



Marko M. Mäkelä | Napsu Karmitsa | Outi Wilppu

Multiobjective Proximal Bundle Method for Nonsmooth Optimization

TURKU CENTRE *for* COMPUTER SCIENCE

TUCS Technical Report
No 1120, July 2014



Multiobjective Proximal Bundle Method for Nonsmooth Optimization

Marko M. Mäkelä

University of Turku, Department of Mathematics and Statistics
FI-20014 Turku, Finland
`makela@utu.fi`

Napsu Karmitsa

University of Turku, Department of Mathematics and Statistics
FI-20014 Turku, Finland
`napsu@karmitsa.fi`

Outi Wilppu

University of Turku, Department of Mathematics and Statistics
FI-20014 Turku, Finland
`omwilp@utu.fi`

TUCS Technical Report

No 1120, July 2014

Abstract

We present a proximal bundle method for finding weakly Pareto optimal solutions to constrained nonsmooth programming problems with multiple objectives. The method is a generalization of proximal bundle approach for single objective optimization. The multiple objective functions are treated individually without employing any scalarization. The method is globally convergent and capable of handling several nonconvex locally Lipschitz continuous objective functions subject to nonlinear (possibly nondifferentiable) constraints. Under some generalized convexity assumptions, we prove that the method finds globally weakly Pareto optimal solutions. Concluding, some numerical examples illustrate the properties and applicability of the method. In addition, we give a collection of multiobjective test problems.

Keywords: Bundle Methods, multiobjective optimization, nonsmooth optimization

TUCS Laboratory

Turku Optimization Group (TOpGroup)

1 Introduction

Nonsmooth (nondifferentiable) optimization problems arise in very many fields of applications, for example, in optimal shape design (see, e.g., [2, 5, 13]), economics [21] and mechanics [19]. On the other hand, instead of one criterion the applications typically have several, often conflicting objectives. During the last three decades the rapid development has been characteristic to the areas of nonsmooth (see, e.g., [1, 4, 6, 9, 10, 12, 18, 24]) and multiobjective optimization (see, e.g., [16, 17, 20, 25], separately. Conversely the consideration of both of these approaches in the same framework, i.e. nonsmooth multiobjective optimization, is much less frequent. Thus there exists an increasing demand to be able to solve efficiently optimization problems with several, possible nonsmooth, objective functions.

In this paper we present a proximal bundle based method for constrained nonconvex nonsmooth programming problems with multiple objectives. The method generalizes the proximal bundle approach for single objective optimization [8] by employing the ideas presented in [7, 17, 26]. We can prove, that under some generalized convexity assumptions [15] the method can find globally weakly Pareto optimal solutions. Unlike the most multicriteria optimization methods the multiple objective functions are treated individually without employing any scalarization. The method is readily implementable and descent, i.e., the value of each objective function is expected to get an improvement at each iteration.

The paper is organized as follows. Chapter 2 contains some preliminary concepts and results of nonsmooth and multiobjective optimization theory. The algorithm of the multicriteria proximal bundle (MPB) method is described in Chapter 3. Some convergence results are presented in Chapter 4. Finally, Chapter 5 is devoted to some numerical examples illustrating the properties and applicability of the method.

2 Preliminaries

Let us consider a nonsmooth multiobjective optimization problem of the form

$$\begin{cases} \text{minimize} & \{f_1(x), \dots, f_k(x)\} \\ \text{subject to} & x \in S, \end{cases} \quad (1)$$

where

$$S = \{\mathbf{x} \in \mathbb{R}^n \mid g_j(x) \leq 0, j = 1, \dots, m\}.$$

The objective functions $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ and the constraint functions $g_j : \mathbb{R}^n \rightarrow \mathbb{R}$ are supposed to be *locally Lipschitz continuous* (not necessarily smooth nor convex). For a locally Lipschitz continuous function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ the *Clarke generalized*

directional derivative at \mathbf{x} in the direction $\mathbf{d} \in \mathbb{R}^n$ is defined by

$$f^\circ(\mathbf{x}; \mathbf{d}) = \limsup_{\substack{\mathbf{y} \rightarrow \mathbf{x} \\ t \downarrow 0}} \frac{f(\mathbf{y} + t\mathbf{d}) - f(\mathbf{y})}{t}$$

and the Clarke subdifferential of f at \mathbf{x} by

$$\partial f(\mathbf{x}) = \{\boldsymbol{\xi} \in \mathbb{R}^n \mid f^\circ(\mathbf{x}; \mathbf{d}) \geq \boldsymbol{\xi}^T \mathbf{d} \text{ for all } \mathbf{d} \in \mathbb{R}^n\},$$

which is a nonempty, convex and compact subset of \mathbb{R}^n . Note, that if a locally Lipschitz continuous function attains its local minimum at \mathbf{x}^* , then

$$0 \in \partial f(\mathbf{x}^*). \quad (2)$$

For a finite maximum of locally Lipschitz continuous functions we have the following subderivation rule.

Theorem 2.1. [13] Let $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ be locally Lipschitz continuous at \mathbf{x} for all $i = 1, \dots, m$. Then the function

$$f(\mathbf{x}) = \max [f_i(\mathbf{x}) \mid i = 1, \dots, m]$$

is locally Lipschitz continuous at \mathbf{x} and

$$\partial f(\mathbf{x}) \subseteq \text{conv} \{\partial f_i(\mathbf{x}) \mid f_i(\mathbf{x}) = f(\mathbf{x}), i = 1, \dots, m\}, \quad (3)$$

where conv denotes the convex hull of a set.

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is weakly semismooth if the classical directional derivative

$$f'(\mathbf{x}, \mathbf{d}) = \lim_{t \downarrow 0} \frac{f(\mathbf{x} + t\mathbf{d}) - f(\mathbf{x})}{t}$$

exists for all \mathbf{x} and \mathbf{d} , and

$$f'(\mathbf{x}, \mathbf{d}) = \lim_{t \downarrow 0} \boldsymbol{\xi}(\mathbf{x} + t\mathbf{d})^T \mathbf{d},$$

where $\boldsymbol{\xi}(\mathbf{x} + t\mathbf{d}) \in \partial f(\mathbf{x} + t\mathbf{d})$.

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is f° -pseudoconvex, if it is locally Lipschitz continuous and for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

$$f(\mathbf{y}) < f(\mathbf{x}) \quad \text{implies} \quad f^\circ(\mathbf{x}; \mathbf{y} - \mathbf{x}) < 0$$

and f° -quasiconvex, if

$$f(\mathbf{y}) \leq f(\mathbf{x}) \quad \text{implies} \quad f^\circ(\mathbf{x}; \mathbf{y} - \mathbf{x}) \leq 0.$$

Note, that a convex function is always f° -pseudoconvex, which again is f° -quasiconvex [15]. Next we present two important properties of f° -pseudoconvex functions.

Theorem 2.2. [15] An f° -pseudoconvex function f attains its global minimum at \mathbf{x}^* , if and only if

$$0 \in \partial f(\mathbf{x}^*).$$

Theorem 2.3. [1] Let $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ be f° -pseudoconvex for all $i = 1, \dots, m$. Then the function

$$f(\mathbf{x}) = \max [f_i(\mathbf{x}) \mid i = 1, \dots, m]$$

is also f° -pseudoconvex.

Note, that for an f° -quasiconvex function f the level set $\text{lev}_\alpha f := \{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) \leq \alpha\}$ is a convex set for all $\alpha \in \mathbb{R}$ [15].

A vector \mathbf{x}^* is said to be a *global Pareto optimum* of (1), if there does not exist $\mathbf{x} \in S$ such, that

$$f_i(\mathbf{x}) \leq f_i(\mathbf{x}^*) \text{ for all } i = 1, \dots, k \text{ and } f_j(\mathbf{x}) < f_j(\mathbf{x}^*) \text{ for some } j.$$

Vector \mathbf{x}^* is said to be a *global weak Pareto optimum* of (1), if there does not exist $\mathbf{x} \in S$ such, that

$$f_i(\mathbf{x}) < f_i(\mathbf{x}^*) \text{ for all } i = 1, \dots, k.$$

Vector \mathbf{x}^* is a *local (weak) Pareto optimum* of (1), if there exists $\delta > 0$ such, that \mathbf{x}^* is a global (weak) Pareto optimum on $B(\mathbf{x}^*; \delta) \cap S$. Trivially every Pareto optimal point is weakly Pareto optimal.

