g_k(x, x^0, A_k) > 0),$$

is true, the inequality $\phi_m(p) > 0$ tells us that the lower bound on the stability radius as well as the stability radius itself are always positive.

First, we prove that $\rho_m(p,q) \geq \phi_m(p)$. Let $A' \in \Omega_{pq}(\phi_m(p))$ be a perturbing matrix with cuts $A_k \in \mathbf{R}^{n\times n}$, $k \in N_m$. Then according to the definition of the number $\phi_m(p)$, for any index $k \in N_m$ and any solution $x \notin C_m(A)$ there exists a solution $x^0 \in X\backslash\{x\}$ such that

$$\frac{g_k(x, x^0, A_k)}{\|\tilde{x} - \tilde{x^0}\|_{p^*}} \geq \phi_m(p) > \|A'\|_{pq} \geq \|A'_k\|_p,$$

due to (1). Therefore, using (4) we deduce

$$g_k(x, x^0, A_k + A'_k) = g_k(x, x^0, A_k) + g_k(x, x^0, A'_k) \geq$$

$$g_k(x, x^0, A_k) - \|A'_k\|_p \|\tilde{x} - \tilde{x^0}\|_{p^*} > 0, \;\; k \in N_m,$$

i.e. $x \notin C_m(A + A')$, Thus, any solution that is not extremum in the problem $Z_m(A)$ stays so in the problem $Z_m(A + A')$. So, we conclude that for any perturbing matrix $A' \in \Omega_{pq}(\phi_m(p))$ the inclusion holds $C_m(A + A') \subseteq C_m(A)$, and hence $\rho_m(p,q) \geq \phi_m(p)$.

Further, we prove that

$$\rho_m(p,q) \leq n^{\frac{2}{p}} m^{\frac{1}{q}} \phi_m(\infty). \tag{8}$$

According to the definition of number $\phi_m(\infty)$, there exist an index $s \in N_m$ and solution $x^0 = (x^0_1, x^0_2, \ldots, x^0_n) \notin C_m(A)$ such that for any solution $x \in X\backslash\{x^0\}$ we have

$$g_s(x^0, x, A_s) \leq \phi_m(\infty)\|\tilde{x^0} - \tilde{x}\|_1 > 0. \tag{9}$$

Setting $\varepsilon > n^{\frac{2}{p}} m^{\frac{1}{q}} \phi_m(\infty)$, we define the elements a^0_{ijk} of any cut A^0_k, $k \in N_m$, of the perturbing matrix A^0 according to the formula

$$a^0_{ijk} = \begin{cases} \delta & \text{if } \; x^0_i x^0_j = 0, \; k \in N_m, \\ -\delta & \text{if } \; x^0_i x^0_j = 1, \; k \in N_m, \end{cases}$$

where

$$\phi_m(\infty) < \delta < \frac{\varepsilon}{n^{\frac{2}{p}} m^{\frac{1}{q}}}. \tag{10}$$

Then according to (6) and (7), we get

$$\|A^0_k\|_p = \delta n^{\frac{2}{p}}, \; k \in N_m,$$

$$\|A^0\|_{pq} = \delta n^{\frac{2}{p}} m^{\frac{1}{q}},$$

$$A^0 \in \Omega_{pq}(\varepsilon).$$

In addition, due to the construction of matrix A^0_k, for any solution $x \neq x^0$ and any $k \in N_m$ we have

$$g_k(x^0, x, A^0_k) = (x^0 - x)^T A^0_k (x^0 - x) = \sum_{i\in N_n} \sum_{j \in N_n} a^0_{ijk}(x^0_i x^0_j - x_i x_j) = -\delta\|\tilde{x^0} - \tilde{x}\|_1. \tag{11}$$

Using (9) and (10), we continue

$$g_s(x^0, x, A_s + A_s^0) = g_s(x^0, x, A_s) + g_s(x^0, x, A_s^0) \leq (\phi_m(\infty) - \delta)\|\tilde{x^0} - \tilde{x}\|_1 < 0$$

i.e. $x^0 \in C_m(A + A^0)$.

Summarizing, for any $\varepsilon > n^{\frac{2}{p}} m^{\frac{1}{q}} \phi_m(\infty)$, we can guarantee the existence of the perturbing matrix $A^0 \in \Omega_{pq}(\varepsilon)$ such that the solution, which is not extremum in the original problem $Z_m(A)$, becomes an extremum in the perturbed problem $Z_m(A + A^0)$. Thus, the formula

$$\forall \varepsilon > n^{\frac{2}{p}} m^{\frac{1}{q}} \phi_m(\infty) \ \ \exists A^0 \in \Omega_{pq}(\varepsilon) \ \ (C_m(A + A^0) \not\subseteq C_m(A)).$$

So, inequality (8) holds.

