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On the priority vector associated with a fuzzy preference relation and a multiplicative preference relation

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Abstract

We propose two straightforward methods for deriving the priority vector associated with a fuzzy preference relation. Then, using transformations between multiplicative preference relations and fuzzy preference relations, we study the relationships between the priority vectors associated with these two types of preference relations.

Keywords: pairwise comparison matrix, fuzzy preference relation, priority vector.

JEL classification: C02
AMS classification: 91B08

Introduction

The two most popular ways for eliciting the expert's preferences by pairwise comparisons between alternatives are *multiplicative preference relations* and *fuzzy preference relations*. Multiplicative preference relations have been widely used in many well-known decision making approaches, such as, for example, Saaty's Analytic Hierarchy Process (AHP) [16]. Fuzzy preference relations have been first introduced in fuzzy sets theory as an extension of crisp (ordinal) preference relations, in order to provide a more flexible tool to represent expert's preferences [2, 12, 15]. Later, they have been widely used in decision processes as a cardinal representation of preferences equivalent to, and interchangeable with, multiplicative preference relations. A large number of methods for deriving weights have been proposed in the framework of multiplicative preference relations. Two well-known examples are Saaty's eigenvector method [16] and the geometric mean method [4]. Other methods are based on some optimization models and a comparative study is [3]. Many methods have also been proposed for deriving weights from fuzzy preference relations and some of them will be briefly recalled in section 2. The aim of this paper is to propose two straightforward methods for deriving the weight vector associated with a fuzzy preference relation and to study the relationship between the weight vectors associated with multiplicative and fuzzy preference relations. Some very simple transformations between the different types of weights are derived and discussed in section 2.2.

1 Multiplicative and Fuzzy Preference Relations

We assume that the reader is familiar with multiplicative and fuzzy preference relations, so that we only recall the main ideas. Let $\Lambda = \{A_1, \ldots, A_n\}$ be a set of alternatives. A multiplicative preference relation, *MPR* in the following, is represented by a matrix $A = (a_{ij})_{n\times n}$ whose entries a_{ij} estimate the ratios w_i/w_j between the preference intensities (weights) of alternatives A_i and A_j. Saaty's ratio scale is used, $a_{ij} \in \{\frac{1}{9}, \frac{1}{8}, \ldots, \frac{1}{2}, 1, 2, \ldots, 8, 9\}$ and multiplicative reciprocity is assumed, $a_{ij}a_{ji} = 1,\ \forall i, j$. If the following multiplicative transitivity condition

$$a_{ij} = a_{ik}a_{kj} \quad i, j, k = 1, \ldots, n \tag{1}$$

is satisfied, A is called *consistent*. If $A = (a_{ij})$ is consistent, then a positive vector $w = (w_1, \ldots, w_n)$ exists such that

$$a_{ij} = w_i/w_j \quad i, j = 1, \ldots, n\ . \tag{2}$$

A fuzzy preference relation, *FPR* in the following, is a nonnegative relation $R : \Lambda \times \Lambda \to [0, 1]$ represented by a matrix $R = (r_{ij})_{n\times n}$, where $r_{ij} := R(A_i, A_j)$. Additive reciprocity is assumed, $r_{ij} + r_{ji} = 1\ \forall i, j$. Analogously to the *MPR*, if the following transitivity condition

$$(r_{ij} - 0.5) = (r_{ik} - 0.5) + (r_{kj} - 0.5) \quad i, j, k = 1, \ldots, n. \tag{3}$$

is satisfied, R is called *additively* consistent. Tanino [18] proves that $R = (r_{ij})$ is additively consistent, i.e. (3) holds, if and only if a nonnegative vector

$u = (u_1, \ldots, u_n)$ exists with $|u_i - u_j| \leq 1$ such that

$$r_{ij} = 0.5 + 0.5(u_i - u_j) \quad i, j = 1, \ldots, n \ . \tag{4}$$

Components u_i are unique up to addition of a constant. Tanino [18] also states an alternative kind of consistency for *FPR* which is called *multiplicative.* A *FPR* $R = (r_{ij})$ with $r_{ij} \neq 0$ is multiplicatively consistent if and only if the following condition of transitivity holds

$$\frac{r_{ik}}{r_{ki}} = \frac{r_{ij} r_{jk}}{r_{kj} r_{ji}} \quad i, j, k = 1, \ldots, n \ . \tag{5}$$

If (5) holds, then a positive vector $v = (v_1, \ldots, v_n)$ exists such that

$$r_{ij} = \frac{v_i}{v_i + v_j} \quad i, j = 1, \ldots, n. \tag{6}$$

Components v_i are unique up to multiplication by a positive constant. Let's also note that property (5) formally corresponds to a property already introduced in [17].

