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Sami Hokuni | Kaisa Joki | Sari Yli-Sipilä | Marko M.Mäkelä |
Yury Nikulin

Minimum spanning tree problem with fuzzy intervals

Turku Centre *for* Computer Science

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Sami Hokuni
University of Turku, Department of Mathematics and Statistics
Vesilinnantie 5, 20014 Turku, Finland
samhok@utu.fi

Kaisa Joki
University of Turku, Department of Mathematics
Vesilinnantie 5, 20014 Turku, Finland
kijoki@utu.fi

Sari Yli-Sipilä
University of Turku, Department of Mathematics and Statistics
Vesilinnantie 5, 20014 Turku, Finland
ssylsi@utu.fi

Marko M.Mäkelä
University of Turku, Department of Mathematics and Statistics
Vesilinnantie 5, 20014 Turku, Finland
makela@utu.fi

Yury Nikulin
University of Turku, Department of Mathematics and Statistics
Vesilinnantie 5, 20014 Turku, Finland
yurnik@utu.fi

Abstract

We present the results of computational experiments performed to study empirically advantages and disadvantages of modeling uncertainty using the concepts of fuzzy intervals. The minimum spanning tree problem and its robust counterpart are chosen as target models in our investigation.

1   Preface

We consider the special case of a minimum spanning tree problem where the edge costs (weights) are not fixed but take their values from some intervals. No stochastic distribution is given inside intervals. The interval function is defined as the sum of interval weights over all edges of feasible spanning tree. This problem was first mentioned in [8], where some questions concerning solvability and computational complexity were studied. Contrary to the classical minimum spanning tree problem which can be easily solved by the algorithms of Kruskal (1956) [9] or Prim (1957) [11], minimum spanning trees of the interval variant depend on weights realization and the optimal objective value generally is not unique. Therefore, the authors of [8] proposed to introduce the relation on the set of intervals, which gives the possibility to transform the problem into a special bicriteria counterpart. The Pareto set of the counterpart, which can be generated by standard multiobjective methods, is taken to be the solution of the interval problem. It was shown that the counterpart problem is intractable, and it follows that the interval problem is also very hard to solve.

The special interest motivated by transportation problems and telecommunications applications induces not to solve the interval spanning tree problem itself, but to hedge against the worst case realization (scenario) of problem parameters, which can be interpreted as given with uncertainty. Playing against worst case scenario is commonly known as robust optimization (see, e.g. [7], [10]). As it was indicated in [7], in many cases the robust equivalent of a polynomially solvable problem becomes NP-hard.

The robust spanning tree problem was originally formulated in [7] for the case where edge costs are taken from some set of scenarios. It was proven that the problem is NP-hard [2] if the number of scenarios is bounded. Furthermore, strong NP-hardness of the problem for unbounded number of scenarios has been shown.

The direct evidence of the NP-hardness of the robust spanning tree problem with interval data was presented in [1], while the basic theoretical background has been presented in [12]. Two different types of robustness were introduced: absolute and relative robustness. It was proven that the absolute robust spanning tree problem can be easily resolved, whereas the relative robust spanning tree problem is very hard to solve. A reformulation of the last problem as a specific mixed integer program was presented. The concepts of weak and strong edges were introduced as well as polynomial time algorithms for their recognition were described. It was shown how these concepts can be efficiently used in a preprocessing stage for solving the relative robust spanning tree problem.

Since the robust spanning tree problem has a lot of applications in road and telecommunication networks design, there is an obvious practical need to

solve large scale instances with simple and fast procedures in order to find a good approximation of optimal solution. Therefore, it is of special interest to attack the problem with powerful metaheuristics the choice of which can be motivated by previous success of these procedures developed for the problems dealing with similar tree structures.

The first attempt to solve heuristically the robust spanning tree problem (and other similar robust optimization problems) is developed in [5], where the 2-approximation polynomial time algorithm with guaranteed performance ratio 2 is described. The idea of the algorithm is the following: 1) the scenario in which the costs of all elements (edges) are at midpoints of their corresponding cost intervals has to be defined 2) a standard optimization algorithm for the optimization problem with fixed scenario has to be applied (the polynomial time algorithms of Kruskal or Prim can be used to solve minimum spanning tree problem with fixed scenario). Obviously the proposed approximation algorithm has polynomial time complexity since it exploits polynomial time algorithms to solve once the problem with fixed edge costs. The algorithm is very fast but in worst performance case it may produce a solution with total cost value twice worse as optimal solution has.

Soft computing ideology contains a bunch of new and innovative approaches to analyze intelligent systems and networks under uncertainty. Soft computing methods applied to real world problems offer another degree of robustness and tractability for solutions. As a modeling tool soft computing approaches utilize fuzzy set theory and fuzzy logic as well as macro heuristic and approximate reasoning. Many deterministic models created to analyze uncertainty can be naturally extended by their fuzzy counterparts where the methodology of soft computing could be potentially beneficial. However dealing with such models may create extra computational difficulties which have to be scrutinized on their efficiency. In this paper we present such empirical analysis on efficiency of utilizing the concept of fuzzy intervals and the usage of corresponding soft computing methods for the minmax regret spanning tree problem.

2 A comprehensive background on closed and fuzzy intervals

We start this study by presenting some theory of closed and fuzzy intervals. Especially closed intervals play a crucial role, because solving a fuzzy interval problem can be turned into solving a closed interval problem. In this research we solve minimum spanning tree problems and we determine a set $E = \{e_1, \ldots, e_n\}$ that consists of edges of a given undirected graph and a set V that consists of the number of nodes. Furthermore, every feasible solution has $V - 1$ edges and we denote the set of feasible solutions by Φ which in

this case consists of all possible spanning trees of considered graph.

Theory and algorithms used in this research are mainly borrowed from [3], [4], [6]. We first formulate a general modeling pattern which is valid for an arbitrary discrete optimization (over a finite family of subsets of the given ground set E) problem $\mathcal{P}$ and then specify details regarding the spanning tree topology.

2.1   Closed interval problem

Suppose that the values of the weights of e in problem $\mathcal{P}$ are only known to belong to the closed intervals $\hat{w}_e = [\underline{w}_e, \overline{w}_e]$ where lower bound $\underline{w}_e \geq 0$ for all $e \in E$. If $\underline{w}_e = \overline{w}_e$ then the value of the weight of e is *precise* and in this case the interval $\hat{w}_e$ is called *degenerate*. A particular realization of the weights $S = (w_e^S)_{e \in E}$ such that $w_e^S \in \hat{w}_e$ for all $e \in E$ is called a *scenario*. Thus every scenario represents a certain state of the world, that is a configuration of the weights, which may occur with a positive, but perhaps unknown probability. We will denote the set of all scenarios by Γ, that is the Cartesian product of all the uncertainty intervals, namely $\Gamma = \times_{e \in E} \hat{w}_e$. Among the scenarios we will distinguish the extreme ones, in which every weight takes either the extreme value of the lower bound $\underline{w}_e$ or the upper bound $\overline{w}_e$. Let $A \subseteq E$ be a given subset of E. We will use S_A^+ to denote the extreme scenario in which the elements $e \in A$ have weights $\overline{w}_e$ and all the other elements have weights $\underline{w}_e$. Similarly, we define scenario S_A^-, in which the elements $e \in A$ have weights $\underline{w}_e$ and all the other elements have weights $\overline{w}_e$.

Now the weight of a given solution $X \in \Phi$ under a scenario $S \in \Gamma$ is defined as follows:
$$F(X, S) = \sum_{e \in X} w_e^S.$$

Let us denote by $F^*(S)$ the weight of the optimal solution under a scenario S, that is
$$F^*(S) = \min_{X \in \Phi} F(X, S).$$

In order to obtain the value of $F^*(S)$ we must solve the deterministic combinatorial optimization problem $\mathcal{P}$ for the fixed scenario $S \in \Gamma$.

Now the *regret* of a solution X under scenario $S \in \Gamma$ is defined as follows:
$$\delta_X(S) = F(X, S) - F^*(S). \tag{1}$$

By means of this, the *maximal regret* of a given solution X is defined as follows:
$$Z(X) = \max_{S \in \Gamma}\{F(X, S) - F^*(S)\} = \max_{S \in \Gamma} \delta_X(S). \tag{2}$$

It is clear that $Z(X) \geq 0$ for all solutions $X \in \Phi$. A scenario S that maximizes the right hand side of (2) is called the *worst case scenario for* X and is

denoted as S_X^+. Using results of [4], [12], we can express the maximal regret of a given solution X in the following way:

$$Z(X) = F(X, S_X^+) - F^*(S_X^+) = \delta_X(S_X^+). \tag{3}$$

In this study we focus on the following minmax regret combinatorial optimization problem which is a robust counterpart of $\mathcal{P}$:

$$\text{MINMAX REGRET } \mathcal{P} : \min_{X \in \Phi} Z(X). \tag{4}$$

A solution of closed interval problem $\mathcal{P}$ can be classified into *possibly optimal* and *necessarily optimal*. If a solution $X \in \Phi$ is possibly optimal then it has to be an optimal solution to the problem $\mathcal{P}$ for at least one scenario $S \in \Gamma$. A solution $X \in \Phi$ is necessarily optimal if it is an optimal solution to the problem $\mathcal{P}$ for all scenarios $S \in \Gamma$. Obviously, a necessarily optimal solution is the best possible solution to the problem. Unfortunately, usually it does not exist. However, every solution X must fulfill possible optimality which is now the minimum requirement.

When we have a deviation of solution X in (1) under scenario S we can calculate the optimality of solution. If the minimum deviation is

$$\underline{\delta}_X = \min_{S \in \Gamma} \delta_X(S) = \delta_X(S_X^-) = 0$$

then the solution is possibly optimal. On the other hand, if the maximum deviation is

$$\overline{\delta}_X = \max_{S \in \Gamma} \delta_X(S) = \delta_X(S_X^+) = 0$$

then the solution is necessarily optimal. Furthermore every necessarily optimal solution is a solution of minmax regret optimization problem $\mathcal{P}$. The value of the objective function is then 0.

Similarly, we can calculate possibly and necessarily optimality for every edge. Generally, this is a computationally difficult problem. We can solve it efficiently only if $\mathcal{P}$ is a matroidal (a family of subsets over the ground set E must be independent) problem such as minimum spanning tree problem.

2.2 Fuzzy interval problem

2.2.1 Notations

Fuzzy intervals have imprecise boundaries and existence of an element to this interval is defined by a membership function. An element can therefore be included in this range also partially. A fuzzy interval is now defined as a set of two elements

$$\tilde{a} = \{(x, \mu_{\tilde{a}}(x)) \,|\, x \in \mathbb{R}\},$$

where

- $\tilde{a}$ is a fuzzy interval;

- x is an element from this range;

- $\mu_{\tilde{a}} : \mathbb{R} \to [0,1]$ is a membership function of $\tilde{a}$;

- $\mu_{\tilde{a}}(x)$ is a degree of membership for an element.

The degree of membership $\mu_{\tilde{a}}(x)$ is the possibility, in which an element x is included into a fuzzy interval. If the degree of membership of an element is 1 then this element is included completely into this fuzzy interval. Similarly, an element is not included at all into the fuzzy interval if the degree of membership for this element is 0. If the degree of membership is from an open range $(0,1)$ then this element is partially included into this fuzzy set. In this case the degree of membership expresses how strongly an element is included into this fuzzy interval. However, the degree of membership is not equivalent with probability and fuzzy intervals are not probability distributions. A closed interval can be interpreted as a special case of a fuzzy interval, where the degree of membership for each element is either 1 or 0.

A λ-cut of a fuzzy interval $\tilde{a}$ is a subset of $\mathbb{R}$ which is defined by a formula

$$\tilde{a}^{\lambda} = \{x \, : \, \mu_{\tilde{a}}(x) \geq \lambda\}, \ \lambda \in (0,1].$$

In addition we also assume that all the elements with a strictly positive degree of membership are included into the set $\tilde{a}^0$. We can show that $\tilde{a}^{\lambda}$ is a closed interval for all $\lambda \in [0,1]$. Therefore a λ-cut can be written as a closed interval $\tilde{a}^{\lambda} = [\underline{a}(\lambda), \overline{a}(\lambda)]$. Now we can see fuzzy interval as a family of closed intervals, when parametrized by the value of $\lambda \in [0,1]$.

The membership function $\mu_{\tilde{a}}$ can now be expressed by using a family of λ-cuts of a fuzzy interval $\tilde{a}$. This is obtained by writing

$$\mu_{\tilde{a}}(x) = \sup\{\lambda \in [0,1] \, : \, x \in \tilde{a}^{\lambda}\},$$

where $\mu_{\tilde{a}}(x) = 0$ for all $x \notin \tilde{a}^0$.