The *contingent cone* and *polar cone* of set $S \in \mathbb{R}^n$ at point \mathbf{x} are defined respectively as

$$\begin{aligned} K_S(\mathbf{x}) &= \{\mathbf{d} \in \mathbb{R}^n \mid \text{there exist } t_i \downarrow 0 \text{ and } \mathbf{d}_i \rightarrow \mathbf{d} \text{ with } \mathbf{x} + t_i \mathbf{d}_i \in S\} \\ S^\leq &= \{\mathbf{d} \in \mathbb{R}^n \mid \mathbf{s}^T \mathbf{d} \leq 0, \text{ for all } \mathbf{s} \in S\}. \end{aligned}$$

The closure of a set S is denoted by $\text{cl } S$. A set $C \subset \mathbb{R}^n$ is a *cone* if $\lambda \mathbf{x} \in C$ for all $\lambda \geq 0$ and $\mathbf{x} \in C$. We also denote

$$\text{ray } S = \{\lambda \mathbf{x} \mid \lambda \geq 0, \mathbf{x} \in S\} \quad \text{and} \quad \text{cone } S = \text{ray conv } S.$$

In other words $\text{ray } S$ is the smallest cone containing S and the conic hull $\text{cone } S$ the smallest convex cone containing S . Furthermore, let

$$F(\mathbf{x}) = \bigcup_{i=1}^k \partial f_i(\mathbf{x})$$

and

$$G(\mathbf{x}) = \bigcup_{j \in J(\mathbf{x})} \partial g_j(\mathbf{x}), \text{ where } J(\mathbf{x}) = \{j \mid g_j(\mathbf{x}) = 0\}.$$

For the optimality condition we pose the following *constraint qualification*

$$G^\leq(\mathbf{x}) \subseteq K_S(\mathbf{x}). \quad (4)$$

Now we can present the following generalized KKT optimality conditions.

Theorem 2.4. [15] *If \mathbf{x}^* is a local weak Pareto optimum of (1) and the constraint qualification (4) is valid, then*

$$0 \in \text{conv } F(\mathbf{x}^*) + \text{cl cone } G(\mathbf{x}^*). \quad (5)$$

Moreover, if f_i are f° -pseudoconvex for all $i = 1, \dots, k$ and g_j are f° -quasiconvex for all $j = 1, \dots, m$, then the condition (5) is sufficient for \mathbf{x}^ to be a global weak Pareto optimum of (1).*

A feasible point $x^* \in S$ is called a *stationary point* for problem (1), if it satisfies the necessary optimality condition (5).

3 Multiobjective Proximal Bundle Method

In this section we develop the MPB (Multiobjective Proximal Bundle) method. The original proximal bundle method of [8] for nonsmooth convex and unconstrained single objective optimization was generalized to handle nonconvex and constrained problems in [13]. The MPB method is a further extension into a multiobjective case. The strategy of handling several objective functions is based on the ideas presented in [7, 17, 26]. The idea, in brief, is to move into a direction where the values of all the objective functions improve simultaneously.

3.1 Direction finding

The MPB method is not directly based on employing any scalarizing function. Some kind of scalarization is, however, needed in deriving the minimization method for all the objective functions. Theoretically, we utilize the *improvement function* $H : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ defined by

$$H(x, y) = \max [f_i(x) - f_i(y), g_j(x) \mid i = 1, \dots, k, j = 1, \dots, m].$$

Now we obtain the following connection between the improvement function and the problem (1).

Theorem 3.1. *A necessary condition for $x^* \in \mathbb{R}^n$ to be a global weak Pareto optimum of (1) is that*

$$x^* = \text{argmin}_{x \in \mathbb{R}^n} H(x, x^*). \quad (6)$$

Moreover, if f_i are f° -pseudoconvex for all $i = 1, \dots, k$, g_j are f° -quasiconvex for all $j = 1, \dots, m$ and the constraint qualification (4) is valid, then the condition (6) is sufficient for \mathbf{x}^ to be a global weak Pareto optimum of (1).*

Proof. Suppose first, that $x^* \in \mathbb{R}^n$ is a global weak Pareto optimum of (1). Since $x^* \in S$ we have $g_j(x^*) \leq 0$ for all $j = 1, \dots, m$, thus $H(x^*, x^*) = 0$. If x^* would not be a global minimizer of $H(\cdot, x^*)$, there would exist $y^* \in \mathbb{R}^n$ such that

$$H(y^*, x^*) < H(x^*, x^*) = 0.$$

Then we have $g_j(y^*) < 0$ for all $j = 1, \dots, m$, in other words, $y^* \in S$. Furthermore, $f_i(y^*) < f_i(x^*)$ for all $i = 1, \dots, k$, which contradicts the global weak Pareto optimality of x^* .

Suppose next that (6) holds true. Suppose also that f_i are f° -pseudoconvex for all $i = 1, \dots, k$, g_j are f° -quasiconvex for all $j = 1, \dots, m$ and (4) is valid. Since x^* is a global minimizer of $H(\cdot, x^*)$ by (2), Theorem 2.3 and Lemma 2.10 of [14] we have

$$\begin{aligned} 0 \in \partial H(x^*, x^*) &= \text{conv} \left\{ \bigcup_{i=1}^k \partial f_i(x^*) \cup \bigcup_{j \in J(x^*)} \partial g_j(x^*) \right\} \\ &= \text{conv} \{F(x^*) \cup G(x^*)\} \\ &\subseteq \text{conv} \{ \text{conv} F(x^*) \cup \text{conv} G(x^*) \} \\ &= \lambda \text{conv} F(x^*) + (1 - \lambda) \text{conv} G(x^*), \quad \text{where } \lambda \in [0, 1]. \end{aligned}$$

Then for $\lambda \in (0, 1]$ we have

$$\begin{aligned} 0 &\in \text{conv} F(x^*) + \frac{1 - \lambda}{\lambda} \text{conv} G(x^*) \\ &\subseteq \text{conv} F(x^*) + \text{ray conv} G(x^*) \\ &= \text{conv} F(x^*) + \text{cone} G(x^*) \\ &\subseteq \text{conv} F(x^*) + \text{cl cone} G(x^*). \end{aligned}$$

Thus, Theorem 2.4 implies that x^* is a global weak Pareto optimum of (1). \square

Let x^h be the current approximation to the solution of (1) at the iteration h . Then, by Theorem 3.1, we seek for the search direction d^h as a solution of

$$\begin{cases} \text{minimize} & H(x^h + d, x^h) \\ \text{subject to} & d \in \mathbb{R}^n. \end{cases} \quad (7)$$

Since (7) still is a nonsmooth problem, we must approximate it somehow. Let us assume for a moment that the problem (1) is convex. We suppose that, at the iteration h besides the current iteration point x^h , we have some auxiliary points $y^j \in \mathbb{R}^n$ from the past iterations and subgradients $\xi_{f_i}^j \in \partial f_i(y^j)$ for $j \in J^h =$

$\{1, \dots, h\}$, $i = 1, \dots, k$, and $\xi_{g_l}^j \in \partial g_l(y^j)$ for $j \in J^h$, $l = 1, \dots, m$. We linearize the objective and the constraint functions at the point y^j by

$$\begin{aligned}\bar{f}_{i,j}(x) &= f_i(y^j) + (\xi_{f_i}^j)^T(x - y^j) \quad \text{for all } i = 1, \dots, k, j \in J^h, \quad \text{and} \\ \bar{g}_{l,j}(x) &= g_l(y^j) + (\xi_{g_l}^j)^T(x - y^j) \quad \text{for all } l = 1, \dots, m, j \in J^h.\end{aligned}$$

Now we can define a convex piecewise linear approximation to the improvement function by

$$\hat{H}^h(x) = \max[\bar{f}_{i,j}(x) - f_i(x^h), \bar{g}_{l,j}(x) \mid i = 1, \dots, k, l = 1, \dots, m, j \in J^h]$$

and we get an approximation to (7) by

$$\begin{cases} \text{minimize} & \hat{H}^h(x^h + d) + \frac{1}{2}u^h\|d\|^2 \\ \text{subject to} & d \in \mathbb{R}^n, \end{cases} \quad (8)$$

where $u^h > 0$ is some weighting parameter. The penalty term $\frac{1}{2}u^h\|d\|^2$ is added to guarantee the existence and uniqueness of a solution to (8) and also to keep the approximation local enough. Notice that (8) still is a nonsmooth problem, but due to its minmax-nature it is equivalent to the following (smooth) quadratic problem

$$\begin{cases} \text{minimize} & v + \frac{1}{2}u^h\|d\|^2 \\ \text{subject to} & -\alpha_{f_i,j}^h + (\xi_{f_i}^j)^T d \leq v, \quad i = 1, \dots, k, j \in J^h \\ & -\alpha_{g_l,j}^h + (\xi_{g_l}^j)^T d \leq v, \quad l = 1, \dots, m, j \in J^h, \end{cases} \quad (9)$$

where

$$\begin{aligned}\alpha_{f_i,j}^h &:= f_i(\mathbf{x}^h) - \bar{f}_{i,j}(\mathbf{x}^h), \quad i = 1, \dots, k, j \in J^h, \quad \text{and} \\ \alpha_{g_l,j}^h &:= -\bar{g}_{l,j}(\mathbf{x}^h), \quad l = 1, \dots, m, j \in J^h,\end{aligned}$$

are so-called *linearization errors*.

