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Abstract

A strategic game with a finite number of players where initial coefficients (costs) of linear payoff functions are subject to perturbations is considered. We define robust solution as a feasible solution which for a given set of realizations of uncertain parameters guarantees the minimum value of the worst-case relative regret among all feasible solutions. For different (either Pareto or Nash) equilibria principles considered, appropriate definitions of the worst-case relative regret are specified. It is shown that these definitions are closely related to the concept of accuracy function being recently intensively studied in the literature. We also present the concept of robustness tolerances of a single cost vector associated with a strategy choice of a player. The tolerance is defined as the maximum level of perturbation of the cost vector which does not destroy the game solution robustness. We present formulae allowing the calculation of the robustness tolerance with respect to a chosen equilibrium obtained for some initial costs. The results are illustrated with several numerical examples.

Keywords: Nash equilibrium, Pareto equilibrium, robust measure, worst-case relative regret, accuracy functions.

1 Introduction

While solving practical optimization problems, it is necessary to take into account various kinds of uncertainty due to lack of input data, inadequacy of mathematical models to real processes, rounding off, calculating errors etc. It is known that in many cases initial data as a link between a reality and a model can not be defined explicitly. The initial data is defined with a certain error, generally depend on many parameters and require to be specified during the problem solving process. In practice any problem can not be properly posed and solved without at least implicit use of the results of stability analysis and related issues of parametric analysis. Therefore widespread use of discrete optimization models in the last decades inspired many specialists to investigate various aspects of ill-posed problems theory and, in particular, the stability issues.

The implications of enhanced optimization methods have in some areas been lead to the situation that optimal or near-optimal solutions have become ”too good”. For example, in design of a communication network a network configuration can now be made so good (with respect to the original objective optimization) that there is hardly any possibility left in the network to accommodate for potential disruptions and possible contingency in terms of e.g. routing delays. Similar problems are faced nowadays in many other areas where deterministic models do not properly reflect possible uncertainty of input parameters. In practice, it usually leads to undesirable situations where optimality (sometimes even feasibility) of solutions is very sensitive to some possible realizations of problem parameters. Thus, chasing for solution optimality, we lose its robustness and vice versa.

As a consequence, two lines of research within the operations research and mathematical optimization community have been initiated:

- Post-optimal and parametric analysis investigate how an optimal solution found behave in response to initial data (problem parameters) changes. A general sensitivity and stability analysis methodology is used based on analyzing the properties of the point-to-set mapping which specifies the optimality principle of the problem. Such research methods have been studied in great detail and covered e.g. in the literature on optimization problems with a continuous set of feasible solutions. Numerous articles are devoted to analysis of conditions when a problem solution possesses a certain property of invariance to the problem parameters perturbations (see, e.g. [11, 34, 39, 40]).

- Robust Optimization – instead of producing an optimal solution for a normal situation, which is described by deterministic models but rarely occurs in practice, and where recovery to optimality can be complicated, the aim is to produce solutions that optimize an additionally constructed objective. The objective must assure that the optimal solution will remain feasible under worst case realization of uncertain problem input parameters. Worst-case optimization is also known as robust optimization, and optimal solutions of worst case optimization are often referred to as robust solutions (see e.g. [16]).

The main drawback of all classical single objective models is that they do not take into account the real multiple criteria nature of real-life problems. It is well-known that under multiobjective framework a solution which is optimal with respect to one single objective might be arbitrarily bad with respect to the others and thus will be unacceptable for a decision maker. Thus, many problems arising in optimization, management and decision making should be ultimately considered under multicriteria framework due to existing of several conflicting goals or interests. Therefore recent interest of applied mathematicians and operations research scientists in multicriteria optimization problems keeps very high. It is confirmed by the intensive publishing activity (see e.g. monographs [9, 24, 38] and bibliography [10]).

The main difficulty while studying stability of discrete optimization problems is discrete models complexity, because even small changes of initial data make a model behave in an unpredictable manner. There are a lot of papers (see e.g. [5, 12, 13, 17, 35, 36, 37]) devoted to analysis of scalar and vector (multicriteria) discrete optimization problems sensitivity to parameters perturbations.