Let $\boldsymbol{u}$ be some unknown real-valued variable and $\tilde{u}$ be a fuzzy interval which is related to $\boldsymbol{u}$. If $\mathcal{A}$ is a subset of $\mathbb{R}$ then the possibility and the necessity of $\boldsymbol{u} \in \mathcal{A}$ is defined as follows

$$\Pi(\boldsymbol{u} \in \mathcal{A}) = \sup_{x \in \mathcal{A}} \mu_{\tilde{u}}(x)$$

$$N(\boldsymbol{u} \in \mathcal{A}) = 1 - \Pi(\boldsymbol{u} \notin \mathcal{A}) = \inf_{x \notin \mathcal{A}}(1 - \mu_{\tilde{u}}(x)).$$

In this study we use trapezoidal fuzzy intervals. One example of those is shown in the figure 1. Now every trapezoidal fuzzy interval can be described by a quadruple $(\underline{a}, \overline{a}, \alpha, \beta)$, where $\underline{a} \leq \overline{a}$ and $\alpha, \beta \geq 0$. It can also be represented as a family of λ-cuts, where

$$\tilde{a}^{\lambda} = [\underline{a} - (1-\lambda)\alpha, \ \overline{a} + (1-\lambda)\beta], \ \lambda \in [0,1]. \tag{5}$$

From now on, we denote the fuzzy intervals of edges $e \in E$ by $\tilde{w}_e$.
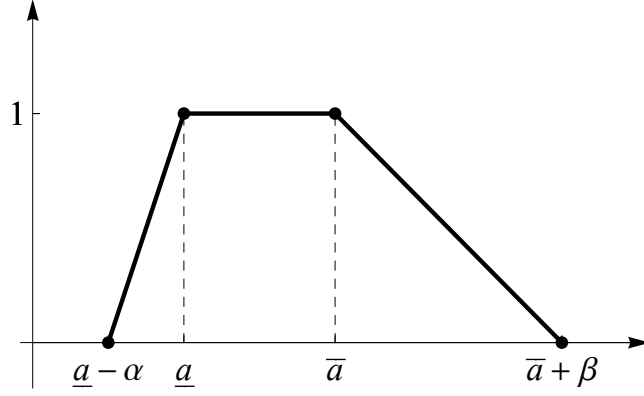
Figure 1: Trapezoidal fuzzy interval $(\underline{a}, \overline{a}, \alpha, \beta)$

2.2.2 Degree of optimality and the problem to be solved

Possible and necessary optimality of a closed interval solution X can be generalized into the degree of possible and necessary optimality of a solution X under fuzzy weights. This degree is a real number from interval $[0, 1]$. With these we can obtain both degrees for the solution X by using fuzzy deviations.

From now on, we use the marking $\mathcal{P}^\lambda$ on a combinatorial optimization problem where $\lambda \in [0, 1]$ and weights of edges are λ-cuts of fuzzy intervals. Therefore in a problem $\mathcal{P}^\lambda$ the weight of the edge $e \in E$ is $\tilde{w}_e^\lambda$. We can denote the set of all scenarios by $\Gamma^\lambda = \times_{e \in E} \tilde{w}_e^\lambda$. A fuzzy interval problem can be transformed into the closed interval problem by giving the variable λ some specific value and we can acquire both minimum and maximum deviations of solution X. This can be done by using formulas

$$\underline{\delta}_X(\lambda) = \min_{S \in \Gamma^\lambda} \delta_X(S) = \delta_X(S_X^-) \quad \text{and}$$
$$\overline{\delta}_X(\lambda) = \max_{S \in \Gamma^\lambda} \delta_X(S) = \delta_X(S_X^+).$$

With these deviations we can define the degrees of both possible and necessary optimality. The degree of possible optimality of solution X is obtained by

$$\Pi(X \text{ optimal}) = \sup\{\lambda \in [0, 1] : \underline{\delta}_X(\lambda) = 0\}. \tag{6}$$

The degree of necessary optimality of solution X is obtained by

$$N(X \text{ optimal}) = 1 - \inf\{\lambda \in [0, 1] : \overline{\delta}_X(\lambda) = 0\}. \tag{7}$$

Proofs of these formulas can be found in [4]. Now we can see that the function $\underline{\delta}_X(\lambda)$ is non-decreasing and function $\overline{\delta}_X(\lambda)$ is non-increasing. Therefore

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degrees of optimality can be defined with a given accuracy ϵ by using binary search.

Before solving fuzzy interval problems we have to choose the objective function. In this study we consider the degree of necessary optimality of solution X and our aim is to maximize its value. Therefore the problem to be solved can be written in a form

$$\text{MOST NEC } \mathcal{P} : \max_{X \in \Phi} N(X \text{ optimal}). \tag{8}$$

The value of $N(X \text{ optimal})$ can be acquired by using (7). Thus the problem MOST NEC $\mathcal{P}$ is equivalent to the optimization problem

$$\begin{aligned}
\min \quad & \lambda \\
\text{s.t.} \quad & \overline{\delta}_X(\lambda) = 0 \\
& X \in \Phi \\
& 0 \leq \lambda \leq 1.
\end{aligned} \tag{9}$$

Our goal is to find the smallest value of $\lambda \in [0, 1]$ for which there exists a solution X that fulfills the condition $\overline{\delta}_X(\lambda) = 0$. Now X is the optimal solution for the problem MOST NEC $\mathcal{P}$ and value of objective function is $N(X \text{ optimal}) = 1 - \lambda^*$. If the problem (9) has no feasible solutions then $N(X \text{ optimal}) = 0$ for all solutions $X \in \Phi$.

Degree of necessary optimality is a extremely strict condition because it requires the solution X to be optimal for all weight realizations of the problem $\mathcal{P}^\lambda$. In most cases this solution does not exist and therefore $N(X \text{ optimal}) = 0$ for all $X \in \Phi$. Because of this we need to define a weaker condition called the degree of necessary soft optimality.

Let the function $\mu_Z : [0, \infty) \to [0, 1]$ be non-increasing and its value is $\mu_Z(0) = 1$. This function models the fuzzy goal of the deviation δ_X and it can be defined in any way such that the previous conditions are fulfilled. In other words $\mu_Z(\delta_X)$ defines the degree that can have the acceptable amount of deviation. In this study we use the piecewise linear function μ_Z shown in the figure 2. We can clearly see that $\delta_X = 0$ is the best possible deviation.

Now we can define the degree of necessary soft optimality of solution X by using the formula

$$N(X \text{ soft optimal}) = \inf_S \max\{1 - \pi(S), \mu_Z(\delta_X(S))\}, \tag{10}$$

where $\pi(S) = \min_{e \in E} \mu_{\tilde{w}_e}(w_e^S)$ and $w_e^S \in \tilde{w}_e$ is some realization of the weights of edges under some certain scenario. Especially when the problem MOST NEC $\mathcal{P}$ has no feasible solution we can move on to solve the problem

$$\text{MOST NEC SOFT } \mathcal{P} : \max_{X \in \Phi} N(X \text{ soft optimal}), \tag{11}$$
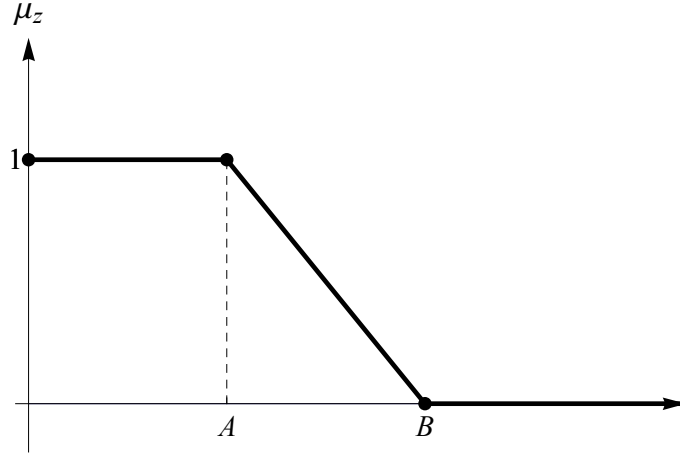
Figure 2: Function μ_Z which represents fuzzy goal of deviation δ_X

which is the generalization of the problem MOST NEC $\mathcal{P}$. We can also represent the problem MOST NEC SOFT $\mathcal{P}$ by using the equivalent form

$$
\begin{aligned}
\min \quad & \lambda \\
\text{s.t.} \quad & \overline{\delta}_X(\lambda) \leq \mu_Z^{-1}(1 - \lambda) \\
& X \in \Phi \\
& 0 \leq \lambda \leq 1,
\end{aligned} \tag{12}
$$

where μ_Z^{-1} is the pseudo-inverse function of μ_Z. We can define this pseudo-inverse function as

$$
\mu_Z^{-1}(y) = \sup\{x \, : \, \mu_Z(x) \geq y\}, \quad \text{when } y \in [0, 1].
$$

3   Algorithms

In this section we present algorithms that have been used to solve fuzzy MIN-MAX REGRET problems. Before solving problems we transform them to a mixed integer linear programming problem. All algorithms were implemented by C++ and GLPK was used to solve optimization problems mentioned in the algorithms.

3.1   Kruskal's algorithm

By using Kruskal's algorithm we can solve deterministic minimum spanning tree problems easily. The algorithm is shown below.

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Algoritm 1. Kruskal's algorithm

Require: Matroidal combinatorial optimization problem.
Ensure: Optimal solution $X \in \Phi$.

1: Arrange edges in ascending order such that $w_{e_1} \leq w_{e_2} \leq \ldots \leq w_{e_n}$.
2: $X \leftarrow \emptyset$
3: **for** $i \leftarrow 1$ **to** n **do**
4: **if** $X \cup e_i$ does not form a cycle **then** $X \leftarrow X \cup e_i$
5: **end for**
6: **return** X

3.2   Algorithms MOST NEC and MOST NEC SOFT

In this study we use two possible algorithms for solving fuzzy optimization problems. In both algorithms we fix some specific value for the variable λ and therefore we obtain the problem $\mathcal{P}^\lambda$, which consists of closed intervals. By solving $\mathcal{P}^\lambda$ we get the solution Y for this problem. Next we need the regret $Z(Y)$ for the solution Y, and depending on this regret we either increase or decrease the value of λ. These phases will be repeated until the difference between old λ and new λ is small enough.

In the algorithm MOST NEC we solve two deterministic problems and that solution which gives smaller maximal regret will be set as Y. In the algorithm MOST NEC SOFT we first transform a MINMAX REGRET problem into a certain form of a mixed integer linear programming problem (MILP). This form is represented in [4]. By solving this MILP problem we obtain the solution Y. Next we represent an approximation algorithm AMU, which is needed in forthcoming algorithms, and after that we formulate the algorithms designed to solve fuzzy interval problems.

Algoritm 2. AMU

Require: Problem MINMAX REGRET $\mathcal{P}$.
Ensure: A 2-approximate solution Y.

1: **for all** $e \in E$ **do**
2: $w_E^{S_1} \leftarrow \frac{1}{2}(\underline{w}_e + \overline{w}_e)$
3: $w_e^{S_2} \leftarrow \overline{w}_e$
4: **end for**
5: $Y_1 \leftarrow$ optimal solution of scenario S_1 by applying **Kruskal's** algorithm
6: $Y_2 \leftarrow$ optimal solution of scenario S_2 by applying **Kruskal's** algorithm
7: **if** $Z(Y_1) \leq Z(Y_2)$ **then return** Y_1 **else return** Y_2.

Algoritm 3. MOST NEC

Require: Problem $\mathcal{P}$ with fuzzy weights, accuracy ϵ.
Ensure: The most necessarily optimal solution $X \in \Phi$ with accuracy ϵ.

1:   Form the problem $\mathcal{P}^1$ and compute solution Y by applying **AMU**.

2:   **if** $Z(Y) > 0$ in $\mathcal{P}^1$ **then return** any $X \in \Phi$ ($N(X$ optimal$) = 0$ for all $X \in \Phi$).

3:   $\lambda_1 \leftarrow 0.5;\ \lambda_2 \leftarrow 0;\ k \leftarrow 1$

4:   **while** $|\lambda_1 - \lambda_2| \geq \epsilon$ **do**

5:       $\lambda_2 \leftarrow \lambda_1$

6:       Form the problem $\mathcal{P}^{\lambda_1}$ and compute solution Y by applying **AMU**.

7:       **if** $Z(Y) = 0$ in $\mathcal{P}^{\lambda_1}$ **then** $X \leftarrow Y;\ \lambda_1 \leftarrow \lambda_1 - \frac{1}{2^{k+1}}$

8:       **else** $\lambda_1 \leftarrow \lambda_1 + \frac{1}{2^{k+1}}$

9:       $k \leftarrow k + 1$

10: **end while**

11: **return** X ($N(X$ optimal$) = 1 - \lambda_1)$)

Algoritm 4. MOST NEC SOFT

Require: Problem $\mathcal{P}$ with fuzzy weights, accuracy ϵ. Choose function μ_Z.
Ensure: The necessarily soft optimal solution $X \in \Phi$ with accuracy ϵ.

1:   $\lambda_1 \leftarrow 0.5;\ \lambda_2 \leftarrow 0;\ k \leftarrow 1$

2:   **while** $|\lambda_1 - \lambda_2| \geq \epsilon$ **do**

3:       $\lambda_2 \leftarrow \lambda_1$

4:       Compute Y by solving MINMAX REGRET $\mathcal{P}^{\lambda_1}$.