In the nonconvex case, we replace the linearization errors by *subgradient locality measures*

$$\begin{aligned}\beta_{f_i,j}^h &:= \max[|\alpha_{f_i,j}^h|, \gamma_{f_i}\|\mathbf{x}^h - \mathbf{y}^j\|^2] \\ \beta_{g_l,j}^h &:= \max[|\alpha_{g_l,j}^h|, \gamma_{g_l}\|\mathbf{x}^h - \mathbf{y}^j\|^2],\end{aligned}$$

where $\gamma_{f_i} \geq 0$ for $i = 1, \dots, k$ and $\gamma_{g_l} \geq 0$ for $l = 1, \dots, m$, ($\gamma_{f_i} = 0$ if f_i is convex and $\gamma_{g_l} = 0$ if g_l is convex).

3.2 Line search

Let (d^h, v^h) be a solution of (9). We perform the following two-point line search strategy, which will detect discontinuities in the gradients of the objective functions. We assume that $m_L \in (0, \frac{1}{2})$, $m_R \in (m_L, 1)$ and $\bar{t} \in (0, 1]$ are some fixed line search parameters. First, we search for the largest number $t_L^h \in [0, 1]$ such that

$$\begin{aligned} \max [f_i(x^h + t_L^h d^h) - f_i(x^h) \mid i = 1, \dots, k] &\leq m_L t_L^h v^h, \quad \text{and} \\ \max [g_l(x^h + t_L^h d^h) \mid l = 1, \dots, m] &\leq 0. \end{aligned}$$

If $t_L^h \geq \bar{t}$, we take a *long serious step*:

$$x^{h+1} = x^h + t_L^h d^h \quad \text{and} \quad y^{h+1} = x^{h+1},$$

if $0 < t_L^h < \bar{t}$, then we take a *short serious step*:

$$x^{h+1} = x^h + t_L^h d^h \quad \text{and} \quad y^{h+1} = x^h + t_R^h d^h$$

and if $t_L^h = 0$, we take a *null step*:

$$x^{h+1} = x^h \quad \text{and} \quad y^{h+1} = x^h + t_R^h d^h,$$

where $t_R^h > t_L^h$ is such that

$$-\beta_{f_i, h+1}^{h+1} + (\xi_{f_i}^{h+1})^T d^h \geq m_R v^h.$$

The iteration is terminated when

$$-\frac{1}{2}v^h < \varepsilon_s,$$

where $\varepsilon_s > 0$ is an accuracy parameter supplied by the user.

3.3 Algorithm

Next we aggregate the previous subsections and present the algorithm of the multiobjective proximal bundle method .

Algorithm (MPB)

1. (*Initialization*) Select a feasible starting point $x^1 \in S$, a final accuracy tolerance $\varepsilon_s > 0$, an initial weight $u^1 > 0$, line search parameters $m_L \in (0, \frac{1}{2})$, $m_R \in (m_L, 1)$ and $\bar{t} \in (0, 1]$. Choose the distance measure parameters $\gamma_{f_i} \geq 0$ for $i = 1, \dots, k$ and $\gamma_{g_l} \geq 0$ for $l = 1, \dots, m$, ($\gamma_{f_i} = 0$ if f_i is convex and $\gamma_{g_l} = 0$ if g_l is convex). Set $h := 1$, $y^1 := x^1$ and calculate $\xi_{f_i}^1 \in \partial f_i(y^1)$ for $i = 1, \dots, k$ and $\xi_{g_l}^1 \in \partial g_l(y^1)$ for $l = 1, \dots, m$.

2. (*Direction finding*) Solve the problem (9) in order to get the solution (d^h, v^h) .
3. (*Stopping criterion*) If $-\frac{1}{2}v^h < \varepsilon_s$, then STOP.
4. (*Line search*) Find the step sizes $t_L^h \in [0, 1]$ and $t_R^h \in [t_L^h, 1]$. Set

$$x^{h+1} = x^h + t_L^h d^h \quad \text{and} \quad y^{h+1} = x^h + t_R^h d^h.$$
5. (*Updating*) Set $h := h + 1$, calculate $\xi_{f_i}^h \in \partial f_i(y^h)$ for $i = 1, \dots, k$ and $\xi_{g_l}^h \in \partial g_l(y^h)$ for $l = 1, \dots, m$. Choose $J^h \subseteq \{1, \dots, h\}$ and update the weight u^h . Go to Step 2.

The subgradient aggregation strategy due to [6] is used to bound the storage requirements (i.e., the size of the index set J^h). We use the line search algorithm of [13] to produce the step-sizes t_L^h and t_R^h in Step 4, and a modification of the weight updating algorithm of [8] is used to update the weight u^h in Step 5.

4 Convergence Analysis

Next we give two important convergence results. First we prove, that for f° -pseudoconvex functions the algorithm produces a global weak Pareto optimum of the problem (1), while in more general case it end ups with a substationary point.

Theorem 4.1. *Let f_i and g_j be f° -pseudoconvex and weakly semismooth functions for all $i = 1, \dots, k$ and $j = 1, \dots, m$, and the constraint qualification (4) be valid. If the MPB algorithm stops with a finite number of iterations, then the solution is a global weak Pareto optimum of (1). On the other hand, any accumulation point of an infinite sequence of solutions generated by the MPB algorithm is global weak Pareto optimum of (1).*

Proof. Due to Theorem 2.3 the improvement function H is f° -pseudoconvex. The formulation of the MPB algorithm implies, that it is equivalent to the proximal bundle algorithm applied to unconstrained single objective optimization of H . According to the convergence analysis of the standard proximal bundle algorithm (see, e.g., [8, 24]) if it stops with a finite number of iterations, then the solution x^h is a substationary point of a weakly semismooth H , in other words $0 \in \partial H(x^h, x^h)$. Then by Theorem 2.2 function H attains its global minimum at x^h . Since every f° -pseudoconvex function is also f° -quasiconvex, the first assertion follows from Theorem 3.1. The proof of the case, when MPB generates an infinite sequence of solutions, goes similarly. \square

Note, that in order to guarantee the f° -pseudoconvexity of the improvement function H also the constraint functions g_j are supposed to be f° -pseudoconvex in Theorem 4.1 although only the f° -quasiconvexity was required in Theorem 3.1.

Finally we show, that in more general case the algorithm produces substationary points of the problem (1).

Theorem 4.2. *Let the functions of (1) be weakly semismooth. If the MPB algorithm stops with a finite number of iterations, then the solution is a substationary point (i.e. satisfies the necessary optimality condition (5)). On the other hand, any accumulation point of an infinite sequence of solutions generated by the MPB algorithm is a substationary point.*

Proof. The proof is analogous to that of Theorem 4.1. □

5 Numerical Experiments

The efficiency and the reliability of the method is shown by some numerical experiments. The MPB algorithm was implemented in Fortran 77. The test runs have been performed on an Intel® Core™ 2 Duo CPU E8400 (3.00GHz, 2.99GHz) PC computer.

5.1 General tests

We wanted to test the method in different functions classes. First we formulated several f° -pseudoconvex objective functions. Next we combined f° -pseudoconvex functions with classical convex test examples from [1]. Finally some nonconvex test examples [1] being not f° -pseudoconvex nor f° -quasiconvex were solved. Furthermore, some f° -quasiconvex constraint functions were used in all the test examples. Thus the used function classes were

1. f° -pseudoconvex objective functions
2. f° -pseudoconvex + convex objective functions
3. Non(generalized)convex objective functions.