The present work continues investigations of different aspects of sensitivity analysis for different types of discrete optimization problems with various partial criteria and optimality principles (see e.g. [6, 7, 8, 21, 23, 28, 29]). We consider a multiobjective Boolean linear programming problem which is simply reformulated under game theoretic framework. The game theory terminology is used in order to make basic concepts and definitions stated in a clearer and intuitively more understandable form. A strategic game with a finite number of players in which initial coefficients (costs) of linear payoff functions are subject to perturbations is considered in the present work. We define robust solution as a feasible solution which for a given set of realizations of uncertain parameters guarantees the minimum value of the worst-case relative regret among all feasible solutions. For two different equilibria principles considered, Pareto and Nash equilibria, appropriate definitions of the worst-case relative regret are specified. We show that these definitions are closely related to the concept of accuracy function which has been recently intensively studied in the literature (see e.g. [18, 21, 29]). We also present the concept of robustness tolerance of a single cost vector associated with a strategy choice of a player, which is defined as the maximum level of perturbation of the cost vector which does not destroy the game solution robustness. In this paper we present formulae which allow calculating the robustness tolerances with respect to an equilibrium (in Pareto, lexicographic or Nash senses) obtained for some initial costs. We illustrate the results with several numerical examples.

The paper is organized as follows. In section 2 we formulate the problem in details and define two basic optimality (equilibria) principles. In section 3 we give a short excursus into the topic of robust optimization and define appropriate robustness measures for various optimality principles considered. Section 4 is devoted to the concept of accuracy function as a tool of post-optimal analysis which is used to describe the behavior of optimal solution under uncertainty of initial problem data. We specify analytical expression to calculating accuracy function for the chosen optimality principle. We also show that accuracy functions can straightforward be used to analyze solution robustness. In section 5, we focus on analyzing the case when only one vector cost is uncertain. We present formulae which allow calculating the robustness tolerances. The theoretical results presented in section 5 are illustrated with numerical examples given in section 6. Some concluding remarks and open problems are summarized in section 7.

2 Problem formulation

We consider a strategic game with $m \geq 2$ players. Let X_i be a finite set of (pure) strategies of the player $i \in N_m := \{1, 2, ..., m\}$. We assume that $|X_i| = 2$ for all $i \in N_m$ indicating each player has a choice of 2 antagonistic strategies to play. For simplicity, we define $X_i := \{0, 1\}$ for all $i \in N_m$, that is the choice of each strategy is encoded by means of Boolean variables x_i. The set of all feasible solutions X can be generally defined as a subset of the Cartesian product over all players of their sets of strategies

$$X \subset \prod_{i \in N_m} X_i = \{0, 1\}^m.$$

Observe that now - formally - X is a subset of the set of all ordered m-tuples. We also assume that there exists at least one player j who selects non-zero strategy to play, namely, for whom $x_j = 1$. Thus, $0_{(m)} := (0, 0, ..., 0)^T \notin X$.

A vector of payoff functions (payoff profile)

$$p(C, x) := (p_1(C, x), ..., p_m(C, x))^T$$

consists of individual payoff functions $p_i(C, x)$ for each player $i \in N_m$, which are defined as linear functions on the set of solutions X:

$$p_i(C, x) := C_i x.$$

Here C_i is i-th row of matrix $C = [c_{ij}] \in \mathbf{R}_+^{m \times m}$, $x := (x_1, x_2, ..., x_m)^T$, $x_i \in X_i$, $i \in N_m$.

Note that for each player $i \in N_m$, the individual payoff $p_i(C, x)$ depends on solution x, that is on the strategy chosen by player i as well as the strategies chosen by all the other players. Thus, a set of payoff profiles $PP(C, X)$ is the following

$$PP(C, X) := \{p(C, x) : \ x \in X\}.$$

The game in the normal form consists in the following: the players, using some relations of preference and trying to minimize the individual payoff functions, select their strategies x_i once from the sets X_i and as a result, the solution $x \in X$ is formed. After that, each player i obtains a payoff $p_i(C, x)$. The game terminates at this stage. We will call any such game a game with the matrix C. This is a non-iterative game of m players with complete information, that is in other words all players have information about preferences of other players before selecting strategies. In other words, the matrix C is known to all players before the actual game starts. It is clear that trying to minimize own payoff, each player would prefer strategy 0 to strategy 1, however playing 0 can be prohibited due to the choice of the strategies by other players. Therefore, the choice of which strategy to play is restricted by the strategy choice made by the other players. If the strategy choice made by the other players does not restrict the choice of the players, the (s)he will ultimately select strategy 0 in pursuit of payoff minimization. Thus, each player must correlate own strategy selection with the choice of the other players, in order to reach feasibility, as well as all players are eager to reach some chosen equilibria solutions (either Pareto or Nash).