5:       **if** $Z(Y) \leq \mu_Z^{-1}(1 - \lambda_1)$ in $\mathcal{P}^{\lambda_1}$

6:          **then** $X \leftarrow Y;\ \lambda \leftarrow \lambda_1;\ \lambda_1 \leftarrow \lambda_1 - \frac{1}{2^{k+1}}$

7:       **else** $\lambda_1 \leftarrow \lambda_1 + \frac{1}{2^{k+1}}$

8:       $k \leftarrow k + 1$

9:   **end while**

10: Compute Y by solving MINMAX REGRET $\mathcal{P}^{\lambda_1}$.

11: **if** $Z(Y) \leq \mu_Z^{-1}(1 - \lambda_1)$ in $\mathcal{P}^{\lambda_1}$ **then** $X \leftarrow Y;\ \lambda \leftarrow \lambda_1;$

12: **return** X ($N(X$ soft optimal$) = 1 - \lambda)$)

3.3 Algorithm DETECT POS

To reduce computational time needed to solve a closed interval MINMAX REGRET problem we implemented the following algorithm called DETECT POS. The reason for using this algorithm as preprocessing procedure is pretty clear. Before solving the actual problem we can check for each edge whether or not it is possibly optimal. If some edge is not possibly optimal it cannot be included into the solution of a MINMAX REGRET problem. Therefore values of these edges can be set to zero before we solve the problem, what reduces computational efforts in all the other algorithms.

Algoritm 5. DETECT POS

Require: A matroidal problem $\mathcal{P}$ with interval weights, element $f \in E$.
Ensure: **true** if f is possibly (necessarily) optimal and **false** otherwise.

1: Order the elements of the problem $\mathcal{P}$ with respect to the scenario $S_{\{f\}}^{-}$.
2: $B \leftarrow \emptyset$
3: **for** $i \leftarrow 1$ **to** n **do**
4:      **if** $B \cup \{e_i\}$ does not form a cycle **then**
5:          $B \leftarrow B \cup \{e_i\}$
6:          **if** $e_i = f$ **return true**
7:      **end if**
8: **end for**
9: **return false**

3.4 About solving problems and comparing solutions

Our primary goal is to find out whether or not it is useful in general to use fuzzy intervals in optimization problems. If fuzzy intervals are not worth using then it might be better to model a problem by using only closed intervals. Therefore, when dealing with fuzzy interval problems we first solve MINMAX REGRET problems $\mathcal{P}^1$ and $\mathcal{P}^0$ and calculate regrets of these solutions. In what follows the regret of the problem $\mathcal{P}^1$ is marked with R^1. Similarly, we use the notation R^0 for the regret of the problem $\mathcal{P}^0$. By using R^1 and R^0 we obtain the piecewise linear function μ_Z which is needed in the algorithm **MOST NEC SOFT**. This μ_Z is acquired by setting A to R^1 and B to R^0 in the figure 2. Therefore, function μ_Z is defined as

$$\mu_Z(x) = \begin{cases} 1 & , x \leq A \\ -\frac{1}{B-A}x + \frac{B}{B-A} & , A \leq x \leq B \\ 0 & , x \geq B. \end{cases}$$

By using μ_Z we obtain both the solution of the fuzzy interval problem and the degree of soft necessary optimality $1 - \lambda^*$ of this solution. In addition we gain the regrets of all three solutions (fuzzy interval, $\mathcal{P}^1$ and $\mathcal{P}^0$) on the scenario of the $\mathcal{P}^{\lambda^*}$ problem. In what follows the regret of the solution of a fuzzy interval problem is marked with R_{fuzzy}. Similarly, we use the notation R^1_{fuzzy} for the regret of the solution of a problem $\mathcal{P}^1$ and the notation R^0_{fuzzy} for the regret of the solution of a problem $\mathcal{P}^0$.

However, we need a way to compare the solution of a fuzzy interval problem to solutions of problems $\mathcal{P}^1$ and $\mathcal{P}^0$. For this we count relative regrets of R^1_{fuzzy} and R^0_{fuzzy} from the R_{fuzzy}. This relative regret is acquired with a formula

$$\frac{R^i_{fuzzy} - R_{fuzzy}}{R_{fuzzy}}, \quad i = 0, 1.$$

From now on when talking about how much solutions of $\mathcal{P}^1$ and $\mathcal{P}^0$ differ from the solution of fuzzy interval problem, we mean this relative regret.

Furthermore, when comparing solutions we also focus on edges that are included to these solutions. Firstly, we check if there exists some edges that are included in the solution of the fuzzy interval, but are not included in either of the solution of $\mathcal{P}^0$ or $\mathcal{P}^1$. We also observe if there are any edges that belong in both solutions of problems $\mathcal{P}^0$ and $\mathcal{P}^1$, but do not belong in the solution of fuzzy interval problem. In addition we also look for edges that exist in all three solutions. Finally we do some pairwise comparisons to look for common edges between a fuzzy interval solution and solution of a problem $\mathcal{P}^0$. Similarly, we perform pairwise comparisons between fuzzy interval solution and solution of a problem $\mathcal{P}^1$.

In order to do all these examinations we implemented the following algorithm called EDGE COMPARISON. The only thing it does not do is to count relative regrets of solutions.

Algoritm 6. EDGE COMPARISON

Require: Matroidal problem $\mathcal{P}$.
Ensure: Compares different solutions X, Y and W.

 1: $X \leftarrow$ Solution of the MINMAX REGRET problem $\mathcal{P}^1$
 2: $Y \leftarrow$ Solution of the MINMAX REGRET problem $\mathcal{P}^0$
 3: $A \leftarrow Z(X)$ in the problem $\mathcal{P}^1$
 4: $B \leftarrow Z(Y)$ in the problem $\mathcal{P}^0$
 5: Define a function μ_Z by using variables A and B
 6: Solve MOST NEC SOFT problem by using the function μ_Z
 7: $W \leftarrow$ solution gained at the previous line, $1 - \lambda \leftarrow$ Degree of the necessary optimality of the solution W

8:   $C \leftarrow Z(W)$ in the problem $\mathcal{P}^\lambda$

9:   $A \leftarrow Z(X)$ in the problem $\mathcal{P}^\lambda$

10:   $B \leftarrow Z(Y)$ in the problem $\mathcal{P}^\lambda$

11:   Compare edges of solutions X, Y and W

4   Literature benchmark problems

We chose from literature different graphs benchmarks to examine. To those graphs we selected complete graphs, complete bipartite graphs, King's graphs and grid graphs. For each type of graph we also solved cases of different sizes and each individual graph was solved with at least 30 different weights. These weights were generated randomly from interval [0,100]. Issues mentioned in chapter 3.3 were calculated for each solution. Finally from these observations we obtained minimums, maximums and means. Results are shown in the appendix.

4.1   Grid graphs

In a grid graph two nodes are connected if they are horizontally or vertically next to each other. We will denote by $A \times B$ the size of the graph where A is the number of nodes that lie at the horizontal row and B is the number of nodes that lie at the vertical row. A grid graph $A \times B$ has altogether $A \times B$ nodes and $(A - 1) \times B + (B - 1) \times A$ edges. We chose to research 9 different grid graphs and each was solved with 50 different weights. The smallest considered grid graph had 9 nodes and largest had 24 nodes, which is quite a many in comparison to the other types of graph we solved. It is because a grid graph are really sparse and therefore solving large graphs is faster. The number of nodes and edges of each considered grid graphs benchmark are shown in Table 1.
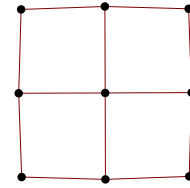
size	3×3	3×4	4×4	6×3	10×2	5×4	7×3	8×3	6×4
nodes	9	12	16	18	20	20	21	24	24
edges	12	17	24	27	28	31	32	37	38
observations	50	50	50	50	50	50	50	50	50

Table 1: Collection of grid graphs benchmarks

Generally about the solutions of fuzzy, $\mathcal{P}^1$ and $\mathcal{P}^0$ problems

On average $75 - 82$ % and even in the worst cases more than 50% of edges belonged to all three solutions. There were also some small problems(less than 20 nodes) where all three solutions were exactly the same. On average 25% of the observations contained one edge that was included into fuzzy solution but was not included into neither the closed interval $\mathcal{P}^1$ solution nor $\mathcal{P}^0$ solution. Furthermore, only $20 - 34$ % of the observations contained $1 - 2$ edges that were included into both closed interval solutions but not into fuzzy one. So it seems like those edges that belonged into both closed interval solutions were quite robust for the fuzzy solution.

Degree of optimality

Degree of necessary soft optimality was about 0.52 in each graph. On the smallest 3×3 graph degrees of necessary soft optimality were on range $[0.42; 0.62]$ and in the larger graphs this range was $[0.45; 0.60]$.

Comparison of the fuzzy and $\mathcal{P}^1$ problems

Fuzzy solution and closed interval $\mathcal{P}^1$ solution were really close to each other because on average over 90% of edges were common for both solutions. Even in the worst cases over 72% of edges were the same. When the amount of nodes increased the amount of exactly same solutions decreased. Fuzzy and $\mathcal{P}^1$ solutions were the same in 23 observations (50 total) for grid graphs containing only 9 nodes. However, on the grid graphs containing more than 16 nodes fuzzy and $\mathcal{P}^1$ solutions were the same in only $5 - 8$ observations.

Small graphs had most of the exactly same solution but these graphs also had most cases where solution of $\mathcal{P}^1$ differed from the fuzzy solution more than 10%. On average the deviation of solutions was nevertheless only $5.5 - 7.3$ %. The largest deviations were on average $20 - 35\%$ but on some problems there were even deviations of 45%. In general on $70 - 80$ % of observations for each type of graph the solution $\mathcal{P}^1$ differed less than 10% from the fuzzy solution. Moreover, on half of observations had the deviation less than 5%.

Comparison of the fuzzy and $\mathcal{P}^0$ problems

On average $85 - 90$ % of edges belonged into both fuzzy solution and $\mathcal{P}^0$ solution. On the worst cases only $60 - 70$ % of edges were the same. There were also completely same solutions but number of them decreased when the number of nodes increased.

The solution of the $\mathcal{P}^0$ problem differed in average $9 - 16$ % from the fuzzy solution. In each graph there also existed some cases where the deviation

was more than 28%. In the 3×3 graph there was one observation where the deviation was even 72%. In each graph about 30% of the observations had the deviation less than 10%. However, in the each graph only $15 - 30$ % of the observations had the deviation less than 5%.

The solution of $\mathcal{P}^1$ seems to be closer to the fuzzy solution and also deviations of $\mathcal{P}^1$ and fuzzy solutions were smaller. However, in each graph there were observations where both $\mathcal{P}^1$ solution and $\mathcal{P}^0$ solution was exactly similar to the fuzzy solution. In $\mathcal{P}^0$ problems there were less these kind of observations than in $\mathcal{P}^1$ problems.

4.2 King's graphs

A King's graph is quite similar as a grid graph but two nodes obliquely next to each other are also connected. A King's graph $A \times B$ has in total of $A \times B$ nodes and $(A-1) \times B + (B-1) \times A + 2 \times (A-1) \times (B-1)$ edges. We chose to research 5 different King's graphs and largest considered graph had now 18 nodes, because solving a King's graph takes a lot longer than solving a grid graph. Table 2 contains information about number of nodes and edges in generated instances.
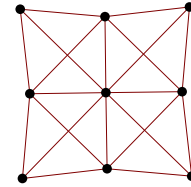
size	3×3	3×4	8×2	4×4	6×3
nodes	9	12	16	16	18
edges	20	29	36	42	47
observations	50	50	50	50	30

Table 2: Collection of King's graphs benchmarks

Generally about the solutions of fuzzy, $\mathcal{P}^1$ and $\mathcal{P}^0$ problems

On average about 65% of the edges belonged to all three solutions, but in the worst case only 35%. All three solutions were exactly the same only in a few cases. About 30% of observations in the graphs having less than 36 edges had $1 - 2$ edges such that they were included into fuzzy solution but neither of closed interval solutions. Some of the closed interval solutions had also $1 - 2$ edges that were not included into the fuzzy solution. About 20% of observations in the graphs having less than 13 nodes and even 60% of observations in the graphs having 18 nodes were like this.

Degree of optimality

The degree of necessary soft optimality was on average 0.54 and generally this degree was from range $[0.44, 0.60]$.

Comparison of the fuzzy and $\mathcal{P}^1$ problems

On average 85% of edges belonged to both fuzzy solution and $\mathcal{P}^1$ solution. Even in the worst case more than 65% of edges were the same. When the amount of nodes in the graphs increased the amount of exactly same solutions decreased significantly. In the 3×3 graph solutions were the same on 36% of cases but in the largest graphs only $4 - 8$ % of cases.

On average the solution of $\mathcal{P}^1$ differed from the fuzzy solution only $4 - 5$ %. In each graph $80 - 90$ % of observations had the deviation less than 10%. Furthermore, the deviation was less than 5% on $50 - 70$ % of observations. There were not many large deviations and on general even in the worst cases deviations were less than 15%. The smallest 3×3 graph had most cases where deviation were greater than 10%.