In all the test cases the number of variables n varied from 2 to 4, the number of objective functions k from 2 to 4, and the number of constraint functions m from 0 to 2. The test problems are given in Appendix A. The numerical results are presented in Table 1, where the first column refers to the above mentioned test classes, the second column tells the number of the test problems in the class. Finally, the last two columns are devoted to the average of the used iterations and function evaluations, respectively. The last line summarizes the overall average numbers. The parameters of MPB were tuned as follows: $\varepsilon_s = 10^{-5}$, $m_L = 0.01$, $m_R = 0.5$, $\bar{t} = 0.01$, $\gamma_{f_i} = 0.5$ (0 for convex objectives) for $i = 1, \dots, k$ and $\gamma_{g_l} = 0.5$ for $l = 1, \dots, m$. The initial weight was chosen by

$$u^1 = \frac{1}{k} \sum_{i=1}^k \|\xi_{f_i}^1\|.$$

Table 1: Computational results

Test class	# Problems	Iterations	Func. calls
1	36	5.1	6.7
2	70	10.4	15.4
3	6	8.7	13.2
All	112	8.6	12.5

In order to summarize the numerical results reported in Table 1 we can state that MPB method seems to be reliable and efficient in all the test classes. The reason why it needed more resources in class 2 with convex problems is the complexity of some single test problems. More details about the computational results are given in Appendix B.

5.2 Numerical example

In order to illustrate the functioning of MPB in more details we consider the following problem

$$\begin{cases} \text{minimize} & f_1(\mathbf{x}) = \sqrt{\|\mathbf{x}\| + 2} \\ & f_2(\mathbf{x}) = \max \{ -x_1 - x_2, -x_1 - x_2 + x_1^2 + x_2^2 - 1 \} \\ \text{subject to} & g(\mathbf{x}) = \max \{ x_1^2 + x_2^2 - 10, 3x_1 + x_2 + 1.5 \} \leq 0, \end{cases}$$

where f_1 is clearly f° -pseudoconvex (see [1]), f_2 is convex and g convex and thus f° -quasiconvex. We used first the starting point $\mathbf{x}^1 = (-0.5, -0.5)$ and the solution iteration by iteration is reported in Table 2.

Table 2: Results of the numerical example

h	x^h	$(f_1(x), f_2(x))$	Accuracy
0	$(-0.5000000, -0.5000000)$	$(1.645329, 1.000000)$	0.5181928
1	$(-0.4153649, -0.3124033)$	$(1.587367, 0.7277682)$	$0.9826704 \cdot 10^{-2}$
2	$(-0.4360219, -0.2067399)$	$(1.575612, 0.6427618)$	$0.3053751 \cdot 10^{-2}$
3	$(-0.4641460, -0.1123331)$	$(1.574022, 0.5764790)$	$0.4805499 \cdot 10^{-3}$
4	$(-0.4622420, -0.1137555)$	$(1.573542, 0.5759975)$	$0.4842027 \cdot 10^{-4}$
5	$(-0.4620497, -0.1138994)$	$(1.573493, 0.5759491)$	$0.4867036 \cdot 10^{-5}$

The numerical results are depicted in Figure 1, where red and blue colors refer to f_1 and f_2 , respectively. Together with the iteration points (black, final solution violet) there can be seen also the unconstrained and constrained optima and contour lines of the objectives. Note, that the Pareto optimal solutions lie on the line segment between red and blue points.

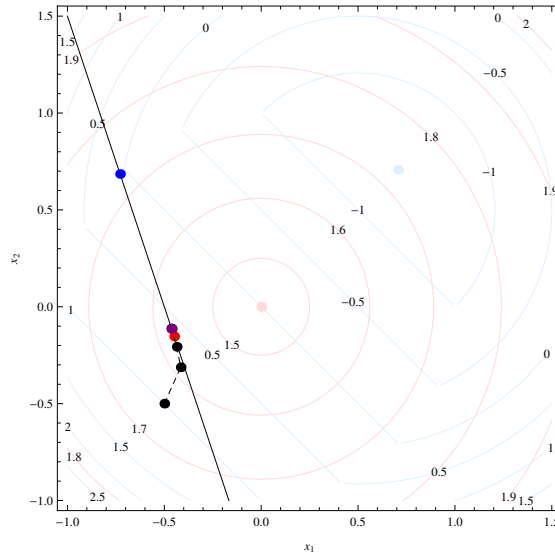


Figure 1: Iteration points of the numerical example

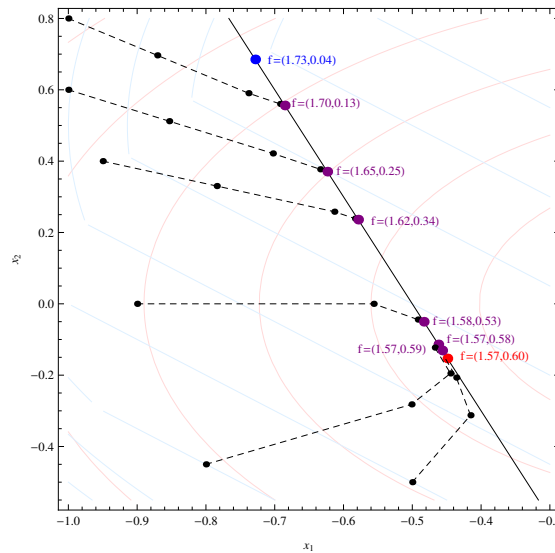


Figure 2: Results with several starting points

In Figure 2 we illustrate the functioning of MPB by starting the optimization from several starting points. According to the character of the method, MPB

projects those points to the Pareto optimal set by using the Chebyshev metric.

6 Conclusions

We have derived a multiobjective version of the proximal bundle method for non-smooth and nonconvex optimization. The objective functions are treated individually without employing any scalarization. The method is globally convergent and descent, and under some generalized convexity assumptions it can be proved to find globally weakly Pareto optimal solutions. This kind of method is needed in many application areas. Especially, it can be used as a part of interactive multiobjective optimization methods producing efficiently (weakly) Pareto optimal counterparts of nonoptimal solutions [16, 17, 20].

Acknowledgements. This work is financially supported by the University of Turku and the Vilho, Yrjö and Kalle Väisälä foundation.

References

- [1] A. Bagirov, N. Karmitsa and M. M. Mäkelä. *Introduction to Nonsmooth Optimization: Theory, Practice and Software*. Springer, 2014.
- [2] N. V. Banichuk and P. J. Neittaanmäki. *Structural Optimization with Uncertainties*. Solid Mechanics and Its Applications, 162. Springer, Berlin, 2010.
- [3] C. Charalambous and A. R. Conn. An efficient method to solve the min-max problem directly. *SIAM Journal of Numerical Analysis*, 15(1):162–187, 1978.
- [4] M. Gaudioso and M. F. Monaco. Quadratic approximations in convex non-differentiable optimization. *SIAM Journal on Control and Optimization*, 29:58–70, 1991.
- [5] J. Haslinger and P. Neittaanmäki. *Finite Element Approximation for Optimal Shape, Material and Topology Design*. John Wiley & Sons, Chichester, 1996.
- [6] K. C. Kiwiel. *Methods of Descent for Nondifferentiable Optimization*. Lecture Notes in Mathematics, 1133. Springer-Verlag, Berlin, 1985.
- [7] K. C. Kiwiel. A descent method for nonsmooth convex multiobjective minimization. *Large Scale Systems*, 8(2):119–129, 1985.