Throughout the paper we will indicate by $A = (a_{ij})$ a multiplicative preference relation and by $R = (r_{ij})$ a fuzzy preference relation, without distinguishing between a preference relation and the corresponding matrix. Moreover, although it should be already clear because of our premises, we want to specify that we consider the range of possible values for a_{ij} and r_{ij} to be bounded while some other authors prefer to deal with open intervals. It is interesting to observe that each *MPR* $A = (a_{ij})$ can be transformed into a *FPR* using the following function $g : [\frac{1}{9}; 9] \rightarrow [0; 1]$ introduced by Fedrizzi [8],

$$r_{ij} = g(a_{ij}) = \frac{1}{2}(1 + \log_9 a_{ij}) \ . \tag{7}$$

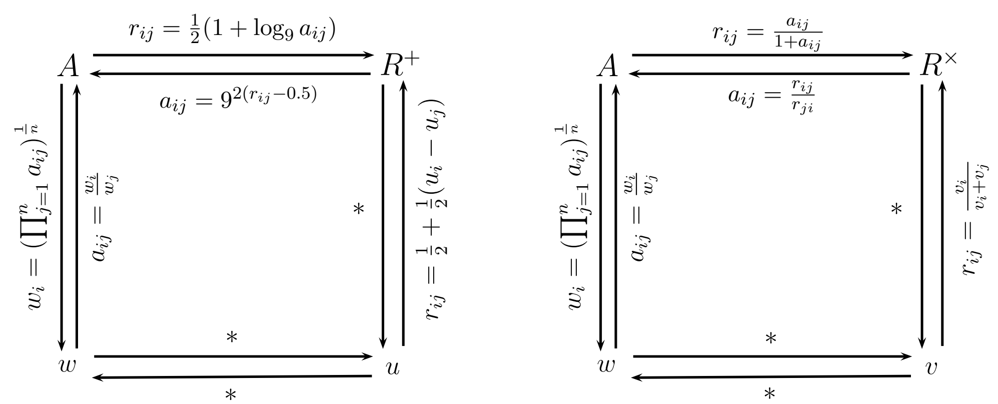
Function (7) transforms the a_{ij} values into the r_{ij} values in such a way that all the relevant properties of $A = (a_{ij})$ are transformed into the corresponding properties for $R = (r_{ij})$. In particular, multiplicative reciprocity is transformed into additive reciprocity and multiplicative consistency (1) is transformed into additive consistency (3). Clearly, the inverse function g^{-1} transforms r_{ij} into a_{ij} with the corresponding properties.

Furthermore, *MPR*s are also related with *FPR*s by the following transformation $f : [\frac{1}{9}; 9] \rightarrow [\frac{1}{10}; \frac{9}{10}]$,

$$r_{ij} = f(a_{ij}) = \frac{a_{ij}}{a_{ij} + 1} \ . \tag{8}$$

Function (8) plays the same role of (7), but it transforms (1) into (5) instead of (3). We will refer to functions (7) and (8) to state the results presented in subsection 2.2. Moreover, we will denote a weight vector by w, u and v referring to (2), (4) and (6) respectively. Finally we observe that a *MPR* is also called pairwise comparison matrix.

To clarify what has already been stated in literature (and simply recalled so far), Figure 1 may be of great help. Additively consistent *FPR*s are denoted by R^+ and they are illustrated in diagram (a). Conversely, multiplicatively consistent *FPR*s are denoted by $R^\times$ and they are exposed in (b). The symbol $*$ indicates that the relation at issue has not been defined in literature yet.



(a) Matrices A, R^+ and corresponding vectors w and u

(b) Matrices A, $R^\times$ and corresponding vectors w and v

Figure 1: Already known transformations

Finally, we observe that given $r_{ik}, r_{kj} \in [0, 1]$, there may not exist $r_{ij} \in [0, 1]$ such that (3) is satisfied. The same result holds for (5) and analogously for (1) referring to *MPR*s. These boundary problems for consistency are well known and unavoidable when using bounded scales. Clearly the problem does not exist when an open scale is used [1] [9]. On the other hand, despite the elegant mathematical results, every unbounded scale yields serious drawbacks in practical applications.