Comparison of the fuzzy and $\mathcal{P}^0$ problems

On average 75% of edges belonged to both $\mathcal{P}^0$ and fuzzy solution. Even in the worst cases more than a half of edges belonged to both solutions. In the small graphs there were also some cases where both solutions were exactly the same. However, in 4×4 and greater graphs there were no such solutions.

On average the solution of $\mathcal{P}^0$ is different from the fuzzy solution about $11 - 15$ %. Now average deviations were rather great and therefore in each graph only $30 - 50$ % of observations had the deviation less than 10%. Only in $10 - 20\%$ of observations the deviation was less than 5% and in the worst cases deviations were from range $30 - 40$ %. After all it seems like $\mathcal{P}^1$ solution is closer to fuzzy solution also on King's graphs.

4.3   Complete bipartite graphs

A complete bipartite graph is a graph where nodes are divided into two sets. Every node in the first set is connected to every node in the second set. If the first set has A nodes and the second set has B nodes we have a complete bipartite graph $A \times B$. It has altogether $A + B$ nodes and $A \times B$ edges. We selected 9 graphs and the largest considered graph had only 14 nodes, because solving time rose significantly after that. Even in the graphs of 14 nodes computation time were huge and so we solved those with only 30 different weights.
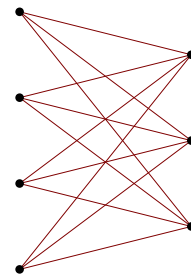
The number of nodes and edges of each considered complete bipartite graphs benchmark are shown in Table 3.

size	4×4	6×3	5×5	6×5	8×4	6×6	8×5	10×4	8×6
nodes	8	9	10	11	12	12	13	14	14
edges	16	18	25	30	32	36	40	40	48
observations	50	50	50	50	50	50	50	30	30

Table 3: Collection of complete bipartite graphs benchmarks

Generally about the solutions of fuzzy, $\mathcal{P}^1$ and $\mathcal{P}^0$ problems

On average 60% of edges belonged to all three solutions. In the worst cases there were only $3-4$ common edges in all three solutions even in the largest graphs. The fuzzy solution consisted mostly of edges that belonged to either $\mathcal{P}^1$ or $\mathcal{P}^0$ solution. For example, in graphs having less than 10 nodes only 20% of fuzzy solutions had $1-2$ edges that does not belonged to $\mathcal{P}^1$ or $\mathcal{P}^0$ solution. Furthermore, on about 25% of cases there were $1-2$ edges that belonged to both $\mathcal{P}^1$ and $\mathcal{P}^0$ solution but not to the fuzzy one.

Degree of optimality

In each graph the degree of necessary soft optimality was on average 0.54. On the graphs having less than 10 nodes the range of necessary soft optimality was the widest and this range was $[0.45; 0.65]$. In the larger graphs the range was a bit narrower, about $[0.47; 0.60]$.

Comparison of the fuzzy and $\mathcal{P}^1$ problems

On average 85% of edges belonged to both fuzzy and $\mathcal{P}^1$ solution. Even in the worst cases more than half of edges were common for both solutions. When the amount of nodes increased the amount of exactly the same solutions decreased. For example in 4×4 graph solutions were the same on 40% of observations but in 8×6 graph solutions were the same on only one case of 30.

Furthermore, solutions of $\mathcal{P}^1$ and fuzzy problem differed from each other on average only $3-6$ %. In small graphs there were observations where the deviation was large. For example in 4x4 graph the largest deviation was even 41.6%. In the graphs having more than 9 nodes the deviations were always less than 15%. Moreover, in each graph the deviation was less than 10% on 80% of observations. Similarly, on $50-70$ % of observations deviation was less than 5%.

Comparison of the fuzzy and $\mathcal{P}^0$ problems

On average 75% of edges were common for both $\mathcal{P}^0$ and fuzzy solution and in the worst cases $45 - 55$ % of edges belonged to both solutions. The amount of exactly same solutions decreased when the amount of nodes increased. However, the amount of exactly same solutions between fuzzy and $\mathcal{P}^0$ problem was always smaller than the amount of exactly same solutions between fuzzy and $\mathcal{P}^1$ problem. In the largest graphs there were no cases where fuzzy solution and $\mathcal{P}^0$ solution would have been exactly the same.

On average the $\mathcal{P}^0$ solution differed from the fuzzy one about $12 - 15$ %. Small graphs had the greatest deviations and in the worst case this deviation was even 62.6%. In the graphs having more than 9 nodes the largest deviation was 30%. Furthermore, in each graph less than half of observations had the deviation less than 10%. After all solution of $\mathcal{P}^1$ appears to be much better than solution of $\mathcal{P}^0$ when they are compared to the fuzzy solution.

4.4 Complete graphs

A graph is complete if it has maximum number of edges, that is every node is connected to all the other nodes. A complete graph with A nodes has in total of $A \times (A-1)/2$ edges. We solved here complete graphs with 6, 8, 10, 12 and 14 nodes and their amount of edges and observations are shown below.
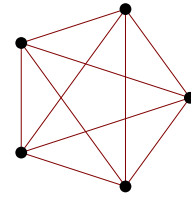
nodes	6	8	10	12	14
edges	15	28	45	66	91
observations	50	50	50	50	50

Table 4: Collection of complete graphs benchmarks

Generally about the solutions of fuzzy, $\mathcal{P}^1$ and $\mathcal{P}^0$ problems

On average $55 - 60$ % of edges belonged to all three solutions. In the graph of 6 nodes there was also one special case where these three solutions had no common edges. In other graphs there were also some cases where all three solutions had only a few common edges. In the graphs having less than 10 nodes, about 20% of observations had $1 - 3$ edges that belonged to the fuzzy solution but did not belong to either $\mathcal{P}^1$ or $\mathcal{P}^0$ solutions. In the larger graphs this number was 40%. Furthermore, in each graph less than 20% of observations had $1 - 2$ edges that belonged to both $\mathcal{P}^0$ and $\mathcal{P}^1$ solutions but not to the fuzzy one.

Degree of optimality

In the graphs having 6 or 8 nodes the degree of necessary soft optimality was on average 0.54 and these values were from range $[0.45; 0.63]$. In the graphs having 10 or 12 nodes this degree was on average $0.52 - 0.53$ and values were from range $[0.46; 0.58]$. In the largest graph with 14 nodes this degree was on average 0.50 and values were from range $[0.45; 0.55]$.

Comparison of the fuzzy and $\mathcal{P}^1$ problems

On average $80 - 87$ % of edges of solutions belonged to both fuzzy and $\mathcal{P}^1$ solution and even in the worst case this number was 54%. In the graphs having less than 10 nodes the fuzzy solution and the $\mathcal{P}^1$ solution were exactly the same on 40% of observations. Deviation of the $\mathcal{P}^1$ and fuzzy solution was on average only $3.5 - 4.5$ % and in the worst case this deviation was only 17%. Furthermore, on 90% of observations the deviation was less than 10% and on 60% of observations this deviation was less than 5%.

Comparison of the fuzzy and $\mathcal{P}^0$ problems

On average $68 - 73$ % of edges belonged to both fuzzy and $\mathcal{P}^0$ solutions. In the worst cases only $20 - 40$ % of edges of solutions were common. In the larger graphs there were only a few cases where these solutions were exactly the same, and in the graph having 10 nodes there were not these cases. In the graph having 6 nodes the largest deviation was 47% but in the larger graphs this deviation was at most only $20 - 30\%$. On average the solution of $\mathcal{P}^0$ differed from the fuzzy solution about $10 - 14$ %. On about $40 - 50$ % of observations the deviation was less than 10% and on $20 - 30$ % of observations this deviation was less than 5%. After all the solution of $\mathcal{P}^1$ seems to be closer to the fuzzy solution than the solution of $\mathcal{P}^0$.

5   Randomly generated problems

In randomly generated problems both graphs and weights of edges were generated randomly. The weights were generated in the same way as in the literature benchmark problems. In this paper we study the differences between sparse and dense graphs. Generated graphs had 8, 10, 12 and 14 nodes. For both densities and number of nodes we generated 10 different graphs and for each graph we generated 20 different sets of weights. Only exceptions were dense graphs having 12 or 14 nodes. For these graphs we generated only 10 sets of weights because computational time grew too large. Therefore, for every sparse graphs and for both dense graphs having 8 and 10 nodes we generated 200 observations and for both dense graphs having 12 and 14 graphs we generated 100 observations.

5.1 Results

Generally about the solutions of fuzzy, $\mathcal{P}^1$ and $\mathcal{P}^0$ problems

Results for both small (8 and 10 nodes) and large (12 and 14 nodes) problems were pretty similar. On average about $55 - 65$ % of edges were common for all three solutions in dense graphs. In sparse graphs this number was $65 - 75$ %. When the amount of nodes and edges decreased the amount of three solutions being exactly the same increased. In dense graphs having 12 or 14 nodes there were no observations where all three solutions would have been exactly the same. Generally in less than half of the observations of all graph types there were $1 - 3$ edges that belonged to the fuzzy solution but neither $\mathcal{P}^1$ nor $\mathcal{P}^0$ solution. However, in dense graphs having 12 or 14 nodes there were some cases where more than half of observations were like this. About 25% of observations of each case in all graph types had $1 - 2$ edges that belonged to both $\mathcal{P}^1$ and $\mathcal{P}^0$ solution but not to the fuzzy one.

Degree of optimality

On average the degree of necessary soft optimality was $0.52 - 0.55$. These values varied from range $[0.45; 0.61]$ but in graphs having 8 nodes this range was a bit wider, about $[0.45; 0.66]$.

Comparison of the fuzzy and $\mathcal{P}^1$ problems

In the dense graphs about $80 - 90$ % of edges belonged to both fuzzy and $\mathcal{P}^1$ solutions and in the sparse graphs this number was $85 - 90$ %. Even in the worst cases more than half of the edges of solutions were common for both densities. Generally in the small graphs all three solutions were exactly the same on about half of observations and in the large graphs on $10 - 20$ % of observations.

In the dense graphs the deviation of $\mathcal{P}^1$ and fuzzy solution was on average $2 - 6$ % and in the sparse graphs $3 - 8$ %. In the small sparse graphs this deviation was in the worst cases even 40% and in the small dense graphs about 20%. In the large sparse graphs deviations was about 20% in the worst cases but only about 10% in the large dense graphs. Furthermore, deviations was less than 5% in $50 - 75$ % of observations. In the dense graphs even $90 - 100$ % of observations had deviation less than 10% and especially in the dense graphs having 12 or 14 nodes almost all observations had deviation less than 10%. In the sparse graphs this deviation was less than 10% in $75 - 90$ % of observations.

Comparison of the fuzzy and $\mathcal{P}^0$ problems

In the dense graphs about $65 - 75$ % of edges belonged to both fuzzy and $\mathcal{P}^0$ solutions and in the sparse graphs this number was $75 - 85$ %. In the worst cases only $30 - 40$ % of edges were common for both solutions. In the small problems these solutions were exactly the same in less than 25% of observations. In the dense graphs having 10, 12 or 14 nodes there were a lot of cases where solutions were not same in any of the observations. In the large sparse graphs solutions were the same on about $5 - 15$ % of observations.

The deviation of the solutions of $\mathcal{P}^0$ and fuzzy was on average $10 - 20$ %. In the large dense graphs the largest deviation was 35%, in the other graphs about 50% and in the sparse graph having 8 nodes even 72%. In general the deviation was less than 10% on $25 - 60$ % of observations. However, in a few large dense graphs this deviation was less than 10% for more then 90% of observations. In general 35% of observations had deviation less than 5% but there were some cases in the large dense graphs where this deviation was more than 5% in all observations. Based on these notions it seems like the solution of $\mathcal{P}^1$ is closer to the fuzzy solution than the solution of $\mathcal{P}^0$.

6 Summary and Conclusions

6.1 Summary of results

Generally about the solutions of fuzzy, $\mathcal{P}^1$ and $\mathcal{P}^0$ problems

In the all different graphs about $55 - 80$ % of edges were common for all three solutions. In the literature benchmark problems it could be seen that the amount of common edges decreased when the graph became more dense. Sparse grid graphs had most of the common edges in all three solutions, on average $75 - 80$ %. In the complete graphs the amount of common edges in all three solutions was smallest, about $55 - 60$ %. Same notices could also be seen on randomly generated graphs. Sparse graphs had the most common edges and dense graphs the least. In the sparse graphs this number was $65 - 75$ % for all amounts of nodes and in the dense graphs it was $55 - 65$ %. It seems like there is no difference in the amounts of common edges in all three solutions between dense and complete graphs.