Below we formulate two classical definitions of equilibria situations. The first equilibrium concept is the well-known Nash equilibrium, which was originally introduced in [30] and [31].

Define $\bar{x}_i = 1$ if $x_i = 0$, and $\bar{x}_i = 0$ otherwise. For any given solution $x^* \in X$, a set of solutions accessible by changing the strategy of player i only is defined as:

$$W_i(x^*) := X_i \times \prod_{j \in N_m \setminus \{i\}} x_j^* = \{(x_1^*, x_2^*, ..., x_i^*, ..., x_m^*), \ (x_1^*, x_2^*, ..., \bar{x}_i^*, ..., x_m^*)\}.$$

Thus, if $(x_1^*, x_2^*, ..., \bar{x}_i^*, ..., x_m^*)$ is feasible, then $W_i(x^*) \cap X$, the set of feasible solutions accessible by changing the strategy of player i, contains two solutions, otherwise a single solution $(x_1^*, x_2^*, ..., x_i^*, ..., x_m^*)$ belongs to $W_i(x^*) \cap X$ only.

A solution $x^* \in X$ is called **Nash equilibrium** in the strategic game with matrix C if for every player $i \in N_m$ the following inequality holds $p_i(C, x^*) \leq p_i(C, x)$ for all $x \in W_i(x^*) \cap X$. In other words, the concept of Nash equilibrium describes a situation in the game, where no player can improve their payoffs by changing own strategies only.

On the other hand, players could be oriented on rather a common benefit than own personal profit, so then the concept of Nash equilibrium transforms into the well-know concept of the Pareto efficiency or Pareto equilibrium [33].

A solution $x^* \in X$ is called **Pareto equilibrium** in the strategic game with matrix C if there exists no solution $x \in X$ such that $p_i(C, x) \leq p_i(C, x^*)$ for all $i \in N_m$, and $p_j(C, x) < p_j(C, x^*)$ for some $j \in N_m$. In other words, the concept of Pareto equilibrium describes a situation in the game, where players can improve own payoffs only at the expense of some other players payoffs.

These two equilibrium cases characterize two polar solutions: either all players behave independently and orient on their own profit only in the case of Nash equilibrium or all players concentrate on mutual interest and common behavior as in the case of Pareto equilibrium. For the game with matrix C, we denote $P^m(C)$ and $N^m(C)$ the set of Pareto and Nash equilibria, respectively.

3 Robust deviations and equilibria

One of the most interesting branches of combinatorial optimization and mathematical programming that has emerged over the past 20 - 30 years is robust optimization. Since the early 1970s there has been an increasing interest in the use of robust optimization models. The theory of robustness deals with uncertainty of problem parameters. The presence of such parameters in optimization models is caused by inaccuracy of initial data, non-adequacy of models to real processes, errors of numerical methods, errors of rounding off and other factors. So it appears to be important to identify classes of models and their solutions which play against the worst-case (in some sense) realization of input parameters. It is commonly accepted fact nowadays that any optimization problem arising in practice can hardly be adequately formulated and solved without usage of results of the theory of robustness.

Authors of most papers devoted to robust optimization attempt to answer to the following closely related questions: How can one represent uncertainty? What is a robust solution? What could be a proper robustness measure? How to calculate robust solutions? How to interpret worst case realization under uncertainty? and many others. Different answers to these questions lead to different research approaches and investigation directions. Bibliographical analysis provides us with a list of contributors who proposed several main avenues in the theory of robustness:

Minmax Regret Optimization

e.g. Averbakh [1], Kouvelis and Yu [16];

Robust Optimization with Ellipsoidal Uncertainty

Ben-Tal, Nemirovski [2];

Worst Case Optimization with Penalties

e.g. Mulvey et al. [27];

Flexible Robust Optimization

Bertsimas and Sim [3];

Absolute and Relative Robustness

e.g. Yaman et al. [42], Montemanni and Gambardella [25, 26], Kasperski [14, 15] and Zielinski [43].

However all the robustness models listed above are primary dealing with the definition of robustness in single objective optimization, without touching at all the multiobjective specific. While moving from a single objective to multiobjective case, the definition of robustness must be accurately tuned to reflect properly the specific of chosen optimality principle.