2 Priority Vectors

Among the large number of methods for deriving weights from *MPR*s, Saaty's eigenvector method [16] and geometric mean method [4] are, as mentioned before, the two best known. Saaty's method suggests to choose, as weight vector w, the normalized principal eigenvector of *MPR* A. On the other hand, according to the geometric mean method, the weights w_i are derived from A by means of

$$w_i = \left(\prod_{k=1}^{n} a_{ik} \right)^{\frac{1}{n}} . \tag{9}$$

Both methods, if applied to a consistent *MPR*, give a vector w satisfying (2). For what concerns *FPR*s, we cite, as examples, the approaches of Fan et al [5], Fan et al. [6], Fan et al. [7], Gong [11], Lipovetsky and Conklin [13], Xu [19], Xu [20], Xu and Da [21], Wang and Fan [22], Wang et al. [23], Wang and Parkan [24]. Some of the methods mentioned above share one of the following two desirable properties: (i) the weight vector u calculated from an additively consistent *FPR* satisfies (4) ; (ii) the weight vector v calculated from a multiplicatively consistent *FPR* satisfies (6). In spite of the large number of proposed methods, they still remain rather complex to be implemented and there is not one method which leads to such vectors u and v, respectively, with a simple formula. Their complexity is sometimes justified by the fact that some of the proposals can be applied to some special cases, e.g. group decisions and incomplete information. Nevertheless, when the single decision maker deals with a complete *FPR* this complexity does not seem to be justified and that is why we aim at finding a simpler approach.

2.1 New methods

As mentioned above, given a consistent *MPR*, the weight vector calculated by the geometric mean method (9) satisfies characterization (2). With the following proposition we give a simple expression of the weight vector u that satisfies the corresponding property (4) for a consistent *FPR*.

Proposition 1. *Given an additively consistent FPR $R^+ = (r_{ij})$, i.e. satisfying (3), the weight vector $u = (u_1 \ldots, u_n)$ defined by*

$$u_i = \frac{2}{n} \sum_{k=1}^{n} r_{ik} \tag{10}$$

is the unique vector, up to an additive constant, that satisfies Tanino's characterization (4).

Proof. By substituting (10) in (4), it is

$$0.5 + 0.5\left(\frac{2}{n}\sum_{k=1}^{n} r_{ik} - \frac{2}{n}\sum_{k=1}^{n} r_{jk}\right) =$$
$$= 0.5 + \frac{1}{n}\left(\sum_{k=1}^{n} r_{ik} - \sum_{k=1}^{n} r_{jk}\right) =$$
$$= 0.5 + \frac{1}{n}\sum_{k=1}^{n}(r_{ik} - r_{jk}).$$

From additive consistency condition (3), it is $(r_{ik} - r_{jk}) = (r_{ij} - 0.5)$. Then,

$$0.5 + \frac{1}{n}\sum_{k=1}^{n}(r_{ij} - 0.5) = 0.5 + \frac{1}{n}(r_{ij} - 0.5)n = r_{ij}.$$

To prove uniqueness, let us rewrite (4) in the form

$$2r_{ij} - 1 = u_i - u_j.$$

Then, by summing with respect to j,

$$2\sum_{j=1}^{n} r_{ij} - n = nu_i - \sum_{j=1}^{n} u_j$$
$$u_i = \frac{2}{n}\sum_{j=1}^{n} r_{ij} - 1 + \frac{1}{n}\sum_{j=1}^{n} u_j.$$

Since $c = -1 + \frac{1}{n}\sum_{j=1}^{n} u_j$ is constant with respect to i, it is

$$u_i = \frac{2}{n}\sum_{j=1}^{n} r_{ij} + c$$

and uniqueness is proved.

□

It may be noted that u_i is nothing else but the arithmetic mean of the entries in the i-th row of R multiplied by 2. Due to the uniqueness of this characterization, we can state that the simple arithmetic mean does not

satisfy Tanino's characterization (4).
Ma et al. [14] propose a consistency improving method which is coherent with (10). We state now a proposition for multiplicatively consistent *FPR* similar to Proposition 1.

Proposition 2. *Given a multiplicatively consistent FPR* $R^{\times} = (r_{ij})$*, i.e. satisfying (5), the weight vector* $v = (v_1 \dots, v_n)$ *defined by*

$$v_i = \left(\prod_{k=1}^{n} \frac{r_{ik}}{r_{ki}} \right)^{\frac{1}{n}} \tag{11}$$

is the unique vector, up to a multiplicative constant, that satisfies Tanino's characterization (6).