In the literature benchmark and randomly generated graphs there were in general $1 - 3$ edges that belonged to the fuzzy solution but did not belong to either $\mathcal{P}^1$ or $\mathcal{P}^0$ solutions. Furthermore, on about $20 - 25$ % of observations there were $1 - 2$ edges that belonged to both $\mathcal{P}^1$ and $\mathcal{P}^0$ solutions but not to the fuzzy one. Those edges that belonged to both $\mathcal{P}^1$ and $\mathcal{P}^0$ solutions were in most cases included also into the fuzzy solution. Now these edges seem to be rather robust with respect to the fuzzy solution.

Degree of optimality

On average the degree of necessary soft optimality was always from range $[0.50; 0.55]$ and the smallest degree was 0.42 and the largest 0.66.

Comparison of the fuzzy, $\mathcal{P}^1$ and $\mathcal{P}^0$ problems

In both literature benchmark and randomly generated graphs on average $80 - 90$ % of edges were common for both fuzzy and $\mathcal{P}^1$ solution and even in the worst cases more than half of edges belonged to both solutions. In the grid graphs the amount of common edges of fuzzy and $\mathcal{P}^1$ solutions was on average always over 90% and even in the worst cases over 72%. Generally fuzzy solution and $\mathcal{P}^0$ solution had less common edges than fuzzy solution and $\mathcal{P}^1$ solution.

In the randomly generated graphs there were some differences in common edges of fuzzy and $\mathcal{P}^0$ solutions between sparse and dense graphs. About $65 - 75$ % of edges were common in the dense graphs but in the sparse graphs $75 - 85$ %. In the graphs from literature benchmarks about 75% of edges belonged to both fuzzy and $\mathcal{P}^0$ solutions. Especially in the grid graphs on average $85 - 90$ % of edges belonged to both fuzzy and $\mathcal{P}^0$ solutions and even in the worst cases this number was $60 - 70$ %. In the all other types of graphs only about $20 - 40$ % of edges were common for both solutions. Based on these notions we can say that the $\mathcal{P}^1$ solution had mostly more common edges with the fuzzy solution than $\mathcal{P}^0$.

The amount of cases where fuzzy solution and $\mathcal{P}^1$ solution were exactly the same decreased when the amount of nodes increased. The similar notion was also made between fuzzy solution and $\mathcal{P}^0$ solution. Furthermore, the amount of exactly same solutions was always greater in randomly generated sparse graph than in the same-sized dense graphs. Problem $\mathcal{P}^1$ had always more exactly same solutions with the fuzzy one than the problem $\mathcal{P}^0$ had.

The deviation of the $\mathcal{P}^1$ and fuzzy solutions was on average $2 - 8$ % and generally deviations of solutions of dense graphs were slightly smaller than deviations of solutions of sparse graphs. Solutions of $\mathcal{P}^0$ had overall much greater deviation, $10 - 17$ %. In some observations there were deviations over 25% in both sparse graphs and small, dense graphs. The largest deviations of the $\mathcal{P}^0$ and fuzzy solutions were generally over 30% and in some cases even over 60%. In sparse graphs about $70 - 90\%$ of observations had the deviation less than 10% between the $\mathcal{P}^1$ and fuzzy solution and in dense graphs at least 90% of observations had this kind of deviation between $\mathcal{P}^1$ and fuzzy solution. However, $\mathcal{P}^0$ solutions differed only on $20 - 50$ % of observations less than 10% from fuzzy solution. It is now clear that the $\mathcal{P}^1$ solution always differed from the fuzzy solution less than $\mathcal{P}^0$ solution.

About computational times

Problem $\mathcal{P}^1$ was always solved in a few minutes even with the large amount of nodes. Fuzzy problems and $\mathcal{P}^0$ problems were solved faster on sparse graphs than on dense graphs. Grid graphs are very sparse and therefore solving them took generally some seconds and even in the worst cases only minutes. Solving sparse randomly generated graphs, complete graphs having less than 12 nodes and complete bipartite graphs with less than 12 nodes also took only a few minutes. In the dense graphs only the graphs having 10 nodes or less were solved as quickly. Solving times for grid graphs with over 20 nodes and King's graphs, complete bipartite graphs and complete graphs with 12 nodes were dozens of minutes. With King's graphs having $16-18$ nodes, complete graphs and complete bipartite graphs having 14 nodes and randomly generated dense graphs having $12-14$ graphs the average computational times were over an hour. With these graphs solving fuzzy problem or $\mathcal{P}^0$ problem took in the worst cases even 20 hours.

6.2 Conclusions

Solving $\mathcal{P}^1$ problem took only a few minutes even in the largest graphs but solving the corresponding fuzzy problem might have taken hours. Furthermore, the solution of $\mathcal{P}^1$ mostly differed only a little from the fuzzy solution. If a slightly inaccurate solution is wanted, it is reasonable to use the solution of $\mathcal{P}^1$ as the fuzzy solution, especially when dealing with the larger graphs. In the smaller graphs the solution of $\mathcal{P}^1$ might also differ a lot from the fuzzy solution in some observations. Furthermore, solving small fuzzy problems did not take long. So if we want to obtain a definitely accurate solution on smaller graphs, we can solve them as fuzzy problems. The solution of $\mathcal{P}^0$ was in general worse than the solution of $\mathcal{P}^1$ and solving $\mathcal{P}^0$ problem also took sometimes even longer than solving the fuzzy problem. Therefore it is not recommendable to use the solution of $\mathcal{P}^0$ as the fuzzy solution.

Sparse graphs had generally more edges that belonged to all three solutions than dense graphs. On some observations there were also really large deviations between fuzzy and $\mathcal{P}^1$ solutions and between fuzzy and $\mathcal{P}^0$ solutions. In addition, solutions of sparse graphs were obtained in a short time so it is reasonable to solve them as fuzzy problems. In the dense graphs deviations were on average $2-8$ % and on over 90% of observations the deviations were less than 10%. Solving dense graphs as fuzzy problems also took a long time so using the solution of $\mathcal{P}^1$ as the fuzzy solution seems sensible.

On average the degree of necessary soft optimality $1-\lambda^*$ was always from range $[0.50; 0.55]$ and therefore λ^* was between 0.45 and 0.50. It might be reasonable to research whether the solution of $\mathcal{P}^{0.5}$ problem could be used as the solution of fuzzy problem. In addition it would be useful to test if

problems in that case can be solved more quickly and is the deviation of $\mathcal{P}^{0.5}$ and fuzzy much smaller than the deviation of $\mathcal{P}^1$ and fuzzy.

Edges that belonged to both $\mathcal{P}^1$ and $\mathcal{P}^0$ solutions were almost always included into the fuzzy solution. These edges appear to be rather robust and it might be interesting to research if these edges could be directly fixed into the fuzzy solution. After this operation this fixed fuzzy problem would be solved as usual. Especially we could focus on finding out if this modification would reduce solving times and if this modified solution would differ much from the solution of the original fuzzy problem.

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A   Summary table of results

Notations used in the following tables

- X_s = the solution of the fuzzy problem
- X_1 = the solution of $\mathcal{P}^1$ problem
- X_0 = the solution of $\mathcal{P}^0$ problem

- X_s = amount of edges only included into the fuzzy solution
- $X_s \cap X_1 \cap X_0$ = amount of edges included into all three solutions
- $X_1 \cap X_0$, not X_s = amount of edges included into both $\mathcal{P}^0$ and $\mathcal{P}^1$ solutions but not into the fuzzy solution
- $X_s \cap X_1$ = amount of edges included into both fuzzy and $\mathcal{P}^1$ solution
- $X_s \cap X_0$ = amount of edges included into both fuzzy and $\mathcal{P}^0$ solution

- SOFT = degree of necessary soft optimality $(1 - \lambda^*)$
- Rel1 = the deviation of $\mathcal{P}^1$ and fuzzy solution
- Rel2 = the deviation of $\mathcal{P}^0$ and fuzzy solution

Table 5. Grid graphs

size		only X_s	$X_s \cap X_1 \cap X_0$	$X_1 \cap X_0$, not X_s	$X_s \cap X_1$	$X_s \cap X_0$	SOFT	Rel1	Rel2
3x3 (50obs)	Minimums	0	4	0	6	5	0.418945	0.0 %	0.0 %
	Maximums	1	8	1	8	8	0.616211	51.7 %	72.0 %
	Means	0.04	6.38	0.1	7.34	7	0.5138672	7.3 %	15.9 %
	No. of zeros	48		45				23	14
	Observations < 5%							32	16
	Observations < 10%							37	19
4x3 (50obs)	Minimums	0	6	0	8	7	0.448242	0.0 %	0.0 %
	Maximums	1	11	1	11	11	0.573242	17.7 %	51.2 %
	Means	0.08	8.82	0.04	10.4	9.6	0.5075781	6.4 %	14.9 %
	No. of zeros	46		48				17	9
	Observations < 5%							25	15
	Observations < 10%							29	20
4x4 (50obs)	Minimums	0	9	0	11	11	0.457031	0.0 %	0.0 %
	Maximums	1	14	2	15	15	0.578125	30.5 %	37.3 %
	Means	0.12	11.9	0.2	13.74	13.04	0.52789068	6.1 %	13.2 %
	No. of zeros	44		41				11	3
	Observations < 5%							25	8
	Observations < 10%							41	19
6x3 (50obs)	Minimums	0	11	0	13	13	0.474609	0.0 %	0.0 %
	Maximums	2	17	2	17	17	0.599609	35.5 %	28.2 %
	Means	0.26	13.8	0.22	15.46	15.08	0.5207227	6.5 %	8.9 %
	No. of zeros	38		41				7	5
	Observations < 5%							27	19
	Observations < 10%							39	27
10x2 (50obs)	Minimums	0	12	0	14	15	0.453125	0.0 %	0.0 %
	Maximums	1	18	2	19	19	0.560547	46.6 %	41.9 %
	Means	0.02	15.54	0.12	17.34	17.18	0.5094727	7.7 %	10.2 %
	No. of zeros	49		45				7	3
	Observations < 5%							24	20
	Observations < 10%							39	29
5x4 (50obs)	Minimums	0	12	0	15	13	0.461914	0.0 %	0.0 %
	Maximums	1	18	2	19	19	0.581055	27.7 %	36.7 %
	Means	0.28	14.94	0.36	17.38	16.28	0.53218758	6.5 %	15.3 %
	No. of zeros	36		35				8	3
	Observations < 5%							21	9
	Observations < 10%							34	15
7x3 (50obs)	Minimums	0	11	0	15	14	0.447266	0.0 %	0.0 %
	Maximums	1	19	2	20	20	0.581055	22.0 %	35.8 %
	Means	0.16	15.3	0.32	18.26	16.88	0.5163868	5.5 %	13.9 %
	No. of zeros	42		35				8	2
	Observations < 5%							28	5
	Observations < 10%							42	18
6x4 (50obs)	Minimums	0	12	0	18	16	0.474609	0.0 %	0.0 %
	Maximums	1	21	2	23	23	0.581055	19.4 %	29.3 %
	Means	0.24	17.86	0.34	20.94	19.68	0.5241015	7.2 %	14.4 %
	No. of zeros	38		35				6	2
	Observations < 5%							19	7
	Observations < 10%							38	16
8x3 (50obs)	Minimums	0	13	0	17	16	0.479492	0.0 %	0.0 %
	Maximums	2	21	3	23	23	0. 570313	26.9 %	51.6 %
	Means	0.26	17.66	0.4	20.84	19.56	0.52281256	6.3 %	14.3 %
	No. of zeros	38		33				5	2
	Observations < 5%							22	8
	Observations < 10%							40	19

Table 6. King's graphs

size		only X_s	$X_s \cap X_1 \cap X_0$	$X_1 \cap X_0$, not X_s	$X_s \cap X_1$	$X_s \cap X_0$	SOFT	Rel1	Rel2
3x3 (50obs)	Minimums	0	3	0	5	4	0.444336	0.0 %	0.0 %
	Maximums	1	8	2	8	8	0.603516	22.1 %	44.4 %
	Means	0.28	5.34	0.24	6.96	6.1	0.538242	5.1 %	13.2 %
	No. of zeros	36		40				18	7
	Observations $<$ 5%							30	15
	Observations $<$ 10%							41	22
3x4 (50obs)	Minimums	0	4	0	7	6	0.469727	0.0 %	0.0 %
	Maximums	2	11	2	11	11	0.609375	15.2 %	46.8 %
	Means	0.36	7.1	0.24	9.6	8.14	0.534668	4.1 %	14.8 %
	No. of zeros	33		39				9	3
	Observations $<$ 5%							32	9
	Observations $<$ 10%							45	19
8x2 (50obs)	Minimums	0	5	0	10	8	0.483398	0.0 %	0.0 %
	Maximums	2	15	3	15	15	0.597656	15.0 %	34.3 %
	Means	0.4	9.62	0.44	12.84	11.38	0.536094	5.0 %	13.7 %
	No. of zeros	35		34				2	1
	Observations $<$ 5%							26	9
	Observations $<$ 10%							46	21
4x4 (50obs)	Minimums	0	6	0	11	8	0.491211	0.0 %	0.4 %
	Maximums	2	14	1	15	14	0.598633	13.3 %	30.5 %
	Means	0.54	9.56	0.4	12.96	11.06	0.538398	5.1 %	14.8 %
	No. of zeros	25		30				4	0
	Observations $<$ 5%							28	5
	Observations $<$ 10%							44	13
6x3 (30obs)	Minimums	0	8	0	12	9	0.503906	0.0 %	0.0 %
	Maximums	2	14	2	17	17	0.585938	17.0 %	40.3 %
	Means	0.63	11.73	0.4	14.67	13.43	0.541504	4.9 %	11.5 %
	No. of zeros	14		20				2	2
	Observations $<$ 5%							22	7
	Observations $<$ 10%							24	14

