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Abstract

Various type of optimal solutions of multiobjective optimization problems can be characterized by means of different cones. Provided the partial objectives are convex, we derive necessary and sufficient geometrical optimality conditions for strongly efficient and lexicographically optimal solutions by using tangent, contingent and normal cones. Combining new results with previously known ones, we derive two general schemes reflecting structural properties and interconnections of five optimality principles: weakly and properly Pareto optimality, efficiency and strongly efficiency as well as lexicographic optimality.

Keywords: Multiple criteria, strong efficiency, lexicographic optimality, tangent cone, contingent cone, normal cone

1 Introduction

The major goal in multiobjective optimization is to find a compromise between several conflicting objectives which is best-fit to the needs of a decision maker. This compromise is usually refereed to as an optimality principle. Various mathematical definitions of the optimality principle can be derived in several different ways depending on the needs of the solution approaches used. Moreover, sometimes the use of one definition may be advantageous to the use of the other definition due to computational complexity reasons.

The optimality principles considered in this paper are strong efficiency and lexicographic optimality. The new results concerning some structural properties of above mentioned optimality principles are obtained using geometrical cone characterization approach. These results are combined with the results previously known for the sets of efficient, weakly and proper Pareto optimal solutions. As a result, we derive two general schemes reflecting structural properties and interconnections of five different optimality concepts: weakly and properly Pareto optimality, efficiency and strongly efficiency as well as lexicographic optimality.

A solution is Pareto optimal if improvement in some objectives can only be obtained at the expense of some other objective(s). This traditional concept is also known as efficiency, non-dominance or non-inferiority. The set of weakly Pareto optimal solutions contains the Pareto optimal solutions together with solutions where all the objectives cannot be improved simultaneously. On the other hand, Pareto optimal solutions can be divided into properly and improperly Pareto optimal ones. Proper Pareto optimality can be defined in different ways (see, e.g. [4]) but here we use only one of them (according to Henig [1]).

Strong efficiency is generally referred to the solutions which deliver optimality to each objective. Despite feasibility of such solutions is rare, they provide an important information on the lower bound for each objective in the Pareto optimal set. They also play a crucial role in various multiobjective methods and algorithms.

Lexicographic optimality principle is generally applied to the situation where objectives have no equal importance anymore but ordered according to their significance. A rigid arrangements of partial criteria with respect to importance is often used for a wide spectrum of important optimization problems, for example problems of stochastic programming, problems of axiomatic systems of utility theory and so on. Observe also that any scalar constrained optimization problem may be transformed to unconstrained bicriteria lexicographic problem by using as first criterion some exact penalty function for problem constraints, and an original objective function as a second constraint.

The five optimality concepts can be characterized with the help of different geometrical concepts, e.g. the use of cones is a natural choice in the case of convex optimization. Sometimes, exploiting geometrical characterization may be advantageous to using straightforward definitions of optimality due to potential decrease of computational efforts needed.

Important tools of classical convex analysis in the sense of Rockafellar [7] are tangent and normal cones of convex sets. In [5], different characterizations of optimality by using tangent and normal cones were specified in convex case for efficient, weakly and properly Pareto optimal solutions. In this paper, we report about new results on characterization optimality for two well-known classes of optimality which are strong efficiency and lexicographic optimality. This will lead to a more global view at structural properties of five well-know optimality principles in convex case. The results are summarized in two interconnected schemes.

In what follows, we introduce the problem formulation as well as some well-known results in Section 2. The new results concerning the set of strongly efficient solutions are given in Section 3. The lexicographic optimality is a subject of throughout research in Section 4. In section 5, we illustrate the geometrical meaning of the main results for the case of two objectives and different feasible regions specified by means of various norms. The paper is concluded in Section 6.

2 Problem Formulation and Preliminaries

We consider general multiobjective optimization problems of the following form:

$$\min_{x\in S}\ \{f_1(x), f_2(x), \ldots, f_k(x)\},$$

where the *objective functions* $f_i : \mathbf{R}^n \to \mathbf{R}$ for all $i \in I_k := \{1, \ldots, k\}$ are supposed to be continuous. The *decision vector* x belongs to the nonempty *feasible set* $S \subset \mathbf{R}^n$. The image of the feasible set is denoted by $Z \subset \mathbf{R}^k$, i.e. $Z := f(S)$ and it is supposed to be convex. Elements of Z are termed *objective vectors* and they are denoted by $z = f(x) = (f_1(x), f_2(x), \ldots, f_k(x))^T$.

The sum of two sets A and E is defined by $A+E = \{a+e \mid a \in A,\ e \in E\}$. The interior, closure, convex hull and complement of a set A are denoted by int A cl A, conv A and A^C, respectively.

A set A is a *cone* if $\lambda x \in A$ whenever $x \in A$ and $\lambda > 0$. We denote the negative orthant of $\mathbf{R}^k$ by $\mathbf{R}^k_- = \{d \in \mathbf{R}^k \mid d_i \leq 0 \text{ for every } i \in I_k\}$. The positive orthant $\mathbf{R}^k_+$, the *standard ordering cone*, is defined respectively. Note, that $\mathbf{R}^k_-$ and $\mathbf{R}^k_+$ are closed convex cones. Furthermore, an *open ball* with centre x and radius δ is denoted by $B(x;\delta)$.

In what follows, the notation $z < y$ for $z, y \in \mathbf{R}^k$ means that $z_i < y_i$ for every $i \in I_k$ and, correspondingly, $z \leq y$ stands for $z_i \leq y_i$ for every $i \in I_k$.

Simultaneous optimization of several objectives for multiobjective optimization problem is not a straightforward task. Contrary to the the single objective case, the concept of optimality is not unique in multiobjective cases.

Below we give traditional definitions of two well-known and most fundamental principles of optimality.