To define uncertainty in the game theory model described in section 2, we will assume that the set of game solutions X is fixed but the original payoff matrix C^0 can change or it is given with errors. Let $S(C^0)$ be a set of all possible realizations of the matrix C^0, called the scenarios. Let us also assume that $C \in \mathbf{R}_+^{m\times m}$ for any $C \in S(C^0)$, thus we guarantee that $p_i(C, x) > 0$ for all $x \in X$ and $i \in N_m$. This is due to our assumption mentioned in previous section that at least one player always chooses strategy 1 to play, so the game solution $x = 0_{(m)}$ is not feasible. We will follow the approach that define robustness measure as a maximum relative error (worst-case relative regret) of the solution considered over the set of all scenarios. Our aim is construct a new objective that incorporates possible worst realization of uncertain parameters. In [16] one can find examples of different robustness measures and wide discussion on related complexity issues. While dealing with multiobjective case, the definition of robustness measures must be adapted to reflect the specific of the multiple objective optimality principle chosen.

For given $x, \tilde{x} \in X$, fixed index (player) $i \in N_m$ and arbitrary $C \in \mathbf{R}_+^{m\times m}$ denote the relative deviation

$$\Delta_i(C, \tilde{x}, x) := \frac{p_i(C, \tilde{x}) - p_i(C, x)}{p_i(C, x)}. \tag{1}$$

Definition 1 *For any given solution $\tilde{x} \in X$, the worst-case relative regret (or robust deviation in other terminology) of this solution on the set $S(C^0)$ is defined as follows:*

in Pareto equilibrium case:

$$REG_P(S(C^0), \tilde{x}) := \max_{C \in S(C^0)} \max_{x \in X} \min_{i \in N_m} \Delta_i(C, \tilde{x}, x); \tag{2}$$

in Nash equilibrium case:

$$REG_N(S(C^0), \tilde{x}) := \max_{C \in S(C^0)} \max_{i \in N_m} \max_{x \in W_i(\tilde{x}) \cap X} \Delta_i(C, \tilde{x}, x). \tag{3}$$

The difference in $REG_N(S(C^0)$ and $REG_P(S(C^0)$ reflects the difference in Pareto and Nash equilibria principles. While in Pareto case, the given solution $\tilde{x}$ must be compared with all other feasible solutions (including the solution $\tilde{x}$ itself to guarantee that $REG_N(S(C^0), \tilde{x}) = 0$ if $\tilde{x} \in P^m(C^0)$), in the Nash case it is sufficient to compare it with solutions $x \in W_i(\tilde{x}) \cap X$ only. Both $REG_N(S(C^0)$ and $REG_P(S(C^0)$ give quantitative expressions to measure the relative distance how far the solution $\tilde{x}$ from optimality under the worst case scenario, i.e. the scenario which delivers maximum over the set of all possible scenarios $S(C^0)$.

For the sake of brevity, we will use notation $REG_{P,N}(S(C^0), x^*)$ when saying something about robust deviations of both optimality principles at once. In a trivial case, when only one (w.l.o.g. i-th) player could make the game assessment and decides which strategies will be played by the other players, all equilibrium situations transform into standard single objective optimality case, so the single player robust deviation is expressed as

$$REG_i(S(C^0), \tilde{x}) := \max_{C \in S(C^0)} \max_{x \in X} \Delta_i(C, \tilde{x}, x) = \max_{C \in S(C^0)} \frac{p_i(C, \tilde{x}) - \min\limits_{x \in X} p_i(C, x)}{\min\limits_{x \in X} p_i(C, x)}. \tag{4}$$

which is identical to the well-known single objective robustness measure (see e.g. [21]).

Definition 2 *A feasible solution $x^* \in X$ is called* **robust** *for the set of scenarios $S(C^0)$ if and only if it has minimal (among all feasible solutions) robust deviation. This happens if the following inequalities hold*

$$REG_{P,N}(S(C^0), x^*) \leq REG_{P,N}(S(C^0), x) \quad \textit{for every} \;\; x \in X. \tag{5}$$

As we stated before, the main goal of robust optimization is to construct a new objective function which will plays against the worst-case scenario. The robust solution is that one which has the smallest among all solutions robust deviation as the definition above stated. Now we would like to show that the measures which are used to quantify the solution robustness can be used (and was indeed mentioned in literature) also in framework of sensitivity and post-optimal analysis. Developing in parallel for many years, those two theories have very much in common, under the closer comparison.

4 Relative errors and accuracy functions

In [19], [22] it was proposed to measure the quality of solutions by means of the so-called accuracy function. In this paper we introduce similar function by analogy with [29].