We are finally left with a need to demonstrate that $\rho_m(p, q) \leq \gamma_m(p)$. To do that it is sufficient to show that for any index $s \in N_m$ we should have $\rho_m(p, q) \leq \|A_s\|_p$.

Let $x^0 = (x_1^0, x_2^0, \ldots, x_n^0)^T \notin C_m(A)$. Denote

$$\gamma_s = \|A_s\|_p. \tag{12}$$

Assuming $\varepsilon > \gamma_s$, we define a number δ such that

$$0 < \delta n^{\frac{2}{p}} < \varepsilon - \gamma_s. \tag{13}$$

Now consider an auxiliary matrix $U = U(x^0) = [u_{ij}] \in \mathbf{R}^{n \times n}$ with elements

$$u_{ij} = \begin{cases} \delta & \text{if } \ x_i^0 x_j^0 = 0, \\ -\delta & \text{if } \ x_i^0 x_j^0 = 1. \end{cases}$$

Using (7), we deduce

$$\|U\|_p = \delta n^{\frac{2}{p}}. \tag{14}$$

Besides that, for any solution $x \in X \backslash \{x^0\}$ the following is obvious (c.f. (11))

$$(x^0 - x)^T U (x^0 - x) = -\delta \|\tilde{x^0} - \tilde{x}\|_1 < 0. \tag{15}$$

Let $A^0 \in \mathbf{R}^{n \times n \times m}$ be a perturbing matrix with cuts $A_k^0 \in \mathbf{R}^{n \times n}$, $k \in N_m$, defined as follows

$$A_k^0 = \begin{cases} U - A_k & \text{if } \ k = s, \\ 0^{(n \times n)} & \text{if } \ k \neq s, \end{cases}$$

where $0^{(n \times n)}$ is $(n \times n)$-matrix with all zero elements. Then according to (12)-(14), we get

$$\|A^0\|_{pq} = \|A_s^0\|_p = \|U - A_s\|_p \leq \|U\|_p + \|A_s\|_p = \delta n^{\frac{2}{p}} + \gamma_s < \varepsilon,$$

i.e. $A^0 \in \Omega_{pq}(\varepsilon)$.

In addition, due to (15), we get

$$g_s(x^0, x, A_s^0) = (x^0 - x)^T U (x^0 - x) - g_s(x^0, x, A_s) = -\delta \|\tilde{x^0} - \tilde{x}\|_1 - g_s(x^0, x, A_s).$$

Therefore, for every solution $x \in X\backslash\{x^0\}$ we get

$$g_s(x^0, x, A_s + A_s^0) = -\delta\|\tilde{x^0} - \tilde{x}\|_1 < 0,$$

i.e. $x^0 \in C_m(A + A^0)$.

Summarizing, for any $\varepsilon > \gamma_s$ we can guarantee the existence of the perturbing matrix $A^0 \in \Omega_{pq}(\varepsilon)$ such that the solution x^0, which is not an extremum in $Z_m(A)$ ($x^0 \notin C_m(A)$), becomes an extremum in the perturbed problem $Z_m(A + A^0)$ ($x^0 \in C_m(A + A^0)$). Therefore, the following formula is valid

$$\forall\, \varepsilon > \gamma_s \;\; \exists\, A^0 \in \Omega_{pq}(\varepsilon) \;\; (C_m(A + A^0) \nsubseteq C_m(A)).$$

So, $\rho_m(p, q) < \varepsilon$ for any $\varepsilon > \gamma_s$, $s \in N_m$. Thus, $\rho_m(p, q) \leq \gamma_m(p) = \min\{\|A_s\| : s \in N_m\}$.

The Theorem has been proven.

3. Corollaries

Since the equalities

$$\|\tilde{x} - \tilde{x'}\|_1 = \|\tilde{x}\|_1 + \|\tilde{x'}\|_1 - 2(\tilde{x})^T\tilde{x'} = \|x\|_1^2 + \|x'\|_1^2 - 2(x^T x')^2,$$

are evident the following corollary is concluded directly from the Theorem, and it illustrates attainability of the lower and upper bounds for $p = q = \infty$.