- [8] K. C. Kiwiel. Proximity control in bundle methods for convex nondifferentiable Optimization. *Mathematical Programming*, 46:105–122, 1990.
- [9] C. Lemaréchal. Nondifferentiable Optimization. In G. L. Nemhauser, A. H. G. Rinnooy Kan and M. J. Todd, editors, *Optimization*, pages 529–572, North-Holland, Amsterdam, 1989.
- [10] L. Lukšan and J. Vlček. Globally convergent variable metric method for convex nonsmooth unconstrained minimization. *Journal of Optimization Theory and Applications*, 102(3):593–613, 1999.
- [11] L. Lukšan and J. Vlček. Test problems for nonsmooth unconstrained and linearly constrained optimization. Technical Report 798, Institute of Computer Science, Academy of Sciences of the Czech Republic, Prague, 2000.
- [12] M. M. Mäkelä. Survey of bundle methods for nonsmooth optimization. *Optimization Methods and Software*, 17(1):1–29, 2002.
- [13] M. M. Mäkelä and P. Neittaanmäki. *Nonsmooth Optimization: Analysis and Algorithms with Applications to Optimal Control*. World Scientific Publishing Co., Singapore, 1992.
- [14] M. M. Mäkelä, V.-P. Eronen and N. Karmitsa. *On Nonsmooth Optimality Conditions with Generalized Convexities*. TUCS Technical Reports 1056, Turku Centre for Computer Science, Turku, 2012.
- [15] M. M. Mäkelä, V.-P. Eronen and N. Karmitsa. On Nonsmooth Multiobjective Optimality Conditions with Generalized Convexities. In Th. M. Rassias, C. A. Floudas, and S. Butenko, editors, *Optimization in Science and Engineering*, pages 333–357, Springer Verlag, 2014.
- [16] K. Miettinen. *Nonlinear Multiobjective Optimization*. Kluwer Academic Publishers, Boston, 1999.
- [17] K. Miettinen and M. M. Mäkelä. Interactive bundle-based method for nondifferentiable multiobjective optimization: NIMBUS. *Optimization*, 34:231–246, 1995.
- [18] R. Mifflin. A Modification and an extension of Lemarechal’s algorithm for nonsmooth minimization. *Mathematical Programming Study*, 176:77–90, 1982.
- [19] J. J. Moreau, P. D. Panagiotopoulos and G. Strang (Eds.). *Topics in Nonsmooth Mechanics*. Birkhäuser, Basel, 1988.
- [20] H. Mukai. Algorithms for multicriterion optimization. *IEEE Transactions on Automatic Control*, ac-25(2):177–186, 1980.

- [21] J. Outrata, M. Kočvara and J. Zowe. *Nonsmooth Approach to Optimization Problems with Equilibrium Constraints*. Theory, Applications and Numerical Results. Kluwer Academic Publishers, Dordrecht, 1998.
- [22] E. Polak and D. Q. Mayne and J. E. Higgins. Superlinearly convergent algorithm for min-max problems. *Journal of Optimization Theory and Applications*, 69:407–439, 1991.
- [23] J. B. Rosen and S. Suzuki. Constructing of non-linear programmin test problems. *Communications of the ACM*, 8:113, 1965.
- [24] H. Schramm and J. Zowe. A Version of the bundle idea for minimizing a nonsmooth functions: Conceptual idea, convergence analysis, numerical results. *SIAM Journal on Optimization*, 2:121–152, 1992.
- [25] R. E. Steuer. *Multiple Criteria Optimization: Theory, Computation, and Applications*. John Wiley & Sons, New York, 1986.
- [26] S. Wang. Algorithms for Multiobjective and Nonsmooth Optimization. In P. Kleinschmidt, F. J. Radermacher, W. Schweitzer and H. Wildermann, editors, *Methods of Operations Research*, 58, pages 131–142, Athenäum Verlag, Frankfurt am Main, 1989.

Appendix A. Test problems for generalized convex multiobjective optimization

We introduce f° -pseudoconvex objectives and f° -quasiconvex constraints.

Singleobjective problems

In this section we describe singleobjective small-scale nonsmooth unconstrained test problems which can be and have been combined to obtain multiobjective problems (see the next section). The number of variables varies from 2 to 4. The classification used is the same as in the book [1] but here we also have notations $P=f^\circ$ -pseudoconvex and $Q=f^\circ$ -quasiconvex. The used classification is described in Table 3.

Table 3: The classification of the test problems

	Description
E	Exact solution known
G	General objective function
M	Min-max type objective function
P	f° -pseudoconvex objective function
Q	f° -quasiconvex objective function
U	Unconstrained problem
X	Convex problem
Z	Nonconvex problem

f° -Pseudoconvex objectives

1. PC1 (f° -pseudoconvex 1)

Classification: G-U-P-E,
 Dimension: 2,
 Objective function: $f(\mathbf{x}) = \min \{ \|\mathbf{x}\|, \mathbf{x}^2 \}$,
 Optimum point: $\mathbf{x}^* = (0, 0)^T$,
 Optimum value: $f(\mathbf{x}^*) = 0$.

2. PC2 (f° -pseudoconvex 2)

Classification: G-U-P-E,
Dimension: 2,
Objective function: $f(\mathbf{x}) = \ln \{ \|\mathbf{x}\| + 2 \}$,
Optimum point: $\mathbf{x}^* = (0, 0)^T$,
Optimum value: $f(\mathbf{x}^*) = \ln 2$.

3. PC3 (f° -pseudoconvex 3)

Classification: G-U-P-E,
Dimension: 2,
Objective function: $f(\mathbf{x}) = \sqrt{\|\mathbf{x}\| + 2}$,
Optimum point: $\mathbf{x}^* = (0, 0)^T$,
Optimum value: $f(\mathbf{x}^*) = \sqrt{2}$.

4. PC4 (Shifted PC2)

Classification: G-U-P-E,
Dimension: 2,
Objective function: $f(\mathbf{x}) = \ln \{ \|\mathbf{x} + 1\| + 1 \}$,
Optimum point: $\mathbf{x}^* = (-1, -1)^T$,
Optimum value: $f(\mathbf{x}^*) = 0$.

5. PC5 (Shifted PC3)

Classification: G-U-P-E,
Dimension: 2,
Objective function: $f(\mathbf{x}) = \sqrt{\|\mathbf{x} - 2\| + 1}$,
Optimum point: $\mathbf{x}^* = (2, 2)^T$,
Optimum value: $f(\mathbf{x}^*) = 1$.

6. PC6 (f° -pseudoconvex 4)

Classification: G-U-P-E,
Dimension: 4,
Objective function: $f(\mathbf{x}) = \min \{ \|\mathbf{x}\|, \mathbf{x}^2 \}$,
Optimum point: $\mathbf{x}^* = (0, 0, 0, 0)^T$,
Optimum value: $f(\mathbf{x}^*) = 0$.

7. PC7 (f° -pseudoconvex 5)

Classification: G-U-P-E,
Dimension: 4,
Objective function: $f(\mathbf{x}) = \ln \{ \|\mathbf{x}\| + 2 \}$,
Optimum point: $\mathbf{x}^* = (0, 0, 0, 0)^T$,
Optimum value: $f(\mathbf{x}^*) = \ln 2$.

Convex objectives

8. CB3 (Charalambous/Bandler) [3]

Classification: GM-U-X-E,
Dimension: 2,
Objective function: $f(\mathbf{x}) = \max \{ x_1^4 + x_2^2, (2 - x_1)^2 + (2 - x_2)^2, 2e^{x_2 - x_1} \}$,
Starting point: $\mathbf{x}^{(1)} = (2, 2)^T$,
Optimum point: $\mathbf{x}^* = (1, 1)^T$,
Optimum value: $f(\mathbf{x}^*) = 2$.

9. DEM [13]

Classification: QM-U-X-E,
Dimension: 2,
Objective function: $f(\mathbf{x}) = \max \{ 5x_1 + x_2, -5x_1 + x_2, x_1^2 + x_2^2 + 4x_2 \}$,
Starting point: $\mathbf{x}^{(1)} = (1, 1)^T$,
Optimum point: $\mathbf{x}^* = (0, -3)^T$,
Optimum value: $f(\mathbf{x}^*) = -3$.

10. QL [13]

Classification: QM-U-X-E,
Dimension: 2,
Objective function: $f(\mathbf{x}) = \max_{1 \leq i \leq 3} f_i(\mathbf{x})$,
where
 $f_1(\mathbf{x}) = x_1^2 + x_2^2$,
 $f_2(\mathbf{x}) = f_1(\mathbf{x}) + 10(-4x_1 - x_2 + 4)$,
 $f_3(\mathbf{x}) = f_1(\mathbf{x}) + 10(-x_1 - 2x_2 + 6)$,
Starting point: $\mathbf{x}^{(1)} = (-1, 5)^T$,
Optimum point: $\mathbf{x}^* = (1.2, 2.4)^T$,
Optimum value: $f(\mathbf{x}^*) = 7.2$.