Definition 3 *In case of Pareto equilibria for $x^* \in X$ and a given matrix $C \in \mathbf{R}_+^{m\times m}$, the relative error of this solution is defined as:*

$$\varepsilon_P(C, x^*) := \max_{x\in X} \min_{i\in N_m} \Delta_i(C, x^*, x). \tag{6}$$

Similar in case of Nash equilibria for $x^* \in X$ and a given matrix $C \in \mathbf{R}_+^{m\times m}$, the relative error of this solution is defined as:

$$\varepsilon_N(C, x^*) := \max_{i\in N_m} \max_{x\in W_i(x^*)\cap X} \Delta_i(C, x^*, x). \tag{7}$$

The difference in definitions of $\varepsilon_{P,N}(C, x^*)$, reflects the difference in the corresponding definitions of equilibria situations. Notice that in single objective case (one player i only) both $\varepsilon_P(C, x^*)$ and $\varepsilon_N(C, x^*)$ transforms into well-known (c.f. e.g. [20])

$$\varepsilon_i(C, x^*) := \max_{x\in X} \Delta_i(C, x^*, x) = \frac{p_i(C, x^*) - \min\limits_{x\in X} p_i(C, x)}{\min\limits_{x\in X} p_i(C, x)}. \tag{8}$$

So, the calculating of the relative error in single objective case is as hard as solving the original problem.

Observe that for an arbitrary $C \in \mathbf{R}_+^{m\times m}$ we have $\varepsilon_{P,N}(C, x^*) \geq 0$. If $\varepsilon_P(C, x^*) > 0$ ($\varepsilon_N(C, x^*) > 0$), then $x^* \notin P^m(C)$ ($x^* \notin N^m(C)$) and this positive value of the relative error may be treated as a measure of inefficiency of the strategy profile x^* for the game with matrix C. The equality $\varepsilon_N(C, x^*) = 0$ automatically implies that $x^* \in N^m(C)$. So, for the solution x^* to belong to $N^m(C)$ it is necessary and sufficient to have $\varepsilon_N(C, x^*) = 0$.

In the Pareto case the situation is a bit more complicated. The equality $\varepsilon_P(C, x^*) = 0$ formulates in general only necessary condition for x^* to be Pareto equilibrium in the game with matrix C, i.e. $\varepsilon_P(C, x^*) = 0$ does not guarantee that $x^* \in P^m(C)$. Indeed, consider the following two examples.

Example 1. Let $m = 2$ and $\tilde{C} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. Assume that $X = \{x^1, x^2\}$, $x^1 = (0, 1)^T$, $x^2 = (1, 0)^T$. Then $PP(\tilde{C}, X) = \{(2, 1)^T, (1, 2)^T\}$. If we consider the matrix $\bar{C} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$, then $PP(\bar{C}, X) = \{(1, 1)^T, (1, 2)^T\}$. Evidently,

$x^2 \in P^2(\tilde{C})$ and $\varepsilon_P(\tilde{C}, x^2) = 0$, but $x^2 \notin P^2(\bar{C})$ and $\varepsilon_P(\bar{C}, x^2) = 0$.

However, later (in Proposition 1) we will show, that if equality $\varepsilon_P(C', x^*) = 0$ is valid for every matrix C' in some open neighborhood of C (i.e.there is $\phi > 0$ such that $\varepsilon_P(C', x^*) = 0$ for any C', $\| C' - C \| < \phi$, where $\| \cdot \|$ denotes a norm in $\mathbf{R}^{m \times m}$), then this equality provides also sufficient condition for the solution x^* to be Pareto equilibrium in the game with matrix C.

From now we assume that some originally specified matrix $C^0 = \{c^0_{ij}\} \in \mathbf{R}^{m \times m}_+$ defines the original problem data. In the following we are interested in the maximum value of the errors $\varepsilon_P(C, x^*)$ and $\varepsilon_N(C, x^*)$ when the matrix C belongs to some specified set, the so-called set of perturbed matrices. We are interested in relative perturbations of the elements of C^0, and the quality of a given solution x^* is described by the so-called **accuracy function**. The value of the accuracy function for a given $\delta \in [0, 1)$ is equal to the maximum relative error of the solution x^* under the assumption that the weights of the elements are perturbed by no more than $\delta \cdot 100\%$ of their original values specified by matrix C^0. Notice that if we compare two different equilibria for the game with matrix C from the point of view of their accuracy on data perturbation, then the smaller values of the accuracy function are more preferable. Thus, accuracy function may be used to evaluate the quality of the game solutions from the accuracy point of view.