Corollary 1. *The stability radius $\rho_m(\infty, \infty)$ of non-trivial problem $Z_m(A),\ m \in$ **N**, is expressed by the following formula*

$$\rho_m(\infty, \infty) = \phi_m(\infty) = \psi_m(\infty, \infty) = \min_{k \in N_m} \min_{x \notin C_m(A)} \max_{x' \in X\backslash\{x\}} \frac{(x - x')^T A_k (x - x')}{\|x\|_1^2 + \|x'\|_1^2 - 2(x^T x')^2}.$$

The next formula is a particular case of Corollary 1 for the scalar problem $Z_1(A)$, $A \in \mathbf{R}^{n \times n}$.

$$\rho_m(\infty, \infty) = \phi_m(\infty) = \psi_m(\infty, \infty) = \min_{x \notin C_1(A)} \max_{x' \in C_1(A)} \frac{(x - x')^T A_k (x - x')}{\|x\|_1^2 + \|x'\|_1^2 - 2(x^T x')^2}.$$

The next corollary implies that the lower bound for the stability radius specified in the Theorem is also attainable.

Corollary 2. For any given $p, q \in [1, \infty]$ and $m \in \mathbf{N}$, there exists a class of non-trivial problems $Z^m(A)$ such that the stability radius of a problem of the class is expressed by formula

$$\rho_m(p, q) = \phi_m(p).$$

Proof. Due to the Theorem, it suffices to find a class of problems $Z_m(A)$ with $\rho_m(p, q) \leq \phi_m(p)$.

Let $X = \{x^0, x^*\} \subset \mathbf{E}^n$ and $C_m(A) = \{x^0\}$, then $x^* \notin C_m(A)$. According to the definition of the number $\phi_m(p)$, there exists an index $s \in N_m$ such that

$$\phi_m(p)\|\tilde{x^*} - \tilde{x^0}\|_{p^*} = g_s(x^*, x^0, A_s), \tag{16}$$

with $\phi_m(p) > 0$. Setting $\varepsilon > \phi_m(p)$, we fix a the number δ that satisfies the condition

$$\phi_m(p) < \delta < \varepsilon. \tag{17}$$

Due to (5), there exists a matrix $B \in \mathbf{R}^{n\times n}$ such that

$$\|B\|_p = \delta,$$

$$(x^* - x^0)^T B(x^* - x^0) = -\delta \|\tilde{x^*} - \tilde{x^0}\|_{p^*}.$$

Now we define the cuts A_k^0, $k \in N_m$, of the perturbing matrix $A^0 \in \mathbf{R}^{n\times n\times m}$ as follows

$$A_k^0 = \begin{cases} B & \text{if } k = s, \\ 0^{(n\times n)} & \text{if } k \in N_m \backslash \{s\}, \end{cases}$$

where where $0^{(n\times n)}$ is $(n \times n)$-matrix with all zero elements. Then we get

$$g_s(x^*, x^0, A_s^0) = -\delta \|\tilde{x^*} - \tilde{x^0}\|_{p^*}.$$

$$\|A^0\|_{pq} = \|A_s^0\|_p = \|B\|_p = \delta,$$

$$A^0 \in \Omega_{pq}(\varepsilon).$$

Using (16) and (17), we deduce

$$g_s(x^*, x^0, A_s + A_s^0) = g_s(x^*, x^0, A_s) - \delta \|\tilde{x^*} - \tilde{x^0}\|_{p^*} = (\phi_m(p) - \delta)\|\tilde{x^*} - \tilde{x^0}\|_{p^*} < 0.$$

This implies $x^* \in C_m(A + A^0)$. Summarizing, for any $\varepsilon > \phi_m(p)$ we can guarantee the existence of the perturbing matrix $A^0 \in \Omega_{pq}(\varepsilon)$ such that $x^* \notin C_m(A)$ and $x^* \in C_m(A + A^0)$, i.e. the inclusion $C_m(A) \supseteq C_m(A + A^0)$ is not valid. This confirms that $\rho_m(p, q) \leq \varepsilon$ for any $\varepsilon > \phi_m(p)$, and hence $\rho_m(p, q) \leq \phi_m(p)$, Corollary 2 has been proven.

At the end we mention that fact that the results of the Theorem and Corollary 1 could also be formulated for $X \subseteq \mathbf{Z}^n$, but Corollary 2 stays true only for $X \subseteq \mathbf{E}^n$.

References

[1] SHOR N.Z., STETSENKO S.I. *Quadratic extreme problems and non-differential optimization*. Naukova dumka, Kiev, 1989 (in Russian).