Weak Pareto optimality. An objective vector $z^* \in Z$ is *weakly Pareto optimal* if there does not exist another objective vector $z \in Z$ such that $z_i < z_i^*$ for all $i \in I_k$.

Pareto optimality or efficiency. An objective vector $z^* \in Z$ is *Pareto optimal* or *efficient* if there does not exist another objective vector $z \in Z$ such that $z_i \leq z_i^*$ for all $i \in I_k$ and $z_j < z_j^*$ for at least one index j.

Next we define the sets of weakly Pareto, Pareto and properly Pareto optimal solutions by using the opposite of the standard ordering cone. It is trivial to verify that the definitions of weak Pareto optimality and efficiency formulated above are equivalent to those following below.

Definition 1 *The* weakly Pareto optimal set *is*

$$WP(Z) := \{z \in Z \mid (z + \text{int}\, \mathbf{R}^k_-) \cap Z = \emptyset\},$$

the Pareto optimal set *is*

$$PO(Z) := \{z \in Z \mid (z + \mathbf{R}^k_- \setminus \{0\}) \cap Z = \emptyset\},$$

and the properly Pareto optimal set *is defined as*

$$PP(Z) := \{z \in Z \mid (z + C \setminus \{0\}) \cap Z = \emptyset\}$$

for some convex cone C such that $\mathbf{R}^k_- \setminus \{0\} \subset \text{int}\, C$.

Notice that the concept of proper Pareto optimality originates from the idea of prohibiting an unbounded trade-off between objectives but preserving the requirement of Pareto optimality. This limitation can be imposed either analytically or geometrically that will lead to slightly different concepts of proper Pareto optimality. We used the definition of proper Pareto optimality given by Henig in [1], since his definition uses geometrical characterization with help of convex ordering cone.

Obviously we have the following relationships between the different grades of Pareto optimality: $PP(Z) \subset PO(Z) \subset WP(Z)$. Next we define several geometrical basic cones (see e.g. [7]).

Definition 2 *The contingent cone of a set* $Z \subset \mathbf{R}^k$ *at* $z \in Z$ *is defined as*

$$K_z(Z) := \{d \in \mathbf{R}^k \mid \text{there exist } t_j \searrow 0 \text{ and } d_j \to d \text{ such that } z + t_j d_j \in Z\}.$$

The normal cone *of* Z *at* $z \in Z$ *is the polar cone of the contingent cone, that is,*

$$N_z(Z) := K_z(Z)^\circ = \{y \in \mathbf{R}^k \mid y^T d \leq 0 \text{ for all } d \in K_z(Z)\}.$$

The cone of feasible directions *of a set* $Z \subset \mathbf{R}^k$ *at* $z \in Z$ *is denoted by*

$$D_z(Z) := \{d \in \mathbf{R}^k \mid \text{ there exists } t > 0 \text{ such that } z + td \in Z\}.$$

By combining results from [3, 8] we have the following relations.

Lemma 1 *If* $Z \subset \mathbf{R}^k$ *is convex, then*

a) $K_z(Z)$, $N_z(Z)$ *and* $D_z(Z)$ *are convex;*

b) $K_z(Z)$ *and* $N_z(Z)$ *are closed;*

c) $0 \in K_z(Z) \cap N_z(Z) \cap D_z(Z)$*;*

d) $Z \subset z + D_z(Z) \subset z + K_z(Z)$*;*

e) $K_z(Z) = \operatorname{cl} D_z(Z)$.

To the end of this section we collect the geometrical optimality conditions derived in [5] characterizing the different notions of Pareto optimality with the contingent cone and the cone of feasible directions.

Theorem 1 *If* $Z \subset \mathbf{R}^k$ *is convex, then*

a) $z \in PO(Z) \iff D_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} = \emptyset$;

b) $z \in PP(Z) \iff K_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} = \emptyset$;

c) $z \in WP(Z) \iff K_z(Z) \cap \operatorname{int} \mathbf{R}^k_- = \emptyset \iff D_z(Z) \cap \operatorname{int} \mathbf{R}^k_- = \emptyset$.

Similar optimality conditions can be derived for weakly and proper Pareto optimality by using the normal cone [5].

Theorem 2 *If* $Z \subset \mathbf{R}^k$ *is convex, then*

a) $z \in PP(Z) \iff N_z(Z) \cap \operatorname{int} \mathbf{R}^k_- \neq \emptyset$;

b) $z \in WP(Z) \iff N_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} \neq \emptyset$.

3 Strong Efficiency

Let us first define the concept of strong optimality.

Definition 3 *The* strongly efficient set *is defined as*

$$SE(Z) := \{z \in Z \mid (z + (\mathbf{R}^k_+)^C) \cap Z = \emptyset\}.$$

Strongly efficient solutions are sometimes called also *ideal solutions*. This is due to fact that

$$SE(Z) = \bigcap_{i=1}^{k} \arg\min_{x \in S} f_i(x).$$

Clearly we have the following relationships between the different grades of optimality: $SE(Z) \subset PP(Z) \subset PO(Z) \subset WP(Z)$.

In this section we try to derive similar geometrical necessary and sufficient optimality conditions presented in previous section also for strongly efficient solutions. We start by verifying the result in part c) of Theorem 1.

Theorem 3 *Let $Z \subset \mathbf{R}^k$ be convex. Then the next three properties are equivalent.*

a) $z \in SE(Z)$;

b) $K_z(Z) \cap (\mathbf{R}^k_+)^C = \emptyset$;

c) $D_z(Z) \cap (\mathbf{R}^k_+)^C = \emptyset$.