Table 7. Complete bipartite graphs

size		only X_s	$X_s \cap X_1 \cap X_0$	$X_1 \cap X_0$, not X_s	$X_s \cap X_1$	$X_s \cap X_0$	SOFT	Rel1	Rel2
4x4	Minimums	0	3	0	4	3	0.447266	0.0 %	0.0 %
(50obs)	Maximums	2	7	2	7	7	0.651367	41.6 %	62.6 %
	Means	0.24	4.68	0.16	6.16	5.28	0.5370704	5.7 %	14.6 %
	No. of zeros	39		43				21	8
	Observations < 5%							34	14
	Observations < 10%							39	25
6x3	Minimums	0	3	0	5	4	0.453125	0.0 %	0.0 %
(50obs)	Maximums	2	8	1	8	8	0.611328	24.0 %	38.9 %
	Means	0.24	5.22	0.26	6.98	6	0.5418946	5.4 %	14.9 %
	No. of zeros	39		37				13	5
	Observations < 5%							30	8
	Observations < 10%							40	18
5x5	Minimums	0	3	0	6	5	0.492188	0.0 %	0.0 %
(50obs)	Maximums	2	8	2	9	9	0.591797	10.3 %	42.8 %
	Means	0.44	6.06	0.28	8	6.62	0.5461524	3.2 %	14.1 %
	No. of zeros	30		37				13	1
	Observations < 5%							35	11
	Observations < 10%							49	23
6x5	Minimums	0	4	0	7	4	0.496094	0.0 %	0.0 %
(50obs)	Maximums	2	10	2	10	10	0.602539	18.2 %	32.2 %
	Means	0.46	6.5	0.24	8.7	7.34	0.5476367	4.4 %	12.7 %
	No. of zeros	30		39				11	1
	Observations < 5%							31	11
	Observations < 10%							46	19
8x4	Minimums	0	3	0	8	5	0.483398	0.0 %	0.0 %
(50obs)	Maximums	3	10	2	11	11	0.589844	16.5 %	33.3 %
	Means	0.5	7.1	0.26	9.5	8.1	0.54002	4.2 %	13.7 %
	No. of zeros	31		38				9	1
	Observations < 5%							34	5
	Observations < 10%							46	22
6x6	Minimums	0	3	0	7	5	0.479492	0.0 %	0.0 %
(50obs)	Maximums	2	10	2	11	11	0.602539	11.5 %	30.8 %
	Means	0.6	6.84	0.3	9.22	8.02	0.541836	4.4 %	12.4 %
	No. of zeros	25		37				5	1
	Observations < 5%							29	10
	Observations < 10%							47	22
8x5	Minimums	0	5	0	8	6	0.512695	0.0 %	0.0 %
(50obs)	Maximums	2	10	3	12	12	0.586914	11.6 %	33.8 %
	Means	0.5	7.48	0.42	10.12	8.86	0.544336	5.0 %	11.8 %
	No. of zeros	28		32				2	2
	Observations < 5%							28	10
	Observations < 10%							43	23
10x4	Minimums	0	4	0	6	6	0.499023	0.0 %	0.4 %
(30obs)	Maximums	2	11	2	13	12	0.582031	17.8 %	28.7 %
	Means	0.53	8.37	0.33	11	9.83	0.543945	6.2 %	12.5 %
	No. of zeros	17		21				3	0
	Observations < 5%							15	5
	Observations < 10%							24	13
8x6	Minimums	0	4	0	8	6	0.47168	0.0 %	1.6 %
(30obs)	Maximums	3	10	1	13	12	0.577148	13.6 %	31.6 %
	Means	0.7	8.07	0.27	11	9.37	0.54121093	4.4 %	12.6 %
	No. of zeros	15		22				1	0
	Observations < 5%							18	4
	Observations < 10%							26	12

Table 8. Complete graphs

size		only X_s	$X_s \cap X_1 \cap X_0$	$X_1 \cap X_{0,\text{ not }} X_s$	$X_s \cap X_1$	$X_s \cap X_0$	SOFT	Rel1	Rel2
6	Minimums	0	0	0	3	1	0.470703	0.0 %	0.0 %
(50obs)	Maximums	1	5	1	5	5	0.631836	16.4 %	47.1 %
	Means	0.24	3.02	0.18	4.38	3.4	0.53863284	3.5 %	14.0 %
	No. of zeros	38		41				22	8
	Observations < 5%							34	18
	Observations < 10%							44	21
8	Minimums	0	1	0	4	3	0.456055	0.0 %	0.0 %
(50obs)	Maximums	2	7	2	7	7	0.597656	16.9 %	30.1 %
	Means	0.2	4.12	0.16	5.94	4.98	0.53802732	3.5 %	14.0 %
	No. of zeros	42		43				18	4
	Observations < 5%							35	18
	Observations < 10%							44	23
10	Minimums	0	2	0	5	3	0.460938	0.0 %	0.4 %
(50obs)	Maximums	3	8	1	9	8	0.577148	11.3 %	27.0 %
	Means	0.38	4.96	0.2	7.34	6.24	0.52767582	4.0 %	10.6 %
	No. of zeros	34		40				7	0
	Observations < 5%							32	8
	Observations < 10%							47	26
12	Minimums	0	2	0	6	3	0.460938	0.0 %	0.0 %
(50obs)	Maximums	2	9	1	11	11	0.564453	11.3 %	31.4 %
	Means	0.56	6.16	0.18	8.98	7.62	0.51732422	4.2 %	9.7 %
	No. of zeros	27		41				5	1
	Observations < 5%							30	14
	Observations < 10%							48	28
14	Minimums	0	5	0	7	7	0.447266	0.0 %	0.0 %
(50obs)	Maximums	3	11	2	13	13	0.551758	15.8 %	22.6 %
	Means	0.46	7.42	0.06	10.5	9.46	0.503011167	4.5 %	9.7 %
	No. of zeros	32		46				2	1
	Observations < 5%							32	11
	Observations < 10%							44	26

Table 9. Sparse graphs with 8 nodes

number of edges		only X_s	$X_s \cap X_1 \cap X_0$	$X_1 \cap X_0$, not X_s	$X_s \cap X_1$	$X_s \cap X_0$	SOFT	Rel1	Rel2
Case 1	Minimums	0	2	0	5	3	0.484375	0 %	0 %
Edges 15	Maximums	1	7	2	7	7	0.594727	24 %	29 %
(20 obs)	Means	0.05	4.6	0.25	6.35	5.2	0.52861315	3 %	14 %
	No. of zeros	19		16				10	4
	Observations < 5 %							16	5
	Observations < 10 %							19	6
Case 2	Minimums	0	3	0	5	4	0.47168	0 %	0 %
Edges 15	Maximums	1	6	1	7	7	0.629883	24 %	50 %
(20 obs)	Means	0.1	4.45	0.2	5.8	5.55	0.5330567	7 %	14 %
	No. of zeros	18		16				1	4
	Observations < 5 %							9	6
	Observations < 10 %							16	9
Case 3	Minimums	0	4	0	5	4	0.445313	0 %	0 %
Edges 14	Maximums	1	7	0	7	7	0.633789	57 %	44 %
(20 obs)	Means	0.15	5.15	0	6.3	5.7	0.53051765	8 %	13 %
	No. of zeros	17		20				9	4
	Observations < 5 %							13	8
	Observations < 10 %							15	9
Case 4	Minimums	0	2	0	5	3	0.453125	0 %	0 %
Edges 15	Maximums	1	6	1	7	7	0.566406	43 %	72 %
(20 obs)	Means	0.1	4.65	0.15	5.9	5.65	0.52846695	8 %	14 %
	No. of zeros	18		17				3	5
	Observations < 5 %							10	8
	Observations < 10 %							14	10
Case 5	Minimums	0	4	0	4	4	0.483398	0 %	0 %
Edges 14	Maximums	1	7	1	7	7	0.664063	21 %	49 %
(20 obs)	Means	0.15	5.15	0.1	6.25	5.75	0.5380371	6 %	13 %
	No. of zeros	17		18				9	5
	Observations < 5 %							9	5
	Observations < 10 %							14	9
Case 6	Minimums	0	4	0	5	4	0.46582	0.0 %	0 %
Edges 14	Maximums	1	7	1	7	7	0.643555	21 %	52 %
(20 obs)	Means	0.15	5.25	0.05	6.5	5.6	0.5389648	3 %	12 %
	No. of zeros	17		19				12	3
	Observations < 5 %							15	7
	Observations < 10 %							17	10
Case 7	Minimums	0	3	0	4	3	0.474609	0 %	0 %
Edges 15	Maximums	3	6	2	7	7	0.59375	20 %	49 %
(20 obs)	Means	0.3	4.7	0.15	6.25	5.15	0.53642585	4 %	20 %
	No. of zeros	16		18				10	2
	Observations < 5 %							14	3
	Observations < 10 %							17	5
Case 8	Minimums	0	3	0	5	4	0.482422	0 %	0 %
Edges 14	Maximums	1	7	1	7	7	0.612305	21 %	50 %
(20 obs)	Means	0.05	5.25	0.1	6.35	5.85	0.5444826	5 %	16 %
	No. of zeros	19		18				10	7
	Observations < 5 %							12	9
	Observations < 10 %							15	9
Case 9	Minimums	0	4	0	4	4	0.455078	0 %	0 %
Edges 15	Maximums	1	7	2	7	7	0.617188	25 %	44 %
(20 obs)	Means	0.05	4.85	0.15	6.3	5.5	0.5195801	4 %	16 %
	No. of zeros	19		18				11	3
	Observations < 5 %							12	6
	Observations < 10 %							16	8
Case 10	Minimums	0	2	0	4	3	0.473633	0 %	0 %
Edges 14	Maximums	1	7	1	7	7	0.582031	12 %	54 %
(20 obs)	Means	0.1	4.6	0.2	6.1	5.4	0.52529295	4 %	13 %
	No. of zeros	18		16				8	2
	Observations < 5 %							13	7
	Observations < 10 %							19	11

Table 10. Dense graphs with 8 nodes

number of edges		only X_s	$X_s \cap X_1 \cap X_0$	$X_1 \cap X_0$, not X_s	$X_s \cap X_1$	$X_s \cap X_0$	SOFT	Rel1	Rel2
Case 1	Minimums	0	2	0	5	3	0.495117	0 %	0 %
Edges 23	Maximums	2	6	1	7	7	0.589844	15 %	46 %
(20 obs)	Means	0.4	4	0.25	5.8	4.8	0.5324219	5 %	15 %
	No. of zeros	13		15				4	1
	Observations < 5 %							14	2
	Observations < 10 %							16	6
Case 2	Minimums	0	2	0	5	4	0.507813	0 %	0 %
Edges 21	Maximums	1	6	1	7	7	0.591797	13 %	32 %
(20 obs)	Means	0.2	4.7	0.1	6.15	5.35	0.5509766	4 %	13 %
	No. of zeros	16		18				6	2
	Observations < 5 %							13	5
	Observations < 10 %							17	7
Case 3	Minimums	0	2	0	5	3	0.473633	0 %	0 %
Edges 22	Maximums	2	6	1	7	7	0.600586	11 %	45 %
(20 obs)	Means	0.35	4.45	0.15	6	5.1	0.5409668	4 %	14 %
	No. of zeros	14		17				5	3
	Observations < 5 %							13	4
	Observations < 10 %							18	9
Case 4	Minimums	0	2	0	4	3	0.501953	0 %	1 %
Edges 22	Maximums	1	6	1	7	6	0.567383	18 %	33 %
(20 obs)	Means	0.2	3.6	0.2	5.75	4.65	0.5339844	5 %	14 %
	No. of zeros	16		16				2	0
	Observations < 5 %							10	3
	Observations < 10 %							19	5
Case 5	Minimums	0	2	0	4	3	0.505859	0 %	0 %
Edges 20	Maximums	2	7	1	7	7	0.6689945	27 %	45 %
(20 obs)	Means	0.25	4.35	0.2	6.1	5	0.55805655	5 %	14 %
	No. of zeros	16		16				7	3
	Observations < 5 %							13	7
	Observations < 10 %							16	8
Case 6	Minimums	0	3	0	5	4	0.496094	0 %	0 %
Edges 20	Maximums	1	7	1	7	7	0.597656	11 %	31 %
(20 obs)	Means	0.15	5.1	0.05	6.15	5.8	0.54824225	5 %	10 %
	No. of zeros	17		19				7	5
	Observations < 5 %							9	8
	Observations < 10 %							17	12
Case 7	Minimums	0	2	0	4	3	0.473633	0 %	0 %
Edges 20	Maximums	2	6	1	7	7	0.600586	10 %	39 %
(20 obs)	Means	0.35	3.8	0.25	5.95	4.5	0.53129885	4 %	18 %
	No. of zeros	14		15				4	1
	Observations < 5 %							15	4
	Observations < 10 %							20	7
Case 8	Minimums	0	2	0	5	2	0.475586	0 %	0 %
Edges 20	Maximums	2	7	2	7	7	0.589844	19 %	41 %
(20 obs)	Means	0.4	4.4	0.2	5.9	5.1	0.54184585	5 %	17 %
	No. of zeros	13		17				4	3
	Observations < 5 %							11	5
	Observations < 10 %							17	8
Case 9	Minimums	0	1	0	5	3	0.511719	0 %	0 %
Edges 20	Maximums	1	7	1	7	7	0.586914	13 %	37 %
(20 obs)	Means	0.15	4.55	0.15	6.35	5.05	0.5448732	2 %	13 %
	No. of zeros	17		17				11	2
	Observations < 5 %							17	4
	Observations < 10 %							19	9
Case 10	Minimums	0	2	0	5	3	0.46875	0 %	0 %
Edges 23	Maximums	1	7	1	7	7	0.615234	22 %	40 %
(20 obs)	Means	0.2	4.5	0.1	6.1	5.2	0.54609375	5 %	16 %
	No. of zeros	16		18				5	1
	Observations < 5 %							13	4
	Observations < 10 %							17	6