Proof. In conformity with the Proof of Proposition 1, we take the right hand side of (6) and substitute v_i and v_j thanks to (11)

$$\frac{v_i}{v_i + v_j} = \frac{1}{1 + \frac{v_j}{v_i}} = \frac{1}{1 + \frac{(\prod_{k=1}^{n} \frac{r_{jk}}{r_{kj}})^{\frac{1}{n}}}{(\prod_{k=1}^{n} \frac{r_{ik}}{r_{ki}})^{\frac{1}{n}}}} = \frac{1}{1 + (\prod_{k=1}^{n} \frac{r_{jk} r_{ki}}{r_{kj} r_{ik}})^{\frac{1}{n}}}$$

at this point, due to the multiplicative transitivity condition we know that $\frac{r_{jk} r_{ki}}{r_{kj} r_{ik}} = \frac{r_{ji}}{r_{ij}}$

$$\frac{1}{1 + (\prod_{k=1}^{n} \frac{r_{ji}}{r_{ij}})^{\frac{1}{n}}} = \frac{1}{1 + \frac{r_{ji}}{r_{ij}}} = \frac{r_{ij}}{r_{ij} + r_{ji}} = r_{ij}.$$

To prove uniqueness, let us rewrite (6) in the form

$$v_i \left(\frac{1 - r_{ij}}{r_{ij}} \right) = v_j.$$

Then, by multiplying with respect to j and exploiting the additive reciprocity,

$$\prod_{j=1}^{n} \left(v_i \frac{r_{ji}}{r_{ij}} \right) = \prod_{j=1}^{n} v_j$$

$$v_i = \left(\prod_{j=1}^{n} \frac{r_{ij}}{r_{ji}} \right)^{\frac{1}{n}} \times \left(\prod_{j=1}^{n} v_j \right)^{\frac{1}{n}}.$$

Since $c = (\prod_{j=1}^{n} v_j)^{\frac{1}{n}}$ is constant with respect to i, it is

$$v_i = \left(\prod_{j=1}^{n} \frac{r_{ij}}{r_{ji}} \right)^{\frac{1}{n}} \times c$$

and uniqueness is proved.

□

Let us only highlight that we derive the explicit form of the priority vectors involved in Tanino's two characterization theorems. While Tanino's characterizations give a nice interpretation of the weights, formulas (10) and (11) give simple expressions of those weights. Furthermore, they can clearly be applied in the non–consistent case too, as it is common practice with (9). The analogy with (9) will be better clarified in the following subsection.

2.2 Transformations between weight vectors

Let us now investigate the relationships between weight vectors u, v and w given by (10), (11) and (9) respectively. Keeping in mind that u is unique up to addition of a constant, while w and v are unique up to a multiplication by a constant, the following propositions hold.

Proposition 3. *Let $A = (a_{ij})$ be a consistent MPR and $R^+ = (r_{ij})$ the corresponding FPR obtained by applying (7) to A. If u and w are given by (10) and (9) respectively, then, up to addition of a constant,*

$$u_i = \log_9 w_i \ , \quad i = 1, \dots, n \ . \tag{12}$$

Proof. Let us consider equation (10) and substitute r_{ik} with the aid of (7).

We obtain

$$\begin{aligned}
u_i &= \frac{2}{n}\sum_{k=1}^{n}\frac{1}{2}(1+\log_9 a_{ik}) \\
&= \frac{1}{n}\sum_{k=1}^{n} 1 + \frac{1}{n}\sum_{k=1}^{n}\log_9 a_{ik} \\
&= 1 + \frac{1}{n}\log_9 \prod_{k=1}^{n} a_{ik} \\
&= 1 + \log_9 \left(\prod_{k=1}^{n} a_{ik}\right)^{\frac{1}{n}}
\end{aligned}$$

which, according to (9), can be rewritten

$$u_i = 1 + \log_9 w_i$$

Finally, since u is unique up to addition of a constant, we can write

$$u_i = \log_9 w_i.$$

□

Proposition 4. *Let $A = (a_{ij})$ be a consistent MPR and $R^{\times} = (r_{ij})$ the corresponding FPR obtained by applying (8) to A. If v and w are given by (11) and (9) respectively, then they are equal up to a multiplication by a constant,*

$$v_i = w_i \ , \quad i = 1, \dots, n \ . \tag{13}$$

Proof. Substituting $\frac{r_{ij}}{r_{ji}}$ for a_{ij} due to the inverse of (8) in (11), we obtain the right hand side of equation (9). □

According to propositions 3 and 4, clearly we obtain also $u_i = \log_9 v_i$ and $u_i = 2g(w_i)$.