For a given $\delta \in [0, 1)$, consider a set of perturbed matrices

$$\Theta_\delta(C^0) := \{C \in \mathbf{R}^{m \times m}_+ : \ |c_{ij} - c^0_{ij}| \leq \delta \cdot c^0_{ij}, \ i \in N_m, \ j \in N_m\}. \qquad (9)$$

Definition 4 *For $x^* \in X$ and $\delta \in [0, 1)$, the value of the accuracy function in Pareto case is defined as:*

$$A_P(C^0, x^*, \delta) := \max_{C \in \Theta_\delta(C^0)} \varepsilon_P(C, x^*). \qquad (10)$$

For $x^* \in X$ and $\delta \in [0, 1)$, the value of the accuracy function in Nash case is defined as:

$$A_N(C^0, x^*, \delta) := \max_{C \in \Theta_\delta(C^0)} \varepsilon_N(C, x^*). \qquad (11)$$

Notice that these definitions imply equivalence between accuracy functions and corresponding robust deviations, respectively. However, recall, that the robust deviation measures were used as a tool of constructing a new robust optimization counterpart problem and to find a robust solution, whereas accuracy functions are used as a tool of post-optimal analysis to express numerically the quality of the given solution under possible perturbations of initial data. Thus, we get

$$A_{P,N}(C^0, x^*, \delta) = REG_{P,N}(S(C^0), x^*),$$

if the set of scenarios $S(C^0)$ is defined as the set of perturbed matrices $\Theta_\delta(C^0)$ according to (9). This means that the properties of accuracy functions can be used in the solution robustness analysis. It is also easy to check that $A_{P,N}(C^0, x^*, \delta) \geq 0$ for each $\delta \in [0, 1)$.

In single objective case (one player i makes a decision only), we have for $x^* \in X$ and $\delta \in [0, 1)$, the value of the accuracy function is defined as follows:

$$A_i(C^0, x^*, \delta) := \max_{C_i \in \Theta_\delta(C_i^0)} \varepsilon_i(C, x^*). \tag{12}$$

Moreover,

$$A_i(C^0, x^*, \delta) = REG_i(\Theta_\delta(C_i^0), x^*),$$

where

$$\Theta_\delta(C_i^0) := \{C_i \in \mathbf{R}_+^{1\times m} : \ |c_{ij} - c_{ij}^0| \leq \delta \cdot c_{ij}^0, \ j \in N_m\}. \tag{13}$$

Denote

$$\Theta'_\delta(C^0) := \{C \in \mathbf{R}_+^{m\times m} : \ |c_{ij} - c_{ij}^0| < \delta \cdot c_{ij}^0, \ i \in N_m, \ j \in N_m\}. \tag{14}$$

Obviously, $\Theta'_\delta(C^0) \subseteq \Theta_\delta(C^0)$. The following statement is true.

Proposition 1 *For $x^* \in X$ and $\delta \in [0, 1)$, we have $x^* \in P^m(C)$ for any $C \in \Theta'_\delta(C^0)$ if and only if $A_P(C^0, x^*, \delta) = 0$.*

Proof. Let $\delta \in [0, 1)$. If $x^* \in P^m(C)$ for any $C \in \Theta'_\delta(C^0)$, then - directly from the definition of the relative error, we have $\varepsilon_P(C, x^*) = 0$ for any $C \in \Theta'_\delta(C^0)$. Consider now the case $C \in \Theta_\delta(C^0)\backslash\Theta'_\delta(C^0)$. Even if x^* loses efficiency for such matrix C, then the relative error $\varepsilon_P(C, x^*)$ is still equal to 0, because - due to continuity of payoffs as linear functions - for any $x \in X$ there exists $j \in N_m$ such that $p_j(C, x) = p_j(C, x^*)$. Thus, $\varepsilon_P(C, x^*) = 0$ which means that $A_P(C^0, x^*, \delta) = 0$.