Table 11. Sparse graphs with 10 nodes

number of edges		only X_s	$X_s \cap X_1 \cap X_0$	$X_1 \cap X_0$	$X_1 \cap X_0$, not X_s	$X_s \cap X_1$	$X_s \cap X_0$	SOFT	Rel1	Rel2
Case 1 Edges 19 (20 obs)	Minimums	0	4		0	6	4	0.470703	0 %	0 %
	Maximums	1	8		1	9	9	0.574219	22 %	30 %
	Means	0.35	6.2		0.2	8.05	6.8	0.5291993	6 %	15 %
	No. of zeros	13			16				7	1
	Observations < 5 %								11	5
	Observations < 10 %								15	7
Case 2 Edges 19 (20 obs)	Minimums	0	4		0	7	5	0.504883	0 %	0 %
	Maximums	1	8		2	9	9	0.567383	30 %	51 %
	Means	0.15	6.2		0.2	7.65	7.4	0.53872075	9 %	11 %
	No. of zeros	17			17				1	4
	Observations < 5 %								7	7
	Observations < 10 %								11	12
Case 3 Edges 20	Minimums	0	3		0	6	5	0.487305	0 %	3 %
	Maximums	1	8		2	9	8	0.586914	18 %	36 %
	Means	0.4	5.4		0.6	7.7	6.3	0.53798835	6 %	17 %
	No. of zeros	12			10				4	0
	Observations < 5 %								10	2
	Observations < 10 %								16	6
Case 4 Edges 19 (20 obs)	Minimums	0	3		0	7	4	0.464844	0 %	0 %
	Maximums	2	8		2	9	9	0.581055	24 %	42 %
	Means	0.25	5.95		0.2	7.95	6.75	0.5194825	6 %	14 %
	No. of zeros	16			17				7	2
	Observations < 5 %								10	6
	Observations < 10 %								16	8
Case 5 Edges 18 (20 obs)	Minimums	0	4		0	6	5	0.452148	0 %	0 %
	Maximums	1	9		1	9	9	0.596668	30 %	46 %
	Means	0.1	6.1		0.25	7.8	7.2	0.5410645	7 %	16 %
	No. of zeros	18			15				7	3
	Observations < 5 %								10	7
	Observations < 10 %								15	10
Case 6 Edges 19 (20 obs)	Minimums	0	4		0	6	5	0.467773	0 %	0 %
	Maximums	2	8		1	9	9	0.571289	13 %	34 %
	Means	0.25	6.2		0.1	7.95	7	0.52524425	5 %	17 %
	No. of zeros	16			18				7	1
	Observations < 5 %								11	3
	Observations < 10 %								16	4
Case 7 Edges 17 (20 obs)	Minimums	0	4		0	6	5	0.488281	0 %	0 %
	Maximums	1	9		1	9	9	0.574219	12 %	36 %
	Means	0.15	6.4		0.2	8.2	7.05	0.5302735	3 %	13 %
	No. of zeros	17			16				7	3
	Observations < 5 %								13	8
	Observations < 10 %								19	8
Case 8 Edges 18 (20 obs)	Minimums	0	4		0	6	6	0.492188	0 %	0 %
	Maximums	1	9		1	9	9	0.606445	22 %	26 %
	Means	0.25	6.55		0.3	7.75	7.55	0.54375005	7 %	10 %
	No. of zeros	15			14				4	4
	Observations < 5 %								10	8
	Observations < 10 %								14	12
Case 9 Edges 20 (20 obs)	Minimums	0	4		0	6	5	0.474609	0 %	0 %
	Maximums	2	9		3	9	9	0.588867	11 %	29 %
	Means	0.2	6.1		0.25	8.15	6.75	0.53178705	4 %	14 %
	No. of zeros	17			18				9	2
	Observations < 5 %								11	3
	Observations < 10 %								17	5
Case 10 Edges 21 (20 obs)	Minimums	0	3		0	6	5	0.459961	0 %	0 %
	Maximums	1	9		2	9	9	0.597656	35 %	38 %
	Means	0.4	6.05		0.35	7.75	6.9	0.5473633	5 %	14 %
	No. of zeros	12			14				6	2
	Observations < 5 %								15	5
	Observations < 10 %								17	11

Table 12. Dense graphs with 10 nodes

number of edges		only X_s	$X_s \cap X_1 \cap X_0$	$X_1 \cap X_0$, not X_s	$X_s \cap X_1$	$X_s \cap X_0$	SOFT	Rel1	Rel2
Case 1 Edges 32 (20 obs)	Minimums	0	1	0	5	3	0.492188	0 %	3 %
	Maximums	2	7	2	9	8	0.597656	7 %	27 %
	Means	0.4	5.35	0.2	7.8	6.15	0.54409195	2 %	14 %
	No. of zeros	13		17				8	0
	Observations < 5 %							17	1
	Observations < 10 %							20	7
Case 2 Edges 31 (20 obs)	Minimums	0	3	0	6	3	0.504883	0 %	3 %
	Maximums	2	8	1	9	8	0.607422	9 %	52 %
	Means	0.55	5.3	0.15	7.8	5.95	0.54692395	2 %	17 %
	No. of zeros	11		17				5	0
	Observations < 5 %							18	1
	Observations < 10 %							20	9
Case 3 Edges 34 (20 obs)	Minimums	0	4	0	6	4	0.511719	0 %	0 %
	Maximums	2	9	1	9	9	0.571289	12 %	39 %
	Means	0.4	5.8	0.25	7.6	6.8	0.54340825	5 %	11 %
	No. of zeros	13		15				2	1
	Observations < 5 %							13	5
	Observations < 10 %							19	12
Case 4 Edges 32 (20 obs)	Minimums	0	4	0	6	4	0.481445	0 %	0 %
	Maximums	2	8	1	9	9	0.575195	16 %	39 %
	Means	0.3	5.45	0.15	7.9	6.25	0.53256845	3 %	10 %
	No. of zeros	15		17				7	2
	Observations < 5 %							17	7
	Observations < 10 %							19	12
Case 5 Edges 34 (20 obs)	Minimums	0	3	0	5	5	0.50293	0 %	0 %
	Maximums	1	8	2	9	9	0.572266	18 %	34 %
	Means	0.4	5.25	0.15	7	6.85	0.54419434	6 %	11 %
	No. of zeros	12		18				1	1
	Observations < 5 %							10	5
	Observations < 10 %							17	11
Case 6 Edges 35 (20 obs)	Minimums	0	2	0	5	3	0.49707	0 %	1 %
	Maximums	2	7	1	9	8	0.580078	14 %	34 %
	Means	0.45	5.2	0.15	7.65	6.1	0.5389649	4 %	15 %
	No. of zeros	12		17				7	0
	Observations < 5 %							10	3
	Observations < 10 %							19	6
Case 7 Edges 30 (20 obs)	Minimums	0	3	0	5	4	0.510742	0 %	3 %
	Maximums	2	7	2	9	8	0.595703	14 %	42 %
	Means	0.3	5.15	0.3	7.6	6.25	0.54311525	4 %	15 %
	No. of zeros	15		15				3	0
	Observations < 5 %							14	2
	Observations < 10 %							19	7
Case 8 Edges 34 (20 obs)	Minimums	0	3	0	6	4	0.499023	0 %	0.2 %
	Maximums	2	8	2	9	8	0.581055	12 %	30 %
	Means	0.4	5.85	0.45	7.8	6.65	0.54687505	3 %	12 %
	No. of zeros	14		13				4	0
	Observations < 5 %							15	5
	Observations < 10 %							18	11
Case 9 Edges 32 (20 obs)	Minimums	0	3	0	5	4	0.493164	0 %	0 %
	Maximums	2	7	1	9	9	0.582031	13 %	39 %
	Means	0.55	5.6	0.45	7.45	6.6	0.54243165	3 %	12 %
	No. of zeros	10		11				3	1
	Observations < 5 %							15	4
	Observations < 10 %							19	8
Case 10 Edges 33 (20 obs)	Minimums	0	2	0	5	4	0.481445	0 %	3 %
	Maximums	3	7	1	9	8	0.578125	13 %	31 %
	Means	0.5	5.15	0.2	7.2	6.45	0.53554685	6 %	11 %
	No. of zeros	13		16				1	0
	Observations < 5 %							7	4
	Observations < 10 %							19	11

Table 13. Sparse graphs with 12 nodes

number of edges		only X_s	$X_s \cap X_1 \cap X_0$	$X_1 \cap X_0$	$X_1 \cap X_0$, not X_s	$X_s \cap X_1$	$X_s \cap X_0$	SOFT	Rel1	Rel2
Case 1	Minimums	0	5		0	8	7	0.461914	0 %	0 %
Edges 21	Maximums	1	10		1	11	11	0.59082	15 %	39 %
(20 obs)	Means	0.2	8.05		0.15	9.95	8.9	0.5318848	6 %	13 %
	No. of zeros	16			17				6	1
	Observations < 5 %								11	5
	Observations < 10 %								14	11
Case 2	Minimums	0	5		0	8	6	0.494141	0 %	0 %
Edges 25	Maximums	1	9		2	11	11	0.613281	14 %	49 %
(20 obs)	Means	0.25	7.3		0.2	9.6	8.45	0.54389665	5 %	17 %
	No. of zeros	15			17				3	1
	Observations < 5 %								11	2
	Observations < 10 %								15	5
Case 3	Minimums	0	4		0	6	6	0.470703	0 %	2 %
Edges 22	Maximums	1	10		1	11	10	0.602539	19 %	35 %
(20 obs)	Means	0.25	7.45		0.2	9.55	8.65	0.5362793	6 %	17 %
	No. of zeros	15			16				3	0
	Observations < 5 %								11	3
	Observations < 10 %								16	5
Case 4	Minimums	0	5		0	8	7	0.488281	0 %	0 %
Edges 22	Maximums	1	10		1	11	11	0.614258	16 %	45 %
(20 obs)	Means	0.35	7.7		0.2	9.75	8.6	0.5374024	5 %	16 %
	No. of zeros	13			16				4	1
	Observations < 5 %								12	3
	Observations < 10 %								16	8
Case 5	Minimums	0	5		0	8	5	0.491211	0 %	0 %
Edges 23	Maximums	2	9		2	11	11	0.607422	28 %	36 %
(20 obs)	Means	0.45	7.75		0.3	9.65	8.65	0.532666	7 %	15 %
	No. of zeros	14			15				2	3
	Observations < 5 %								11	5
	Observations < 10 %								14	9
Case 6	Minimums	0	5		0	8	6	0.477539	0 %	0 %
Edges 23	Maximums	2	9		1	11	11	0.564453	21 %	31 %
(20 obs)	Means	0.4	7.4		0.2	9.7	8.3	0.5278808	6 %	16 %
	No. of zeros	14			16				5	1
	Observations < 5 %								10	4
	Observations < 10 %								14	6
Case 7	Minimums	0	5		0	7	8	0.473633	0 %	0 %
Edges 21	Maximums	1	10		2	11	11	0.592773	19 %	26 %
(20 obs)	Means	0.3	8.45		0.25	9.85	9.3	0.5338379	4 %	9 %
	No. of zeros	14			16				5	1
	Observations < 5 %								14	6
	Observations < 10 %								18	13
Case 8	Minimums	0	6		0	8	7	0.483398	0 %	2 %
Edges 22	Maximums	2	10		1	11	10	0.576172	27 %	35 %
(20 obs)	Means	0.4	7.55		0.25	9.6	8.55	0.53417965	8 %	16 %
	No. of zeros	13			15				4	0
	Observations < 5 %								6	5
	Observations < 10 %								15	6
Case 9	Minimums	0	4		0	8	5	0.507813	0 %	0 %
Edges 24	Maximums	1	10		1	11	11	0.588867	19 %	48 %
(20 obs)	Means	0.3	7.4		0.25	9.55	8.55	0.5418946	7 %	16 %
	No. of zeros	14			15				1	2
	Observations < 5 %								7	5
	Observations < 10 %								13	8
Case 10	Minimums	0	6		0	8	7	0.46875	0 %	0 %
Edges 25	Maximums	1	11		1	11	11	0.606445	18 %	26 %
(20 obs)	Means	0.35	7.6		0.2	9.65	8.6	0.5355469	4 %	12 %
	No. of zeros	13			16				5	1
	Observations < 5 %								15	4
	Observations < 10 %								18	10