3 Example

Let us now present an example involving pairwise comparison matrices and both additively and multiplicatively consistent fuzzy preference relations. We start considering the following additively consistent fuzzy preference relation

$$R^{+} = \begin{pmatrix} 0.5 & 0.55 & 0.65 & 0.85 \\ 0.45 & 0.5 & 0.6 & 0.8 \\ 0.35 & 0.4 & 0.5 & 0.7 \\ 0.15 & 0.2 & 0.3 & 0.5 \end{pmatrix}.$$

We can derive the priority vector with the aid of (10) and it is easy to verify that relation (4) is satisfied

$$u = \begin{pmatrix} 1.275 \\ 1.175 \\ 0.975 \\ 0.575 \end{pmatrix}.$$

At this point we proceed using the inverse of (12), that is $w_i = 9^{u_i}$. After a multiplication for a proper scalar and taking into account the equality $v = w$ we derive the following two vectors

$$w = v = \begin{pmatrix} 0.394505 \\ 0.316686 \\ 0.204070 \\ 0.084739 \end{pmatrix},$$

which are respectively associated to the following two matrices

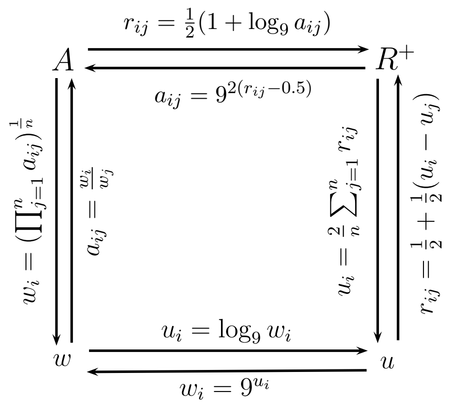
$$A = \begin{pmatrix} 1 & 1.24573 & 1.93318 & 4.65554 \\ 0.802742 & 1 & 1.55185 & 3.73719 \\ 0.517282 & 0.644394 & 1 & 2.40822 \\ 0.214798 & 0.267581 & 0.415244 & 1 \end{pmatrix},$$

$$R^{\times} = \begin{pmatrix} 0.5 & 0.554711 & 0.659073 & 0.823182 \\ 0.445289 & 0.5 & 0.608127 & 0.788905 \\ 0.340927 & 0.391873 & 0.5 & 0.706592 \\ 0.176818 & 0.211095 & 0.293408 & 0.5 \end{pmatrix}.$$

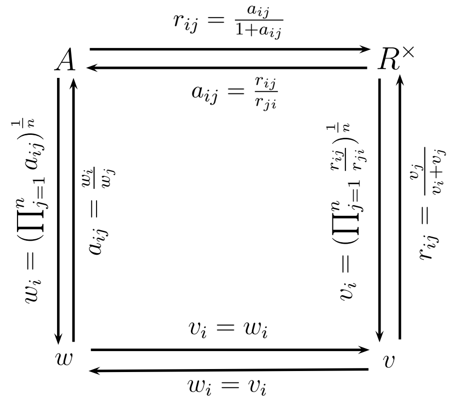
To conclude, it can be verified that v can be derived directly from $R^{\times}$ by using (11)

4 Conclusion and Remarks

By way of summarizing, we want to present in Figure 2 the same diagrams presented above but completed with the relations that we have been introducing in this paper. As already stressed, some of them are particularly



(a) Matrices A, R^+ and corresponding vectors w and u



(b) Matrices A, $R^\times$ and corresponding vectors w and v

Figure 2: Complete diagrams of transformations

interesting because with the aid of them it is possible to estimate priority vectors in a rapid, but also reliable and fully justified, way.

Tanino [18] demonstrates the existence of vectors u and v satisfying (4) and (6) respectively. With (10) and (11), we provide the simplest representation of such vectors, which can be considered to be the counterpart of (9) for *FPR* satisfying (3) and (5) respectively. Note that if we interpret $(r_{ij} - 0.5)$ to be the intensity of preference of A_i over A_j, then additive consistency (3) is the right type of consistency to be chosen and weights u_i are given on an interval scale. Conversely, if r_{ij}/r_{ji} indicates the ratio of the preference intensity for A_i to that for A_j, then multiplicative consistency (5) has to be chosen and weights v_i are given on a ratio scale [18].