Proof. Let us start by showing that a) implies b). Let $z \in SE(Z)$ and suppose to the contrary that $K_z(Z) \cap (\mathbf{R}^k_+)^C \neq \emptyset$. Then there exist $d \in (\mathbf{R}^k_+)^C$ and $d_j \to d$, $t_j \searrow 0$ such that $z + t_j d_j \in Z$. Because $(\mathbf{R}^k_+)^C$ is open, there exists $m > 0$ such that $d_j \in (\mathbf{R}^k_+)^C$ for every $j \geq m$. On the other hand, $(\mathbf{R}^k_+)^C$ is a cone and $t_j > 0$, thus $t_j d_j \in (\mathbf{R}^k_+)^C$ for every $j \geq m$. In other words, $(z + (\mathbf{R}^k_+)^C) \cap Z \neq \emptyset$. This contradicts the definition of strong efficiency of z. Thus, b) follows.

The property b) implies c) because of Lemma 1, part e).

Finally, we prove that c) implies a). Let us suppose that $D_z(Z) \cap (\mathbf{R}^k_+)^C = \emptyset$. If $z \notin SE(Z)$, then there exists $d \in (\mathbf{R}^k_+)^C$ such that $z + d \in Z$. By choosing $t := 1$, we have $d \in D_z(Z)$, which contradicts (iii). Thus, $z \in SE(Z)$. This ends the proof.

As a consequence we get the characterization of strongly efficient solutions with the standard ordering cone.

Corollary 1 *Let $Z \subset \mathbf{R}^k$ be convex. Then the next three properties are equivalent.*

a) $z \in SE(Z)$*;*

b) $K_z(Z) \cap \mathbf{R}^k_+ = K_z(Z)$*;*

c) $D_z(Z) \cap \mathbf{R}^k_+ = D_z(Z)$.

Proof. By Theorem 3 part b) we have

$$\begin{aligned} z \in SE(Z) &\iff K_z(Z) \cap (\mathbf{R}^k_+)^C = \emptyset \\ &\iff K_z(Z) \subset ((\mathbf{R}^k_+)^C)^C = \mathbf{R}^k_+ \\ &\iff K_z(Z) \cap \mathbf{R}^k_+ = K_z(Z). \end{aligned}$$

The equivalence between parts a) and c) can be proved analogously. This ends the proof.

Besides using contingent cones, strongly efficiency can alternatively be characterized by employing normal cones, cf. Theorem 2.

However, in order to prove the next theorem, we must state a lemma.

Lemma 2 *Let* C_1 *and* C_2 *be cones in* $\mathbf{R}^k$ *such that* $0 \in C_1 \cap C_2$*. If* C_1 *and* C_2 *are*

a) *closed, then* $C_1 + C_2$ *is also closed.*

b) *convex, then* $C_1 + C_2 = \text{conv}\,(C_1 \cup C_2)$.

Theorem 4 *If* $Z \subset \mathbf{R}^k$ *is convex, then* $z \in SE(Z)$ *if and only if*

$$N_z(Z) \cap \mathbf{R}^k_- = \mathbf{R}^k_-.$$

Proof. Let $z \in SE(Z)$, then by Theorem 3 we have $K_z(Z) \cap (\mathbf{R}^k_+)^C = \emptyset$. Suppose to the contrary that $N_z(Z) \cap \mathbf{R}^k_- \neq \mathbf{R}^k_-$, in other words, $\mathbf{R}^k_-$ is not a subset of $N_z(Z)$. Since both sets are closed (Lemma 1, part b)), there exists $y \in \text{int}\,\mathbf{R}^k_-$ such that $y \notin N_z(Z)$. This means, that $y_i < 0$ for all $i \in I_k$ and there exist $d \in K_z(Z)$ such that $y^T d > 0$. Then, there exist $j \in I_k$ such that $d_j < 0$, in other words $d \in (\mathbf{R}^k_+)^C$. Thus we have $d \in K_z(Z) \cap (\mathbf{R}^k_+)^C$, which contradicts Theorem 3 part b).

As far as sufficiency is concerned, let us assume that $N_z(Z) \cap \mathbf{R}^k_- = \mathbf{R}^k_-$. Then we have

$$(N_z(Z) \cap \mathbf{R}^k_-)^\circ = (\mathbf{R}^k_-)^\circ = \mathbf{R}^k_+. \tag{1}$$

On the other hand, by Corollary 16.4.2 in [7] together with Lemma 1, parts b) and c), and Lemma 2 parts a) and b) we get

$$\begin{aligned} (N_z(Z) \cap \mathbf{R}^k_-)^\circ &= \text{cl}\,(N_z(Z)^\circ + (\mathbf{R}^k_-)^\circ) \\ &= \text{cl}\,(K_z(Z) + \mathbf{R}^k_+) \\ &= K_z(Z) + \mathbf{R}^k_+ \\ &= \text{conv}\,(K_z(Z) \cup \mathbf{R}^k_+). \end{aligned} \tag{2}$$

$$
\begin{array}{ccccc}
 & & D_z(Z) \cap \mathbf{R}^k_+ = D_z(Z) & & \\
 & & \Updownarrow & & \\
z \in SE(Z) & \Longleftrightarrow & K_z(Z) \cap \mathbf{R}^k_+ = K_z(Z) & \Longleftrightarrow & N_z(Z) \cap \mathbf{R}^k_- = \mathbf{R}^k_- \\
\Downarrow & & \Downarrow & & \Downarrow \\
z \in PP(Z) & \Longleftrightarrow & K_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} = \emptyset & \Longleftrightarrow & N_z(Z) \cap \text{int } \mathbf{R}^k_- \neq \emptyset \\
\Downarrow & & \Downarrow & & \\
z \in PO(Z) & \Longleftrightarrow & D_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} = \emptyset & & \Downarrow \\
\Downarrow & & \Downarrow & & \\
z \in WP(Z) & \Longleftrightarrow & D_z(Z) \cap \text{int } \mathbf{R}^k_- = \emptyset & \Longleftrightarrow & N_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} \neq \emptyset \\
 & & \Updownarrow & & \\
 & & K_z(Z) \cap \text{int } \mathbf{R}^k_- = \emptyset & &
\end{array}
$$

Figure 1: Collection of the relationships with properly Pareto optimality.