Table 14. Dense graphs with 12 nodes

number of edges		only X_s	$X_s \cap X_1 \cap X_0$	$X_1 \cap X_{0,\,\text{not }X_s}$	$X_s \cap X_1$	$X_s \cap X_0$	SOFT	Rel1	Rel2
Case 1	Minimums	0	3	0	8	4	0.503906	0 %	3 %
Edges 43	Maximums	1	9	1	11	9	0.59082	12 %	27 %
(10 obs)	Means	0.7	6.3	0.2	9.2	7.4	0.5322264	4 %	10 %
	No. of zeros	3		8				1	0
	Observations < 5 %							5	4
	Observations < 10 %							9	7
Case 2	Minimums	0	5	0	8	6	0.5	0.4 %	1 %
Edges 44	Maximums	2	8	1	10	10	0.563477	12 %	31 %
(10 obs)	Means	0.6	6.5	0.3	8.9	8	0.5269533	4 %	13 %
	No. of zeros	5		7				0	0
	Observations < 5 %							7	3
	Observations < 10 %							8	4
Case 3	Minimums	0	4	0	6	4	0.495117	0 %	5 %
Edges 43	Maximums	2	8	2	11	9	0.564453	10 %	20 %
(10 obs)	Means	1	5.9	0.2	9.1	6.8	0.5359375	4 %	15 %
	No. of zeros	3		9				1	0
	Observations < 5 %							7	0
	Observations < 10 %							10	1
Case 4	Minimums	0	4	0	7	5	0.452148	0 %	9 %
Edges 51	Maximums	3	8	0	11	9	0.570313	5 %	18 %
(10 obs)	Means	0.7	5.9	0	9.2	7	0.5183594	3 %	13 %
	No. of zeros	5		10				1	0
	Observations < 5 %							9	0
	Observations < 10 %							10	1
Case 5	Minimums	0	5	0	7	6	0.530273	1 %	6 %
Edges 44	Maximums	2	8	1	10	10	0.55957	7 %	28 %
(10 obs)	Means	0.7	6.5	0.2	9.1	7.7	0.5444334	4 %	13 %
	No. of zeros	4		7				0	0
	Observations < 5 %							8	0
	Observations < 10 %							10	3
Case 6	Minimums	0	5	0	8	6	0.490234	0 %	1 %
Edges 46	Maximums	1	10	1	11	10	0.595703	26 %	31 %
(10 obs)	Means	0.5	7	0.1	9.7	7.8	0.5458983	4 %	13 %
	No. of zeros	5		9				2	0
	Observations < 5 %							9	2
	Observations < 10 %							9	4
Case 7	Minimums	0	4	0	8	6	0.517578	0 %	5 %
Edges 46	Maximums	2	9	2	11	9	0.578125	5 %	30 %
(10 obs)	Means	0.3	6.9	0.3	10.1	7.5	0.5454103	2 %	13 %
	No. of zeros	8		8				5	0
	Observations < 5 %							9	0
	Observations < 10 %							10	2
Case 8	Minimums	0	4	0	8	6	0.50293	0 %	2 %
Edges 47	Maximums	0	8	1	11	10	0.56543	9 %	32 %
(10 obs)	Means	0	6.4	0.1	9.5	7.9	0.5291992	3 %	14 %
	No. of zeros	10		9				2	0
	Observations < 5 %							9	2
	Observations < 10 %							10	4
Case 9	Minimums	0	5	0	7	6	0.521484	0 %	5 %
Edges 44	Maximums	1	7	1	11	9	0.572266	12 %	20 %
(10 obs)	Means	0.5	6.6	0.1	9.3	7.8	0.5439453	4 %	12 %
	No. of zeros	5		9				1	0
	Observations < 5 %							8	0
	Observations < 10 %							9	5
Case 10	Minimums	0	5	0	8	6	0.514648	0 %	0.05 %
Edges 43	Maximums	1	9	1	11	10	0.558594	7 %	27 %
(10 obs)	Means	0.3	6.9	0.1	9.6	8	0.538672	3 %	14 %
	No. of zeros	7		9				1	0
	Observations < 5 %							8	2
	Observations < 10 %							10	3

Table 15. Sparse graphs with 14 nodes

number of edges		only X_s	$X_s \cap X_1 \cap X_0$	$X_1 \cap X_0$	$X_1 \cap X_0$, not X_s	$X_s \cap X_1$	$X_s \cap X_0$	SOFT	Rel1	Rel2
Case 1	Minimums	0	8		0	9	8	0.476563	0 %	0 %
Edges 24	Maximums	2	11		2	13	13	0.574219	12 %	35 %
(20 obs)	Means	0.2	9.25		0.3	11.35	10.7	0.51899435	5 %	14 %
	No. of zeros	17			17				3	3
	Observations < 5 %								9	4
	Observations < 10 %								17	7
Case 2	Minimums	0	8		0	10	9	0.484375	0 %	2 %
Edges 26	Maximums	2	12		1	13	12	0.564453	13 %	39 %
(20 obs)	Means	0.45	9.45		0.4	11.75	10.25	0.5319825	5 %	15 %
	No. of zeros	13			12				4	0
	Observations < 5 %								9	5
	Observations < 10 %								16	7
Case 3	Minimums	0	6		0	10	7	0.499023	0 %	3 %
Edges 24	Maximums	2	11		1	13	12	0.571289	22 %	48 %
(20 obs)	Means	0.3	9.55		0.2	11.5	10.75	0.54350585	7 %	17 %
	No. of zeros	16			16				2	0
	Observations < 5 %								8	6
	Observations < 10 %								15	8
Case 4	Minimums	0	7		0	11	8	0.490234	0 %	0 %
Edges 27	Maximums	1	12		1	13	13	0.612305	15 %	45 %
(20 obs)	Means	0.15	9.7		0.05	12	10.55	0.5509765	6 %	16 %
	No. of zeros	17			19				5	1
	Observations < 5 %								8	3
	Observations < 10 %								16	8
Case 5	Minimums	0	7		0	9	8	0.485352	0 %	0 %
Edges 27	Maximums	2	12		2	13	13	0.601563	14 %	41 %
(20 obs)	Means	0.4	9.35		0.4	11.45	10.5	0.5335938	5 %	13 %
	No. of zeros	13			14				5	1
	Observations < 5 %								10	3
	Observations < 10 %								15	5
Case 6	Minimums	0	6		0	10	8	0.484375	0 %	1 %
Edges 25	Maximums	2	12		2	13	12	0.581055	21 %	29 %
(20 obs)	Means	0.3	9.55		0.3	11.7	10.55	0.5348634	5 %	12 %
	No. of zeros	15			15				2	0
	Observations < 5 %								11	4
	Observations < 10 %								17	10
Case 7	Minimums	0	8		0	10	10	0.46875	0 %	0 %
Edges 24	Maximums	2	12		2	13	13	0.59082	20 %	42 %
(20 obs)	Means	0.5	9.9		0.25	11.4	11	0.53403325	7 %	13 %
	No. of zeros	12			16				4	1
	Observations < 5 %								7	5
	Observations < 10 %								13	12
Case 8	Minimums	0	8		0	10	8	0.488281	0 %	0 %
Edges 24	Maximums	2	11		1	13	12	0.581055	18 %	30 %
(20 obs)	Means	0.15	9.8		0.2	11.75	10.9	0.53339845	7 %	12 %
	No. of zeros	18			16				4	1
	Observations < 5 %								7	6
	Observations < 10 %								14	10
Case 9	Minimums	0	8		0	9	9	0.499023	0 %	0 %
Edges 25	Maximums	1	11		1	13	13	0.568359	13 %	35 %
(20 obs)	Means	0.25	9.7		0.25	11.75	10.7	0.5294434	6 %	13 %
	No. of zeros	15			15				3	2
	Observations < 5 %								8	3
	Observations < 10 %								16	7
Case 10	Minimums	0	7		0	10	9	0.474609	0 %	0 %
Edges 26	Maximums	1	13		2	13	13	0.614258	22 %	42 %
(20 obs)	Means	0.25	9.55		0.1	11.55	10.75	0.54116215	7 %	14 %
	No. of zeros	15			19				5	3
	Observations < 5 %								10	5
	Observations < 10 %								15	9

Table 16. Dense graphs with 14 nodes

number of edges		only X_s	$X_s \cap X_1 \cap X_0$	$X_1 \cap X_0$	$X_1 \cap X_0$, not X_s	$X_s \cap X_1$	$X_s \cap X_0$	SOFT	Rel1	Rel2
Case 1	Minimums	0	4		0	8	7	0.488281	0 %	1 %
Edges 59	Maximums	3	10		4	13	11	0.575195	9 %	24 %
(10 obs)	Means	1.1	7.2		1.1	9.9	9.2	0.5333983	5 %	8 %
	No. of zeros	3			5				1	0
	Observations < 5 %								3	4
	Observations < 10 %								10	7
Case 2	Minimums	0	6		0	9	7	0.484375	0 %	3 %
Edges 61	Maximums	2	10		0	13	11	0.575195	5 %	32 %
(10 obs)	Means	0.6	8.1		0	11.7	8.8	0.5260741	2 %	13 %
	No. of zeros	6			10				2	0
	Observations < 5 %								10	2
	Observations < 10 %								10	5
Case 3	Minimums	0	5		0	7	8	0.484375	0.3 %	2 %
Edges 63	Maximums	2	10		1	12	11	0.567383	9 %	13 %
(10 obs)	Means	1	7.9		0.1	10.4	9.5	0.5308595	4 %	8 %
	No. of zeros	3			9				0	0
	Observations < 5 %								5	4
	Observations < 10 %								10	6
Case 4	Minimums	0	5		0	10	8	0.488281	1 %	4 %
Edges 63	Maximums	1	9		1	12	11	0.548828	9 %	18 %
(10 obs)	Means	0.2	7.3		0.3	10.9	9.2	0.5228515	6 %	13 %
	No. of zeros	8			7				0	0
	Observations < 5 %								3	1
	Observations < 10 %								10	2
Case 5	Minimums	0	6		0	8	7	0.475586	0.3 %	3 %
Edges 63	Maximums	2	11		1	12	11	0.564453	12 %	22 %
(10 obs)	Means	0.5	7.7		0.3	10.8	9.4	0.5172852	4 %	12 %
	No. of zeros	5			6				0	0
	Observations < 5 %								8	3
	Observations < 10 %								9	5
Case 6	Minimums	0	6		0	9	7	0.5	0 %	5 %
Edges 62	Maximums	1	10		1	13	11	0.580078	13 %	20 %
(10 obs)	Means	0.3	7.4		0.3	11.1	9	0.5389648	4 %	14 %
	No. of zeros	7			7				1	0
	Observations < 5 %								9	0
	Observations < 10 %								9	4
Case 7	Minimums	0	6		0	9	7	0.505859	1 %	5 %
Edges 63	Maximums	1	10		2	12	11	0.570313	9 %	19 %
(10 obs)	Means	0.5	8.2		0.3	11	9.7	0.5347656	4 %	11 %
	No. of zeros	5			8				0	0
	Observations < 5 %								8	0
	Observations < 10 %								10	5
Case 8	Minimums	0	4		0	8	8	0.496094	0 %	0 %
Edges 64	Maximums	1	10		2	13	13	0.584961	7 %	21 %
(10 obs)	Means	0.4	8.1		0.5	10.7	10	0.5297852	4 %	10 %
	No. of zeros	6			6				1	1
	Observations < 5 %								6	2
	Observations < 10 %								10	5
Case 9	Minimums	0	5		0	8	8	0.500977	0 %	5 %
Edges 62	Maximums	1	11		1	13	11	0.609375	11 %	24 %
(10 obs)	Means	0.4	8.1		0.1	10.7	10	0.5348634	4 %	10 %
	No. of zeros	6			9				1	0
	Observations < 5 %								7	0
	Observations < 10 %								9	6
Case 10	Minimums	0	5		0	8	7	0.480469	0 %	2 %
Edges 65	Maximums	2	11		1	13	12	0.564453	5 %	36 %
(9 obs)	Means	0.5556	8.3333		0.1111	11.3333	9.4444	0.5387	2 %	13 %
	No. of zeros	5			8				1	0
	Observations < 5 %								7	1
	Observations < 10 %								9	5

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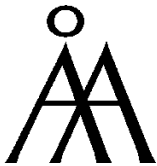
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