[2] KRARUP J. PRUZAN P.M. *Computer-aided Layot Design*, Mathematical Programming Study, 1978, **9**, 75 – 94.

[3] PARDALOS P.M., RODGERS G.P. *A branch and bound algorithm for the maximum clique problem*, Computers and Operation Research, 1992, **19**, 363 – 375.

[4] PARDALOS P.M., XUE J. *The maximum clique problem*, Journal of Global Optimization, 1994, **4**, 301 – 328.

[5] HAMMER P.L. *Pseudo-Boolean Remarks on Balanced Graphs*, International Series of Numerical Mathematics, 1977, **36**, 69 – 78.

[6] EBENEGGER CH., HAMMER P. L., DE WERRA, D. *Pseudo-Boolean functions and stability of graphs. Algebraic and combinatorial methods in operations research*, North-Holland Math. Stud., 1984, **95**, 83 – 97.

[7] GAREY M.R., JOHNSON D.S. *Computers and Intractability: A Guide to the Theory of NP-Completeness. A Series of Books in the Mathematical Sciences*. Calif.: W. H. Freeman and Co., San Francisco, 1979.

[8] PADBERG M. *The boolean quadric polytope - some characteristics, facets and relatives*, Mathematical Programming, 1989, **45**, 139 – 172.

[9] EMELICHEV V.A., KUZMIN K.G., MYCHKOV V.I. *Estimates of stability radius of multicriteria Boolean problem with Hölder metrics in parameter spaces*, Bulletin of the Academy of Sciences of Moldova. Mathematics, 2015, **78**, No.2, 74 – 81.

[10] PODINOVSKI V.V., NOGIN V.D. *Pareto-optimal solutions of multicriteria problems*. FizMatLit, Moscow, 2007 (in Russian).

[11] MOLODTSOV D.A. *Stability of optimality principle*. Nauka, Moscow, 1987 (in Russian).

[12] SHOLOMOV L.A. *Logical methods for the investigation of discrete choice models*. Nauka, Moscow, 1989 (in Russian).

[13] YUDIN D.B. *Computational methods in decision making*. Nauka, Moscow, 1989 (in Russian).

[14] AIZERMAN M.A., ALEKSEROV F.T. *Choice of alternatives: theoretical foundations*. Nauka, Moscow, 1990 (in Russian).

[15] PARETO V. *Manuel d'ecoonomie politique*. Qiard.–Paris. 1909.

[16] SLATER M. *Lagrange Multipliers Revisited: A Contribution to Nonlinear Programming*, Cowles Commission Discussion Paper 80, Mathematics 403, Yale University, 1950.

[17] STEUER R.E. *Multiple Criteria Optimization: Theory, Computation and Application*. John Wiley, New York, 1986.

[18] EMELICHEV V.A., GIRLICH E., NIKULIN Y.V., PODKOPAEV D.P. *Stability and regularization of vector problems of integer linear programming*, Optimization, 2002, **51**, No.4, 645 – 676.

[19] HARDY G.H., LITTLEWOOD J.E., POLYA G. *Inequalities (2nd ed.)*. Cambridge University Press, Cambridge, 1952.

[20] EMELICHEV V.A., KUZMIN K.G. *Stability radius of a vector integer linear programming problem: case of a regular norm in the space of criteria*, Cybernetics and Systems Analysis, 2010, **46**, No.1, 72 – 79.

[21] KOROTKOV V.V., NIKULIN Y.V., EMELICHEV V.A. *Stability of the bicriteria Boolean investment problem subject to extreme optimism and pessimism criteria*, Croatian Operational Research Review, 2015, **6**, No.1, 195 – 205.

[22] EMELICHEV V.A., KOROTKOV V.V. *On stability radius of the multicriteria variant of Markowitz's investment portfolio problem*, Bulletin of the Academy of Sciences of Moldova. Mathematics, 2011, **65**, No.1, 83 – 94.

TURKU
CENTRE *for*
COMPUTER
SCIENCE

Joukahaisenkatu 3-5 A, 20520 TURKU, Finland | www.tucs.fi

University of Turku
Faculty of Mathematics and Natural Sciences
- Department of Information Technology
- Department of Mathematics and Statistics

Turku School of Economics
- Institute of Information Systems Sciences

Åbo Akademi University
- Computer Science
- Computer Engineering

ISBN 978-952-12-3601-3
ISSN 1239-1891