Combining (1) and (2) we get

$$K_z(Z) \cup \mathbf{R}^k_+ \subset \mathbf{R}^k_+,$$

thus $K_z(Z) \subset \mathbf{R}^k_+$, which means that

$$K_z(Z) \cap (\mathbf{R}^k_+)^C = \emptyset,$$

and then by Theorem 3 part b) we have $z \in SE(Z)$. This ends the proof.

The results related to the four optimality concepts and different cones in case Z is convex are collected in Fig. 1.

4 Lexicographic Optimality

We start by giving a standard definition of the concept of lexicographic optimality (see e.g. [4]). An objective vector $z^* \in Z$ is *lexicographically optimal* if for any other objective vector $z \in Z$ one of the following two conditions holds:
1) $z = z^*$
2) $\exists\, i \in I_k :\ (z^*_i < z_i) \wedge (\forall j \in I_{i-1} :\ z^*_j = z_j)$, where $I_0 = \emptyset$.

Next we will give one more equivalent definition of the lexicographic optimality: an objective vector $z^* \in Z$ is lexicographically optimal if

$$\Big\{z \in Z \mid z_i < z_i^*,\ i = \min\{j \in I_k \mid \ z_j \neq z_j^*\}\Big\} = \emptyset.$$

Note that the lexicographic optima may be obtained as a result of the solution of single objective (scalar) problems sequence

$$L^{(i)} = \min\{z_i \mid \ z \in L^{(i-1)}\},$$

where $i \in I_k$, $L^{(0)} = Z$, and z_i denotes i-th objective. Thus $L^{(k)}$ will constitute the set of lexicographically optimal solutions which we define below by using the complement of the lexicographic cone. It is simple to verify that all definitions are equivalent and referred to the following concept of lexicographic optimality.

Definition 4 *The* lexicographically optimal set *is*

$$LO(Z) = \{z \in Z \mid (z + (C^k_{\text{lex}})^C) \cap Z = \emptyset\},$$

where the lexicographic cone *is*

$$C^k_{\text{lex}} := \{0\} \cup \{d \in \mathbf{R}^k \mid \exists\, i \in I_k \textit{ such that } d_i > 0 \textit{ and } d_j = 0\ \forall\, j < i\}.$$

Emphasize the following properties of the lexicographic cone [2]:

a) C^k_{lex} is pointed, i.e. $l(C^k_{\text{lex}}) = C^k_{\text{lex}} \cap -C^k_{\text{lex}} = \{0\}$;

b) C^k_{lex} is not correct, i.e. $\operatorname{cl} C^k_{\text{lex}} + C^k_{\text{lex}} \backslash l(C^k_{\text{lex}}) \not\subset C^k_{\text{lex}}$;

c) C^k_{lex} is not strictly supported, i.e. $C^k_{\text{lex}} \backslash l(C^k_{\text{lex}})$ is not contained in an open homogeneous half space.

Some more properties of C^k_{lex} can be easily verified:

d) C^k_{lex} is neither closed nor open;

e) $(C^k_{\text{lex}})^* := \{y \in \mathbf{R}^k \mid y^T d \geq 0 \text{ for all } d \in C^k_{\text{lex}}\} = \mathbf{R}_+$;

f) $(C^k_{\text{lex}})^\circ := \{y \in \mathbf{R}^k \mid y^T d \leq 0 \text{ for all } d \in C^k_{\text{lex}}\} = \mathbf{R}_-$.

It is evident that we have the following relationships between the different optimalities: $SE(Z) \subset LO(Z) \subset PO(Z) \subset WP(Z)$. However, nothing can be said in general case about the relation of $LO(Z)$ and $PP(Z)$. The example in the next section will illustrate this fact in the case of two objectives.

Now we will formulate the main results concerning lexicographic optimality characterization by means of different cones.

Theorem 5 *If $Z \subset \mathbf{R}^k$ is convex, then $z \in LO(Z)$ if and only if*

$$D_z(Z) \cap (C_{\text{lex}}^k)^C = \emptyset.$$

Proof. Let $z \in LO(Z)$. Let us suppose that there exists $d \in D_z(Z) \cap (C_{\text{lex}}^k)^C$. Then there exists $t > 0$ such that $z + td \in Z$ and $td \in (C_{\text{lex}}^k)^C$. This implies that $z + td \in (z + (C_{\text{lex}}^k)^C) \cap Z$ and, by the definition of $LO(Z)$, this means that $z \notin LO(Z)$. Thus, we must have $D_z(Z) \cap (C_{\text{lex}}^k)^C = \emptyset$.

On the other hand, let us assume that

$$D_z(Z) \cap (C_{\text{lex}}^k)^C = \emptyset. \tag{3}$$

Let us suppose that $z \notin LO(Z)$. Then there exists $d \in (C_{\text{lex}}^k)^C$ such that $z+d \in Z$. This implies (by selecting $t = 1$) that $d \in D_z(Z)$. This contradicts (3), in other words, $z \in LO(Z)$. This ends the proof.

As a consequence we get the characterization of lexicographically optimal solutions with the lexicographic cone.

Corollary 2 *If $Z \subset \mathbf{R}^k$ is convex, then $z \in LO(Z)$ if and only if*

$$D_z(Z) \cap C_{\text{lex}}^k = D_z(Z).$$

Proof. By Theorem 5 we have

$$\begin{aligned} z \in LO(Z) &\iff D_z(Z) \cap (C_{\text{lex}}^k)^C = \emptyset \\ &\iff D_z(Z) \subset ((C_{\text{lex}}^k)^C)^C = C_{\text{lex}}^k \\ &\iff D_z(Z) \cap C_{\text{lex}}^k = D_z(Z). \end{aligned}$$

The proof ends here.