References

[1] J. Barzilai. Consistency measures for pairwise comparison matrices, *J. Multi–Crit. Decis. Anal.*, 7 (1998) 123–132.

[2] Bezdek J. Spillman B and Spillman R., A fuzzy relational space for group decision theory, *Fuzzy Sets and Systems* 1, 255–268 (1978)

[3] Choo. E. U. and Wedley W. C., A common framework for deriving preference values from pairwise comparison matrices, *Computers and Operations Research*, 31, 893–908 (2004)

[4] Crawford G. and Williams C., A note on the analysis of subjective judgement matrices, *Journal of Mathematical Psychology*, 29, 25–40 (1985)

[5] Fan Z.-P., Hu G.-F. and Xiao S.-H, A method for multiple attribute decision-making with the fuzzy preference relation on alternatives *Computers & Industrial Engineering* 46, 321–327 (2004)

[6] Fan Z.-P., Ma J. and Zhang Q., An approach to multiple attribute decision making based on fuzzy preference information on alternatives *Fuzzy Sets and Systems* 131, 101–106 (2002)

[7] Fan Z.-P., Ma J., Jiang Y.-P., Sun Y.-H and Ma L., A goal programming approach to group decision making based on multiplicative preference relations and fuzzy preference relations *European Journal of Operational Research* 174, 311–321 (2006)

[8] Fedrizzi Michele. On a consensus measure in a group MCDM problem, in *Multiperson Decision Making Models using Fuzzy Sets and Possibility Theory*, J. Kacprzyk and M. Fedrizzi (eds.), Kluwer Academic Publisher, Dortrecht, The Netherlands (1990)

[9] P.C. Fishburn. SSB utility theory: an economic perspective, *Mathematical Social Sciences*, 8 (1984) 63–94.

[10] Fodor J. and Roubens M., *Fuzzy preference modelling and multicriteria decision analysis* Kluwer Academic Publisher (1994)

[11] Gong Z.-W., Least-square method to priority of the fuzzy preference relations with incomplete information *International Journal of Approximate Reasoning* 47, 258–264 (2008)

[12] Kacprzyk J.: Group decision making with a fuzzy linguistic majority. *Fuzzy Sets and Systems*, **18**, 105–118 (1986)

[13] Lipovetsky S. and Conklin M. W., Robust estimation of priorities in AHP, *European Journal of Operational Research* 137, 110–122 (2002)

[14] Ma J., Fan Z.-P., Jiang Y.-P., Mao J.-Y., Ma L. A method for repairing the inconsistency of fuzzy preference relations, *Fuzzy Sets and Systems* 157 (2006) 20–33.

[15] Nurmi H., Approaches to collective decision making with fuzzy preference relations, *Fuzzy Sets and Systems* 6, 249–259 (1981)

[16] Saaty T. L., A scaling method for priorities in hierarchical structures, *J. Math Psychology*, 15, 234–281 (1977)

[17] Shimura M., Fuzzy sets concept in rank-ordering objects, *J. Math. Anal. Appl.* 43, 717–733 (1973)

[18] Tanino T., Fuzzy preference orderings in group decision making, *Fuzzy Sets and Systems* 12 (1984) 117–131.

[19] Xu Z.S., Goal programming models for obtaining the priority vector of incomplete fuzzy preference relation, *International Journal of Approximate Reasoning*, 36 (2004) 261–270.

[20] Xu. Z. S., A procedure for decision making based on incomplete fuzzy preference relation, *European Journal of Operational Research*, 164, 206-216 (2005)

[21] Xu Z.S. and Da Q.L., A least deviation method to obtain a priority vector of a fuzzy preference relation, *European Journal of Operational Research*, 164 (2005) 206–216.

[22] Wang Y.-M. and Fan Z.-P, Group decision analysis based on fuzzy preference relations: Logarithmic and geometric least squares methods *Applied Mathematics and Computation*, 194 (2007) 108–119.

[23] Wang Y.-M., Fan Z.-P and Hua Z., A chi-square method for obtaining a priority vector from multiplicative and fuzzy preference relations *European Journal of Operational Research*, 182 (2007) 356–366.

[24] Wang Y.-M. and Parkan C., Multiple attribute decision making based on fuzzy preference information on alternatives: Ranking and weighting, *Fuzzy sets and systems*, 153 (2005) 331–346.