11. LQ [13]

Classification:	QM-U-X-E,
Dimension:	2,
Objective function:	$f(\mathbf{x}) = \max \{ -x_1 - x_2, -x_1 - x_2 + x_1^2 + x_2^2 - 1 \},$
Starting point:	$\mathbf{x}^{(1)} = (-0.5, -0.5)^T,$
Optimum point:	$\mathbf{x}^* = (1/\sqrt{2}, 1/\sqrt{2})^T,$
Optimum value:	$f(\mathbf{x}^*) = -\sqrt{2}.$

12. Mifflin 1 [13]

Classification:	G-U-X-E,
Dimension:	2,
Objective function:	$f(\mathbf{x}) = -x_1 + 20 \max \{ x_1^2 + x_2^2 - 1, 0 \},$
Starting point:	$\mathbf{x}^{(1)} = (0.8, 0.6)^T,$
Optimum point:	$\mathbf{x}^* = (1, 0)^T,$
Optimum value:	$f(\mathbf{x}^*) = -1.$

13. Wolfe [11]

Classification:	G-U-X-E,
Dimension:	2,
Objective function:	$f(\mathbf{x}) = 5\sqrt{9x_1^2 + 16x_2^2},$ when $x_1 \geq x_2 ,$ $f(\mathbf{x}) = 9x_1 + 16 x_2 ,$ when $0 < x_1 \leq x_2 ,$ $f(\mathbf{x}) = 9x_1 + 16 x_2 - x_1^9,$ when $x_1 \leq 0,$
Starting point:	$\mathbf{x}^{(1)} = (3, 2)^T,$
Optimum point:	$\mathbf{x}^* = (-1, 0)^T,$
Optimum value:	$f(\mathbf{x}^*) = -8.$

14. Rosen-Suzuki [23]

Classification:	QM-U-X-E,
Dimension:	4,
Objective function:	$f(\mathbf{x}) = \max_{1 \leq i \leq 4} f_i(\mathbf{x}),$
where	$f_1(\mathbf{x}) = x_1^2 + x_2^2 + 2x_3^2 + x_4^2 - 5x_1 - 5x_2 - 21x_3 + 7x_4,$ $f_2(\mathbf{x}) = f_1(\mathbf{x}) + 10(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1 - x_2 + x_3,$ $\quad -x_4 - 8),$ $f_3(\mathbf{x}) = f_1(\mathbf{x}) + 10(x_1^2 + 2x_2^2 + x_3^2 + 2x_4^2 - x_1 - x_4 - 10),$ $f_4(\mathbf{x}) = f_1(\mathbf{x}) + 10(2x_1^2 + x_2^2 + x_3^2 + 2x_4^2 - x_2 - x_4 - 5),$
Starting point:	$\mathbf{x}^{(1)} = (0, 0, 0, 0)^T,$
Optimum point:	$\mathbf{x}^* = (0, 1, 2, -1)^T,$
Optimum value:	$f(\mathbf{x}^*) = -44.$

Nonconvex objectives

15. Crescent [6]

Classification:	QM-U-Z-E,
Dimension:	2,
Objective function:	$f(\mathbf{x}) = \max \left\{ x_1^2 + (x_2 - 1)^2 + x_2 - 1, -x_1^2 - (x_2 - 1)^2 + x_2 + 1 \right\},$
Starting point:	$\mathbf{x}^{(1)} = (-1.5, 2),$
Optimum point:	$\mathbf{x}^* = (0, 0)^T,$
Optimum value:	$f(\mathbf{x}^*) = 0.$

16. Mifflin 2 [13]

Classification:	G-U-Z-E,
Dimension:	2,
Objective function:	$f(\mathbf{x}) = -x_1 + 2(x_1^2 + x_2^2 - 1) + 1.75 x_1^2 + x_2^2 - 1 ,$
Starting point:	$\mathbf{x}^{(1)} = (-1, -1)^T,$
Optimum point:	$\mathbf{x}^* = (1, 0)^T,$
Optimum value:	$f(\mathbf{x}^*) = -1.$

17. WF [11]

Classification:	GM-U-Z-E,
Dimension:	2,
Objective function:	$f(\mathbf{x}) = \max \left\{ \frac{1}{2} \left(x_1 + \frac{10x_1}{x_1+0.1} + 2x_2^2 \right), \frac{1}{2} \left(-x_1 + \frac{10x_1}{x_1+0.1} + 2x_2^2 \right), \frac{1}{2} \left(x_1 - \frac{10x_1}{x_1+0.1} + 2x_2^2 \right) \right\},$
Starting point:	$\mathbf{x}^{(1)} = (3, 1)^T,$
Optimum point:	$\mathbf{x}^* = (0, 0)^T,$
Optimum value:	$f(\mathbf{x}^*) = 0.$

18. SPIRAL [11]

Classification:	GM-U-Z-E,
Dimension:	2,
Objective function:	$f(\mathbf{x}) = \max \{ f_1(\mathbf{x}), f_2(\mathbf{x}) \},$
where	$f_1(\mathbf{x}) = \left(x_1 - \sqrt{x_1^2 + x_2^2} \cos \sqrt{x_1^2 + x_2^2} \right)^2 + 0.005(x_1^2 + x_2^2),$
	$f_2(\mathbf{x}) = \left(x_2 - \sqrt{x_1^2 + x_2^2} \sin \sqrt{x_1^2 + x_2^2} \right)^2 + 0.005(x_1^2 + x_2^2),$
Starting point:	$\mathbf{x}^{(1)} = (1.411831, -4.79462)^T,$
Optimum point:	$\mathbf{x}^* = (0, 0)^T,$
Optimum value:	$f(\mathbf{x}^*) = 0.$

19. Polak 6 [22]

Classification: PM-U-Z-E,

Dimension: 4,

Objective function: $f(\mathbf{x}) = \max_{1 \leq i \leq 4} f_i(\mathbf{x})$,

where

$$f_1(\mathbf{x}) = (x_1 - (x_4 + 1)^4)^2 + (x_2 - (x_1 - (x_4 + 1)^4)^2 + 2x_3^2 + x_4^2 - 5(x_1 - (x_4 + 1)^4) - 21x_3 + 7x_4,$$

$$f_2(\mathbf{x}) = f_1(\mathbf{x}) + 10((x_1 - (x_4 + 1)^4)^2 + (x_2 - (x_1 - (x_4 + 1)^4)^2 + x_3^2 + x_4^2 + (x_1 - (x_4 + 1)^4) - (x_2 - (x_1 - (x_4 + 1)^4)^4) + x_3 - x_4 - 8),$$

$$f_3(\mathbf{x}) = f_1(\mathbf{x}) + 10((x_1 - (x_4 + 1)^4)^2 + 2(x_2 - (x_1 - (x_4 + 1)^4)^2 + x_3^2 + 2x_4^2 - (x_1 - (x_4 + 1)^4) - x_4 - 10),$$

$$f_4(\mathbf{x}) = f_1(\mathbf{x}) + 10((x_1 - (x_4 + 1)^4)^2 + (x_2 - (x_1 - (x_4 + 1)^4)^2 + x_3^2 + 2(x_1 - (x_4 + 1)^4) - (x_2 - (x_1 - (x_4 + 1)^4)^4) - x_4 - 5),$$

Starting point: $\mathbf{x}^{(1)} = (0, 0, 0, 0)^T$,

Optimum point: $\mathbf{x}^* = (0, 1, 2, -1)^T$,

Optimum value: $f(\mathbf{x}^*) = -44$.

f° -Quasiconvex constraints

20. C1

Dimension: 2,

Constraint: $g(x) = \max \{x_1 + x_2 + 3, x_2 + 0.5\}$,

21. C2

Dimension: 2,

Constraint: $g(x) = \max \left\{ \ln(\sqrt{x_1^2 + x_2^2} + 1) - 1.5, x_1 + x_2 + 3.5 \right\}$,

22. C3

Dimension: 2,

Constraint: $g(x) = \max \{-x_1 - x_2 + 1.5, -x_2 + 0.5\}$,

23. C4

Dimension: 2,

Constraint: $g(x) = \max \{ x_1, x_2 - 6 \},$

24. C5

Dimension: 2,

Constraint: $g(x) = \max \{ 0.2x_1 + x_2, x_1 + 0.2 \},$

25. C6

Dimension: 2,

Constraint: $g(x) = \max \{ x_1 + x_2 - 2, x_1 - 0.9 \},$

26. C7

Dimension: 2,

Constraint: $g(x) = \max \{ -x_1 - x_2 + 0.5, -x_2 + 0.5 \},$

27. C8

Dimension: 2,

Constraint: $g(x) = \max \{ -x_1 - x_2 - 2, -x_2 + 0.5 \},$

28. C9

Dimension: 2,

Constraint: $g(x) = \max \{ x_1^2 + x_2^2 - 10, -3x_1 + x_2 + 2 \},$

29. C10

Dimension:

2,

Constraint:

$$g(x) = \max \{ x_1^2 + x_2^2 - 10, -3x_1 + x_2 + 1 \},$$

30. C11

Dimension:

2,

Constraint:

$$g(x) = \max \{ x_1^2 + x_2^2 - 30, x_1 - 3x_2 + 1 \},$$

31. C12

Dimension:

2,

Constraint:

$$g(x) = \max \{ x_1^2 + x_2^2 - 10, 3x_1 + x_2 + 1.5 \},$$

32. C13

Dimension:

2,

Constraint:

$$g(x) = \max \{ x_1^2 + x_2^2 - 10, 3x_1 - x_2 - 2 \},$$

33. C14

Dimension:

2,

Constraint:

$$g(x) = \max \{ x_1^2 + x_2^2 - 30, -3x_1 + x_2 + 2 \},$$

34. C15

Dimension:

2,

Constraint:

$$g(x) = \max \{ x_1^2 + x_2^2 - 30, 3x_1 - x_2 + 1 \},$$

35. C16

Dimension:

2,

Constraint:

$$g(x) = \max \{ x_1^2 + x_2^2 - 10, 3x_1 + x_2 + 1 \},$$

36. C17

Dimension: 4,
Constraint: $g(x) = \max \{ x_1^2 + x_2^2 + x_3^2 + x_4^2 - 20, x_1 + x_2 + x_3 + x_4 + 4 \},$

Multiobjective problems

In this section we combine the functions introduced in previous section to obtain multiobjective problems with f° -pseudoconvex objectives and f° -quasiconvex constraints ($g(x) \leq 0$). Due to relations between different generalized convexities the constraints may be also convex, pseudoconvex, f° -pseudoconvex, or subdifferentially regular quasiconvex functions. Moreover, also some of the objectives may be convex or pseudoconvex.

In addition to these problems for which we can prove the optimality conditions we introduce some generally nonconvex problems.

All objectives f° -pseudoconvex

37. M1

Dimension: 2,
No. of objectives: 2,
Objective functions: (1), (4)
Starting point: $\mathbf{x}^{(1)} = (-2, -2)^T,$
Constraints: (20) and/or (21).

38. M2

Dimension: 2,
No. of objectives: 2,
Objective functions: (1), (5)
Starting point: $\mathbf{x}^{(1)} = (-2, -2)^T,$
Constraints: (20) and/or (21).

39. M3

Dimension: 2,
No. of objectives: 3,
Objective functions: (1), (4), (5)
Starting point: $\mathbf{x}^{(1)} = (-2, -2)^T,$
Constraints: (20) and/or (21).

40. M4

Dimension: 2,
No. of objectives: 2,
Objective functions: (2), (4)
Starting point: $\mathbf{x}^{(1)} = (-2, -2)^T$,
Constraints: (20) and/or (21).

41. M5

Dimension: 2,
No. of objectives: 2,
Objective functions: (2), (5)
Starting point: $\mathbf{x}^{(1)} = (-2, -2)^T$,
Constraints: (20) and/or (21).

42. M6

Dimension: 2,
No. of objectives: 3,
Objective functions: (2), (4), (5)
Starting point: $\mathbf{x}^{(1)} = (-2, -2)^T$,
Constraints: (20) and/or (21).

43. M7

Dimension: 2,
No. of objectives: 2,
Objective functions: (3), (4)
Starting point: $\mathbf{x}^{(1)} = (-2, -2)^T$,
Constraints: (20) and/or (21).

44. M8

Dimension: 2,
No. of objectives: 2,
Objective functions: (3), (5)
Starting point: $\mathbf{x}^{(1)} = (-2, -2)^T$,
Constraints: (20) and/or (21).

45. M9

Dimension: 2,
No. of objectives: 3,
Objective functions: (3), (4), (5)
Starting point: $\mathbf{x}^{(1)} = (-2, -2)^T$,
Constraints: (20) and/or (21).

f° -pseudoconvex + convex objectives

46. M10–M15

Dimension: 2,
No. of objectives: 2,
Objective functions: (1) and one of next: (8)–(13)
Starting point: Same as that with the convex problem,
Constraints: (22) and/or (28).
Constraints: (26) and/or (29).
Constraints: (23) and/or (30).
Constraints: (24) and/or (31).
Constraints: (25) and/or (32).
Constraints: (22) and/or (33).

47. M16–M21

Dimension: 2,
No. of objectives: 2,
Objective functions: (2) and one of next: (8)–(13)
Starting point: Same as that with the convex problem,
Constraints: (22) and/or (28).
Constraints: (26) and/or (29).
Constraints: (23) and/or (33).
Constraints: (24) and/or (31).
Constraints: (27) and/or (32).
Constraints: (22) and/or (34).

48. M22–M27

Dimension: 2,
No. of objectives: 2,
Objective functions: (3) and one of next: (8)–(13)
Starting point: Same as that with the convex problem,
Constraints: (22) and/or (28).
Constraints: (26) and/or (29).
Constraints: (23) and/or (33).
Constraints: (24) and/or (31).
Constraints: (27) and/or (32).
Constraints: (22) and/or (33).

Remark: One can add problems (4) and (5) to above to have more objectives.

49. M28

Dimension: 4,
No. of objectives: 2,
Objective functions: (6), (14)
Starting point: $(-2, -2, -2, -2)^T$,
Constraints: (36).

50. M29

Dimension: 4,
No. of objectives: 2,
Objective functions: (7), (14)
Starting point: $(-2, -2, -2, -2)^T$,
Constraints: (36).

Nonconvex objectives

51. M30

Dimension: 2,
No. of objectives: 2,
Objective functions: (15), (16)
Starting point: $\mathbf{x}^{(1)} = (-1, -1)^T$,
Constraints: (35).

52. M31

Dimension: 2,
No. of objectives: 2,
Objective functions: (16), (17)
Starting point: $\mathbf{x}^{(1)} = (3, 1)^T$,
Constraints: (33).

53. M32

Dimension: 2,
No. of objectives: 2,
Objective functions: (16), (18)
Starting point: $\mathbf{x}^{(1)} = (-1, -1)^T$,
Constraints: (35).

54. M33

Dimension: 4,
No. of objectives: 2,
Objective functions: (6), (19)
Starting point: $(-2, -2, -2, -2)^T$,

55. M34

Dimension: 4,
No. of objectives: 2,
Objective functions: (7), (19)
Starting point: $(-2, -2, -2, -2)^T$,

Appendix B. Computational results

In the following we describe the results of 112 test problems. In the first two columns the problem is described such that in the first column is listed the used objectives and in the second column the used constraints. We give the objective function values which we obtain with MPB. In addition, the number of needed iterations and the function calls are given.

In Table 4 is described the test problems with f° -pseudoconvex objectives, in Table 5 and in Table 6 the test problems with f° -pseudoconvex and convex objectives and in Table 6 the test problems with nonconvex objectives. In the last line of the Table 6, the average numbers of iterations and function calls are given for all 112 test problems.