The results related to the four optimality concepts and different cones in case Z is convex are collected in Fig. 2.

Sometimes, the lexicographic optimality principle is defined in more general way in order to reflect all possible objective orderings. This will lead to the so-called generalized lexicographic optimality concept which we define below.

Definition 5 *The* generalized lexicographic set $GLO(Z)$ *defined by all $k!$ permutations of objectives is:*

$$GLO(Z) := \bigcup_{s \in S_k} LO_s(Z),$$

where

$$LO_s(Z) := \Big\{ z \in Z \mid \ \big(z = z^*\big) \vee$$

$$\big(\exists i \in I_k : \ (z_{s_i}^* < z_{s_i}) \wedge (\forall j \in I_{s_i - 1} : \ z_{s_j}^* = z_{s_j})\big)\Big\},$$

and S_k is a set of all $k!$ permutations of the numbers $1, 2, ..., k$.

$$
\begin{array}{ccccc}
 & & D_z(Z) \cap \mathbf{R}^k_+ = D_z(Z) & & \\
 & & \Updownarrow & & \\
z \in SE(Z) & \Longleftrightarrow & K_z(Z) \cap \mathbf{R}^k_+ = K_z(Z) & \Longleftrightarrow & N_z(Z) \cap \mathbf{R}^k_- = \mathbf{R}^k_- \\
\Downarrow & & \Downarrow & & \\
z \in LO(Z) & \Longleftrightarrow & D_z(Z) \cap C^k_{\text{lex}} = D_z(Z) & & \\
\Downarrow & & \Downarrow & & \Downarrow \\
z \in PO(Z) & \Longleftrightarrow & D_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} = \emptyset & & \\
\Downarrow & & \Downarrow & & \\
z \in WP(Z) & \Longleftrightarrow & D_z(Z) \cap \text{int } \mathbf{R}^k_- = \emptyset & \Longleftrightarrow & N_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} \neq \emptyset \\
 & & \Updownarrow & & \\
 & & K_z(Z) \cap \text{int } \mathbf{R}^k_- = \emptyset & &
\end{array}
$$

Figure 2: Collection of the relationships with properly Lexicographic optimality.

The elements of the set $LO_s(Z)$ are called lexicographic optima with respect to permutation s of objective order. Notice that $LO_s(Z) = LO(Z)$ if s is identity permutation, i.e. $s = (s_1, s_2, ..., s_k) = (1, 2, ..., k)$. The elements of the set $GLO(Z)$ are called *generalized lexicographic optima*. It is easy to see that any generalized lexicographic optimum belongs to the Pareto set, i.e. the following chain of inclusions holds

$$SE(Z) \subset LO(Z) \subset GLO(Z) \subset PO(Z) \subset WP(Z).$$

Using Theorem 5 and Corollary 2, we obtain the following straightforward results.

Corollary 3 *If $Z \subset \mathbf{R}^k$ is convex, then*

$$GLO(Z) = \bigcup_{s \in S_k} \Big\{ z \in Z \mid \big(D_z(Z) \cap (C^k_{\text{lex}})^C \big)_s = \emptyset \Big\},$$

where $()_s$ means that $D_z(Z)$ and C^k_{lex} are taken respectively for each $s \in S_k$.

Corollary 4 *If $Z \subset \mathbf{R}^k$ is convex, then*

$$GLO(Z) = \bigcup_{s \in S_k} \Big\{ z \in Z \mid \big(D_z(Z) \cap C^k_{\text{lex}} = D_z(Z) \big)_s \Big\}.$$

5 Illustrative example for the case of two objectives

We will illustrate geometrical meaning of the basic results formulated above for proper and lexicographic optimality via the following example in biobjective case.

To construct the example, we will use the following norms in an arbitrary q-dimensional vector space $\mathbf{R}^q$:

- L_1 or *linear* norm
$$||y||_1 := \sum_{i\in I_q} |y_i|,\ y \in \mathbf{R}^q;$$

- L_2 or *Euclidean* norm
$$||y||_2 := \sqrt{\sum_{i\in I_q} (y_i)^2},\ y \in \mathbf{R}^q;$$

- L_∞ or *Chebyshev* norm
$$||y||_\infty := \max_{i\in I_q} |y_i|,\ y \in \mathbf{R}^q.$$

As follows from the results described in previous sections, the various optimal solutions can be characterized with the help of contingent cone, cone of feasible directions, normal cone and lexicographic cone which are depicted for the case of two objectives on Figures 3, 4, 5 and 6, respectively.

Example. Let $z := f(x) = (f_1(x), f_2(x))$, where $f_1(x) = x_1$ and $f_2(x) = x_2$. Assume that the sets of feasible solutions are given as

$$X_1 := \Big\{x \mid\ ||x||_1 \leq 1\Big\},$$

$$X_2 := \Big\{x \mid\ ||x||_2 \leq 1\Big\},$$

$$X_3 := \Big\{x \mid\ ||x||_\infty \leq 1\Big\}.$$

Then, respectively, we have

$$\begin{aligned} Z_1 &:= \Big\{(f_1(x), f_2(x)) :\ x \in X_1\Big\} = \Big\{z \mid\ ||z||_1 \leq 1\Big\}, \\ Z_2 &:= \Big\{(f_1(x), f_2(x)) :\ x \in X_2\Big\} = \Big\{z \mid\ ||z||_2 \leq 1\Big\}, \\ Z_3 &:= \Big\{(f_1(x), f_2(x)) :\ x \in X_3\Big\} = \Big\{z \mid\ ||z||_\infty \leq 1\Big\}. \end{aligned}$$