In order to prove that for $\delta \in [0, 1)$, $A_P(C^0, x^*, \delta) = 0$ implies that $x^* \in P^m(C)$ for any $C \in \Theta'_\delta(C^0)$, suppose that $A_P(C^0, x^*, \delta) = 0$, but there exists a matrix $C' \in \Theta'_\delta(C^0)$, such that $x^* \notin P^m(C')$. We will show that such assumption must lead to a contradiction. Indeed, $x^* \notin P^m(C')$ means that there exist $x \in X$ such that $p_i(C', x^*) \geq p_i(C', x)$ for all $i \in N_m$ and there exists $j \in N_m$ such that $p_j(C', x^*) > p_j(C', x)$. Let $I \subseteq N_m$ be a set of indices for which $p_i(C', x^*) = p_i(C', x)$. Consider matrix $\tilde{C}' \in \mathbf{R}_+^{m\times m}$ with elements

$$\tilde{c}'_{ij} = \begin{cases} c'_{ij} - \phi, & \text{if } i \in I, \quad x_j^* = 0; \\ c'_{ij} + \phi, & \text{if } i \in I, \quad x_j^* = 1; \\ c'_{ij}, & \text{otherwise}, \end{cases} \tag{15}$$

where $\phi > 0$ is taken small enough to satisfy $\tilde{C}' \in \Theta'_\delta(C^0)$. Now it is easy to see that $p_i(\tilde{C}', x^*) > p_i(\tilde{C}', x)$ for every $i \in N_m$, i.e. $\varepsilon_P(\tilde{C}', x^*) > 0$, which implies $A_P(C^0, x^*, \delta) > 0$. Thus we have a contradiction which completes the proof. □

The validity of Proposition 2 follows directly from the observation that $x^* \in N^m(C)$ if and only if $\varepsilon_N(C, x^*) = 0$ for any $C \in \mathbf{R}_+^{m\times m}$.

Proposition 2 *For $x^* \in X$ and $\delta \in [0,1)$, we have $x^* \in N^m(C)$ for any $C \in \Theta_\delta(C^0)$ if and only if $A_N(C^0, x^*, \delta) = 0$.*

For given $x, x^* \in X$, fixed index $i \in N_m$ and $C^0 \in \mathbf{R}_+^{m\times m}$ denote

$$\Xi_i(C^0, x^*, x, \delta) := \frac{C_i^0(x^* - x) + \delta \sum\limits_{j\in N_m} c_{ij}^0 |x_j^* - x_j|}{(1-\delta)C_i^0 x}. \tag{16}$$

The following theorem gives a formulae for calculating value of the accuracy function.

Theorem 1 *The following statements are true.*

(i) For $x^* \in X$ and $\delta \in [0,1)$, the accuracy function can be expressed by the formula:

$$A_P(C^0, x^*, \delta) = \max_{x\in X} \min_{i\in N_m} \Xi_i(C^0, x^*, x, \delta). \tag{17}$$

(ii) For $x^* \in X$ and $\delta \in [0,1)$, the accuracy function can be expressed by the formula:

$$A_N(C^0, x^*, \delta) = \max_{i\in N_m} \max_{x\in W_i(x^*)\cap X} \Xi_i(C^0, x^*, x, \delta). \tag{18}$$

Proof. We prove (i) first. We start with showing that $A_P(C^0, x^*, \delta) \leq \Gamma_P(C^0, x^*, \delta)$, where $\Gamma_P(C^0, x^*, \delta)$ is a right-hand side of (17). Using the property of $max - min$ and $min - max$ operators, we consequently, yield

$$A_P(C^0, x^*, \delta) = \max_{C\in\Theta_\delta(C^0)} \varepsilon_P(C, x^*) = \max_{C\in\Theta_\delta(C^0)} \max_{x\in X} \min_{i\in N_m} \Delta_i(C, x^*, x) \leq$$

$$\leq \max_{x\in X} \min_{i\in N_m} \max_{C\in\Theta_\delta(C^0)} \Delta_i(C, x^*, x)$$

For some fixed game solution $x \in X$ and player $i \in N_m$ the maximum $\Delta_i(C, x^*, x)$ over $C \in \Theta_\delta(C^0)$ is attained at matrix C^* with elements in the i-th row defined as

$$c_{ij}^* = \begin{cases} c_{ij}^0 - \delta \cdot c_{ij}^0 \text{ if } x_j = 1, \\ c_{ij}^0 + \delta \cdot c_{ij}^0 \text{ otherwise,} \end{cases} \tag{19}$$

The elements of the other rows in matrix C^* are the same as in the original matrix C^0. Then $C^* \in \Theta_\delta(C^0)$, and

$$\max_{x\in X} \min_{i\in N_m} \max_{C\in\Theta_\delta(C^0)} \Delta_i(C, x^*, x) = \max_{x\in X} \min_{i\in N_m} \Delta_i(C^*, x^*, x) =$$

$$\max_{x\in X}\min_{i\in N_m}\Xi(C^0,x^*,x,\delta)=\Gamma_P(C^0,x^*,\delta).$$

Thus, we have shown that $A_P(C^0,x^*,\delta)\le\Gamma_P(C^0,x^*,\delta)$ for all $\delta\in[0,1)$. Now it remains to show that $A_P(C^0,x^*,\delta)\ge\Gamma_P(C^0,x^*,\delta)$ for all $\delta\in[0,1)$. For any fixed $x\in X$, consider matrix C^* with elements defined for all rows (players) $i\in N_m$ according to (19).