Table 4: Testproblems 1

All objectives f° -pseudoconvex				
Problems	Const.	$f(x^*)$	Iter.	Func.
(1), (4)		$(1.38795, 2.59165 \cdot 10^{-2})$	4	6
(1), (4)	(20)	$(2.12132, 0.53480)$	9	10
(1), (4)	(21)	$(2.47489, 0.72303)$	7	9
(1), (4)	(20), (21)	$(2.47489, 0.72303)$	7	9
(1), (5)		$(1.06756, 1.66158)$	3	4
(1), (5)	(20)	$(2.12133, 2.43921)$	6	7
(1), (5)	(21)	$(-1.75003, -1.75003)$	4	5
(1), (5)	(20), (21)	$(2.47492, 2.51064)$	4	5
(1), (4), (5)		$(1.29785, 0.11006, 2.26413)$	6	7
(1), (4), (5)	(20)	$(2.12133, 0.53480, 2.43921)$	6	7
(1), (4), (5)	(21)	$(2.47489, 0.72303, 2.51064)$	6	10
(1), (4), (5)	(20), (21)	$(2.47489, 0.72303, 2.51064)$	6	10
(2), (4)		$(1.22032, 2.55821 \cdot 10^{-2})$	2	3
(2), (4)	(20)	$(1.41618, 0.53482)$	5	6
(2), (4)	(21)	$(1.49847, 0.72303)$	5	7
(2), (4)	(20), (21)	$(1.49847, 0.72303)$	5	7
(2), (5)		$(0.82302, 1.88442)$	3	7
(2), (5)	(20)	$(1.41617, 2.43921)$	5	6
(2), (5)	(21)	$(1.49848, 2.51063)$	6	7
(2), (5)	(20), (21)	$(1.49848, 2.51063)$	6	7
(2), (4), (5)		$(1.22434, 1.22036 \cdot 10^{-2}, 2.28699)$	2	3
(2), (4), (5)	(20)	$(1.41617, 0.53481, 2.43921)$	5	6
(2), (4), (5)	(21)	$(1.49848, 0.72303, 2.51063)$	6	7
(2), (4), (5)	(20), (21)	$(1.49848, 0.72303, 2.51063)$	6	7
(3), (4)		$(1.83072, 6.07757 \cdot 10^{-2})$	3	7
(3), (4)	(20)	$(2.03011, 0.53482)$	5	6
(3), (4)	(21)	$(2.11539, 0.72303)$	4	6
(3), (4)	(20), (21)	$(2.11539, 0.72303)$	4	6
(3), (5)		$(1.45832, 1.92398)$	5	10
(3), (5)	(20)	$(2.03010, 2.43921)$	5	6
(3), (5)	(21)	$(2.11539, 2.51063)$	6	7
(3), (5)	(20), (21)	$(2.11539, 2.51063)$	6	7
(3), (4), (5)		$(1.84543, 8.54649 \cdot 10^{-3}, 2.28780)$	4	7
(3), (4), (5)	(20)	$(2.03010, 0.53481, 2.43921)$	5	6
(3), (4), (5)	(21)	$(2.11539, 0.72303, 2.51063)$	6	7
(3), (4), (5)	(20), (21)	$(2.11539, 0.72303, 2.51063)$	6	7

Table 5: Testproblems 2

f° -pseudoconvex + convex objectives				
Problems	Const.	$f(x^*)$	Iter.	Func.
(1), (8)		(0.40072, 4.81979)	7	12
(1), (8)	(22)	(1.40071, 2.03835)	7	13
(1), (8)	(28)	(1.09037, 3.09491)	10	15
(1), (8)	(22), (28)	(1.27785, 2.41485)	8	9
(1),(9)		$(3.05848 \cdot 10^{-4}, -1.74885 \cdot 10^{-2})$	11	19
(1), (9)	(26)	(0.25001, 2.25008)	20	22
(1), (9)	(29)	(0.10649, 1.19629)	11	13
(1), (9)	(26), (29)	(0.50002, 3.00005)	23	24
(1), (10)		(1.33724, 31.88659)	1	2
(1), (10)	(23)	(0.26216, 50.02199)	22	27
(1), (10)	(30)	(0.86445, 40.07434)	14	22
(1), (10)	(23), (30)	(0.26216, 50.02199)	22	27
(1), (11)		$(8.57864 \cdot 10^{-2}, -0.41421)$	13	14
(1), (11)	(24)	$(4.00134 \cdot 10^{-2}, 0.19776)$	10	11
(1), (11)	(31)	(0.22502, 0.59729)	8	9
(1), (11)	(24), (31)	(0.22546, 0.58642)	8	9
(1), (12)		(0.91355, -0.95576)	10	19
(1), (12)	(25)	(0.91813, -0.89998)	16	17
(1), (12)	(32)	(0.96521, -0.83772)	7	8
(1), (12)	(25), (32)	(0.96571, -0.83784)	7	9
(1),(13)		$(1.89739 \cdot 10^{-6}, 1.40885 \cdot 10^{-2})$	4	5
(1), (13)	(22)	(1.06073, 18.69192)	20	22
(1), (13)	(33)	(0.40003, 9.83213)	16	20
(1), (13)	(22),(33)	(1.07529, 18.12512)	18	21
(2), (8)		(1.21084, 2.16705)	11	19
(2), (8)	(22)	(1.21085, 2.16705)	4	5
(2), (8)	(28)	(1.21838, 2.09355)	9	20
(2), (8)	(22), (28)	(1.15902, 2.72730)	7	9
(2),(9)		$(0.69317, -5.66049 \cdot 10^{-5})$	9	19
(2), (9)	(26)	(0.91629, 2.25006)	12	15
(2), (9)	(29)	(0.84452, 1.19089)	9	13
(2), (9)	(26), (29)	(0.99588, 3.00008)	11	14
(2),(10)		(0.92824, 48.42818)	2	3
(2),(10)	(23)	(0.92697, 49.74089)	22	50
(2), (11)		(0.78670, -0.27739)	6	7
(2), (11)	(24)	(0.78877, 0.18360)	9	10
(2), (11)	(31)	(0.90628, 0.58298)	5	6
(2), (11)	(24), (31)	(0.90608, 0.59008)	6	8

Table 6: Testproblems 3

f° -pseudoconvex + convex objectives				
Problems	Const.	$f(\mathbf{x}^*)$	Iter.	Func.
(2), (12)		(1.06085, -0.88883)	11	20
(2), (12)	(27)	(1.08399, -0.81534)	7	8
(2), (12)	(32)	(1.07900, -0.82064)	7	8
(2), (12)	(27), (32)	(1.08399, -0.81534)	7	8
(2), (13)		(0.69314, $1.38206 \cdot 10^{-5}$)	4	5
(2), (13)	(22)	(1.11872, 18.63571)	17	29
(3), (8)		(1.79817, 2.54400)	10	20
(3), (8)	(22)	(1.79817, 2.54400)	4	5
(3), (8)	(28)	(1.82140, 2.28799)	8	12
(3), (8)	(22), (28)	(1.78165, 2.77286)	7	9
(3), (9)		(1.41484, $-1.79013 \cdot 10^{-3}$)	13	35
(3), (9)	(26)	(1.58114, 2.25014)	11	14
(3), (9)	(29)	(1.54299, 0.86318)	9	13
(3), (9)	(26), (29)	(1.64533, 3.00011)	10	13
(3), (10)		(1.73610, 38.35306)	2	3
(3), (10)	(23)	(1.61541, 48.18086)	20	45
(3), (11)		(1.42339, $-3.68438 \cdot 10^{-2}$)	6	7
(3), (11)	(24)	(1.48324, 0.19938)	9	11
(3), (11)	(31)	(1.57349, 0.57595)	5	6
(3), (11)	(24), (31)	(1.57309, 0.58955)	6	8
(3), (12)		(1.69740, -0.88116)	11	26
(3), (12)	(27)	(1.71900, -0.81362)	7	8
(3), (12)	(32)	(1.71400, -0.81897)	7	8
(3), (12)	(27), (32)	(1.71900, -0.81362)	7	8
(1), (13)		(1.41769, $-8.86413 \cdot 10^{-2}$)	22	46
(1), (13)	(22)	(1.74986, 18.50659)	17	30
(1), (13)	(33)	(1.62249, 9.83447)	13	28
(1), (13)	(22), (33)	(1.75365, 18.12519)	11	15
(6), (14)		($8.74976 \cdot 10^{-3}$, -2.15762)	15	27
(6), (14)	(36)	(2.00001, 59.00077)	14	15
(7), (14)		(0.69713, -0.18549)	9	10
(7), (14)	(36)	(1.38629, 59.00154)	9	10
Nonconvex objectives				
Problems	Const.	$f(\mathbf{x}^*)$	Iter.	Func.
(15), (16)		(0.17156, -0.72429)	9	10
(15), (16)	(35)	(0.76838, $-1.33175 \cdot 10^{-2}$)	10	11
(16), (17)		(-0.41712, 3.72341)	9	10
(16), (17)	(33)	(30.75000, 7.33870)	10	32
(16), (18)		(-0.36199, 0.26112)	2	3
(16), (18)	(35)	($-9.29430 \cdot 10^{-3}$, 0.48864)	12	13
Average:			8.6	12.5

TURKU
CENTRE *for*
COMPUTER
SCIENCE

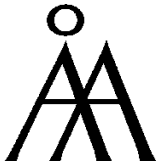
Joukahaisenkatu 3-5 A, 20520 TURKU, Finland | www.tucs.fi



University of Turku

Faculty of Mathematics and Natural Sciences

- Department of Information Technology
 - Department of Mathematics
- Turku School of Economics*
- Institute of Information Systems Sciences



Åbo Akademi University

- Department of Computer Science
- Institute for Advanced Management Systems Research

ISBN 978-952-12-3098-1
ISSN 1239-1891