Using the results of theorems 1, 3 and 5, we get the following.
For Z_1, we have

$$\begin{aligned}
SE(Z_1) &= \Big\{z \in Z_1 \mid \ K_z(Z_1) \cap \mathbf{R}^2_+ = K_z(Z_1)\Big\} = \emptyset && \text{—no Figure;}\\
PP(Z_1) &= \Big\{z \in Z_1 \mid \ K_z(Z_1) \cap \mathbf{R}^2_- \setminus \{0\} = \emptyset\Big\} \\
&= \Big\{z \mid \ |z_1| + |z_2| = 1,\ z_1 \le 0,\ z_2 \le 0\Big\} && \text{—see Figure 7;}\\
PO(Z_1) &= \Big\{z \in Z_1 \mid \ D_z(Z_1) \cap \mathbf{R}^2_- \setminus \{0\} = \emptyset\Big\} \\
&= \Big\{z \mid \ |z_1| + |z_2| = 1,\ z_1 \le 0,\ z_2 \le 0\Big\} && \text{—see Figure 8;}\\
WP(Z_1) &= \Big\{z \in Z_1 \mid \ D_z(Z_1) \cap \operatorname{int} \mathbf{R}^2_- = \emptyset\Big\} \\
&= \Big\{z \mid \ |z_1| + |z_2| = 1,\ z_1 \le 0,\ z_2 \le 0\Big\} && \text{—see Figure 9;}\\
LO(Z_1) &= \Big\{z \in Z_1 \mid \ D_z(Z_1) \cap C^2_{\text{lex}} = D_z(Z_1)\Big\} \\
&= \Big\{(-1,0)\Big\} && \text{—see Figure 10.}
\end{aligned}$$

Next, the corresponding optimality sets for Z_2 are

$$\begin{aligned}
SE(Z_2) &= \Big\{z \in Z_2 \mid \ K_z(Z_2) \cap \mathbf{R}^2_+ = K_z(Z_2)\Big\} = \emptyset && \text{—no Figure;}\\
PP(Z_2) &= \Big\{z \in Z_2 \mid \ K_z(Z_2) \cap \mathbf{R}^2_- \setminus \{0\} = \emptyset\Big\} \\
&= \Big\{z \mid \ z_1^2 + z_2^2 = 1,\ z_1 < 0,\ z_2 < 0\Big\} && \text{—see Figure 11;}\\
PO(Z_2) &= \Big\{z \in Z_2 \mid \ D_z(Z_2) \cap \mathbf{R}^2_- \setminus \{0\} = \emptyset\Big\} \\
&= \Big\{z \mid \ z_1^2 + z_2^2 = 1,\ z_1 \le 0,\ z_2 \le 0\Big\} && \text{—see Figure 12;}\\
WP(Z_2) &= \Big\{z \in Z_2 \mid \ D_z(Z_2) \cap \operatorname{int} \mathbf{R}^2_- = \emptyset\Big\} \\
&= \Big\{z \mid \ z_1^2 + z_2^2 = 1,\ z_1 \le 0,\ z_2 \le 0\Big\} && \text{—see Figure 13;}\\
LO(Z_2) &= \Big\{z \in Z_2 \mid \ D_z(Z_2) \cap C^2_{\text{lex}} = D_z(Z_2)\Big\} \\
&= \Big\{(-1,0)\Big\} && \text{—see Figure 14.}
\end{aligned}$$

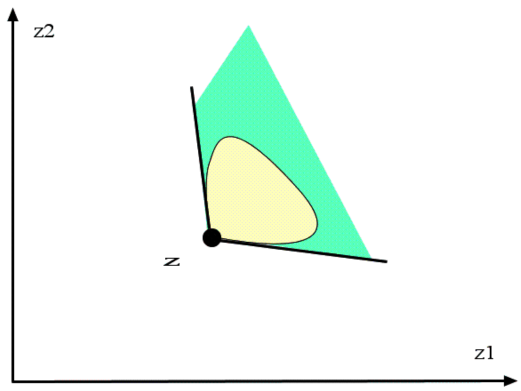

Figure 3: The contingent cone $K_z(Z)$.

Finally, for Z_3, we have

$$\begin{aligned}
SE(Z_3) &= \left\{ z \in Z_3 \mid \; K_z(Z_3) \cap \mathbf{R}^2_+ = K_z(Z_3) \right\} \\
&= \{(-1,-1)\} && \text{—see Figure 15;} \\
PP(Z_3) &= \left\{ z \in Z_3 \mid \; K_z(Z_3) \cap \mathbf{R}^2_- \setminus \{0\} = \emptyset \right\} \\
&= \left\{ (-1,-1) \right\} && \text{—see Figure 16;} \\
PO(Z_3) &= \left\{ z \in Z_3 \mid \; D_z(Z_3) \cap \mathbf{R}^2_- \setminus \{0\} = \emptyset \right\} \\
&= \left\{ (-1,-1) \right\} && \text{—see Figure 17;} \\
WP(Z_3) &= \left\{ z \in Z_3 \mid \; D_z(Z_3) \cap \operatorname{int} \mathbf{R}^2_- = \emptyset \right\} \\
&= \left\{ z \mid z_1 = -1, -1 \le z_2 \le 1 \right\} \\
&= \bigcup \left\{ z \mid z_2 = -1, -1 \le z_1 \le 1 \right\} && \text{—see Figure 18;} \\
LO(Z_3) &= \left\{ z \in Z_3 \mid \; D_z(Z_3) \cap C^2_{\text{lex}} = D_z(Z_3) \right\} \\
&= \left\{ (-1,-1) \right\} && \text{—see Figure 19.}
\end{aligned}$$

Notice, that we have $LO(Z_1) \subsetneq PP(Z_1)$, and $LO(Z_2) \cap PP(Z_2) = \emptyset$, and $LO(Z_3) = PP(Z_3)$.

6 Concluding Remarks

Additionally to previously known cone characterizations of three optimality principles - efficiency, weakly and proper Pareto optimality, we have characterized two other optimality concepts - strongly efficiency and lexicographic optimality in terms of intersections of different cones. The results

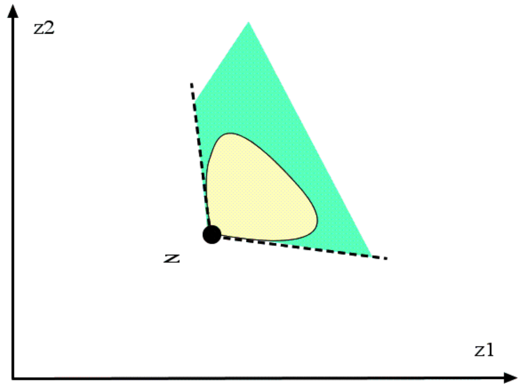

Figure 4: The cone of feasible directions $D_z(Z)$.

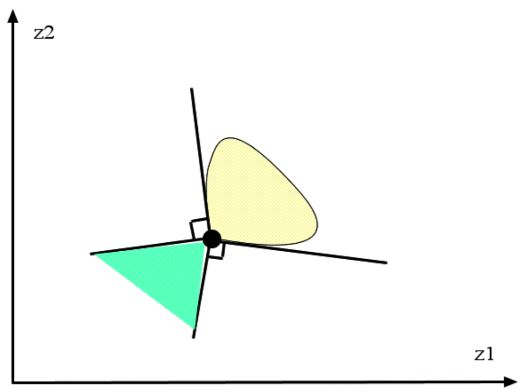

Figure 5: The normal cone $N_z(Z)$.

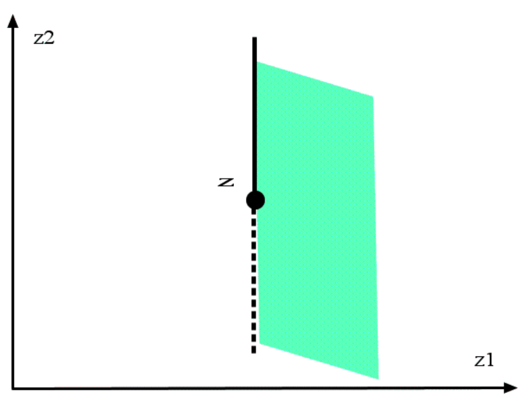

Figure 6: The lexicographic cone C^2_{lex}.

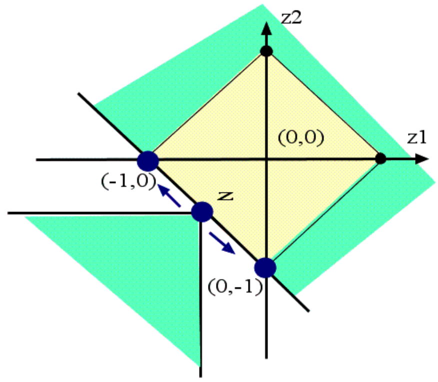

Figure 7: Detection of $PP(Z_1)$ by means of $K_z(Z_1)$ and $\mathbf{R}^2_-$.

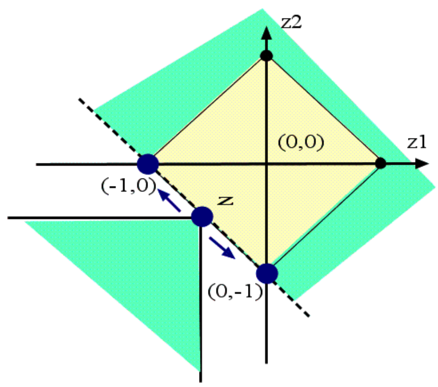

Figure 8: Detection of $PO(Z_1)$ by means of $D_z(Z_1)$ and $\mathbf{R}^2_-$.

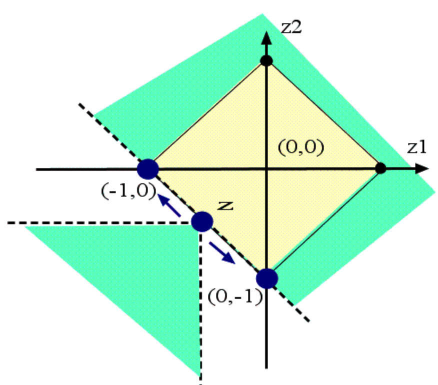

Figure 9: Detection of $WP(Z_1)$ by means of $D_z(Z_1)$ and int$\mathbf{R}^2_-$.

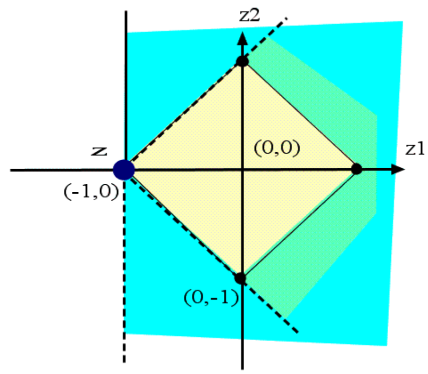

Figure 10: Detection of $LO(Z_1)$ by means of $D_z(Z_1)$ and C^2_{lex}.

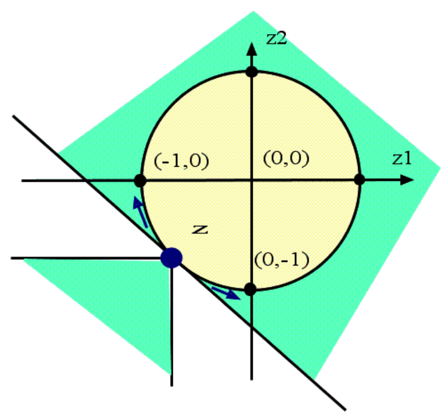

Figure 11: Detection of $PP(Z_2)$ by means of $K_z(Z_2)$ and $\mathbf{R}^2_-$.

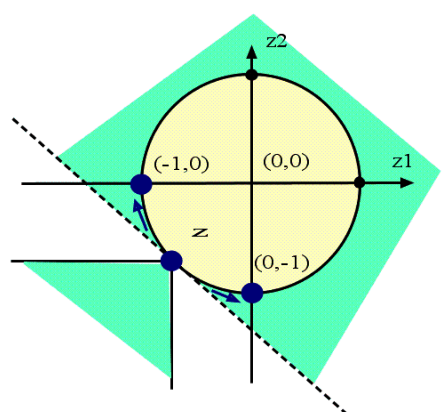

Figure 12: Detection of $PO(Z_2)$ by means of $D_z(Z_2)$ and $\mathbf{R}^2_-$.

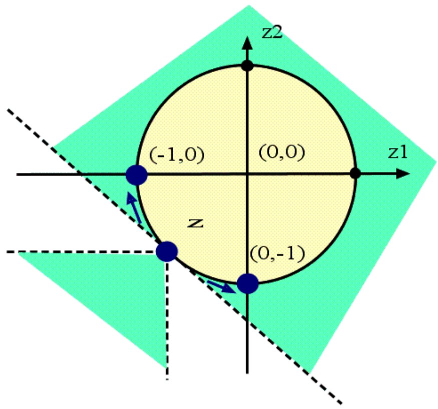

Figure 13: Detection of $WP(Z_2)$ by means of $D_z(Z_2)$ and int $\mathbf{R}^2_-$.

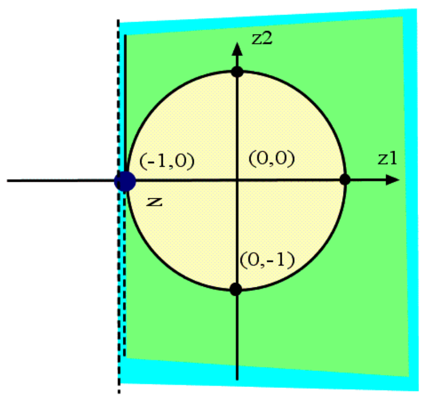

Figure 14: Detection of $LO(Z_2)$ by means of $D_z(Z_2)$ and C^2_{lex}.

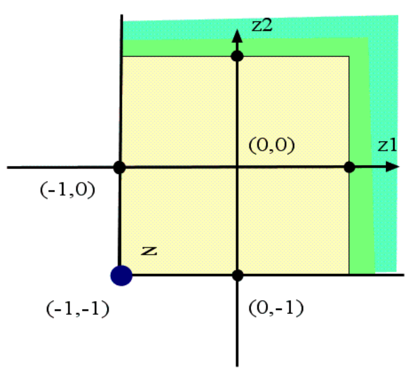

Figure 15: Detection of $SE(Z_3)$ by means of $K_z(Z_3)$ and $\mathbf{R}^2_+$.

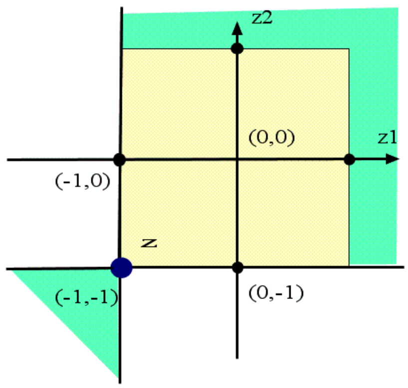

Figure 16: Detection of $PP(Z_3)$ by means of $K_z(Z_3)$ and $\mathbf{R}^2_-$.

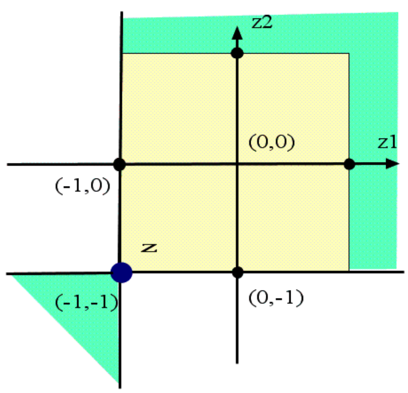

Figure 17: Detection of $PO(Z_3)$ by means of $D_z(Z_3)$ and $\mathbf{R}^2_-$.

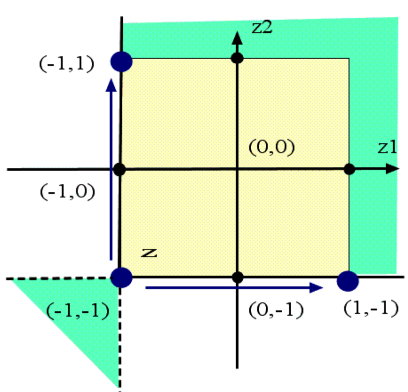

Figure 18: Detection of $WP(Z_3)$ by means of $D_z(Z_3)$ and $\operatorname{int}\mathbf{R}^2_-$.

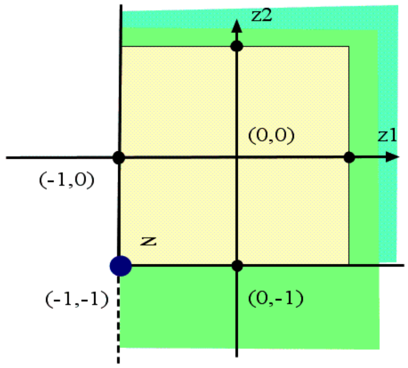

Figure 19: Detection of $LO(Z_3)$ by means of $D_z(Z_3)$ and C^2_{lex}.

were collected and summarized in two figures illustrating the interconnections between different optimality principles. The aim was to point out the differences and similarities between the five optimality principles. As far as the results are concerned, it would be also interesting to investigate in future if similar characterizations can be obtained for the case where partial objectives are non-convex.

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