$$A_P(C^0,x^*,\delta)=\max_{C\in\Theta_\delta(C^0)}\varepsilon_P(C,x^*)\ge\max_{x\in X}\min_{i\in N_m}\Delta_i(C^*,x^*,x)=$$

$$\max_{x\in X}\min_{i\in N_m}\Xi(C^0,x^*,x,\delta)=\Gamma_P(C^0,x^*,\delta).$$

So, we have that $A_P(C^0,x^*,\delta)\ge\Gamma_P(C^0,x^*,\delta)$ for all $\delta\in[0,1)$. Summarizing, we have just proven that $A_P(C^0,x^*,\delta)=\Gamma_P(C^0,x^*,\delta)$ for all $\delta\in[0,1)$

Now we prove (ii). Indeed, using the property of *max* operator, we deduce

$$A_N(C^0,x^*,\delta)=\max_{C\in\Theta_\delta(C^0)}\varepsilon_N(C,x^*)=\max_{i\in N_m}\max_{x\in W_i(x^*)\cap X}\max_{C\in\Theta_\delta(C^0)}\Delta_i(C,x^*,x).$$

For any fixed $x\in X$ and row (player) $i\in N_m$, $\max_{C\in\Theta_\delta(C^0)}\Delta_i(C,x^*,x)$ is attained at matrix C^* with elements defined according to (19). So, we get directly $A_N(C^0,x^*,\delta)=\Gamma_N(C^0,x^*,\delta)$ for all $\delta\in[0,1)$, where $\Gamma_N(C^0,x^*,\delta)$ is a right-hand side of (18). $\square$

From Theorem 1 we get the following result giving us an expression for accuracy function in single objective case

Corollary 1 *Let the choice of strategies be made by player i only. Then for $x^*\in X$ and $\delta\in[0,1)$, the accuracy function can be expressed by the formula:*

$$A_i(C^0,x^*,\delta)=\max_{x\in X}\ \Xi_i(C^0,x^*,x,\delta). \tag{20}$$

Notice that similar result was specified in [18] for single objective generic combinatorial optimization problem

Notice that analytical formula (18) specified in Theorem 1 can be computed relatively easy. At the same time analytical formula (17) specified in Theorem 1 is based on enumerating all feasible solutions, so in general it is hard to be computed. Therefore, we provide some attainable lower and upper bounds for the Pareto accuracy function which are computationally more attractive. Next proposition gives an upper bound for the accuracy function of $x^*\in X$ in the case of Pareto optimality principle.

Proposition 3 *For $x^*\in X$ and $\delta\in[0,1)$,*

$$A_P(C^0,x^*,\delta)\le\frac{2\delta}{1-\delta}+\frac{1+\delta}{1-\delta}\cdot\min_{i\in N_m}A_i(C^0,x^*,0). \tag{21}$$

Proof. Let & denote standard conjunction operator between two Boolean vectors, which is performed componentwise, that is every component of the resulting vector is a conjunction of corresponding Boolean components of the two operating vectors. Using (17), we get

$$
\begin{aligned}
A_P(C^0, x^*, \delta) &= \max_{x\in X}\min_{i\in N_m} \frac{C_i^0(x^*-x)+\delta \sum\limits_{j\in N_m} c_{ij}^0|x_j^*-x_j|}{(1-\delta)C_i^0x} \\
&= \max_{x\in X}\min_{i\in N_m} \frac{(1+\delta)C_i^0x^*-(1-\delta)C_i^0x-2\delta C_i^0(x^*\&x)}{(1-\delta)C_i^0x} \\
&\le \max_{x\in X}\min_{i\in N_m} \frac{(1+\delta)C_i^0x^*-(1-\delta)C_i^0x}{(1-\delta)C_i^0x} \\
&= \max_{x\in X}\min_{i\in N_m} \frac{(1+\delta)C_i^0x^*+(1+\delta)C_i^0x-(1+\delta)C_i^0x-(1-\delta)C_i^0x}{(1-\delta)C_i^0x} \\
&= \tfrac{2\delta}{1-\delta} + \tfrac{1+\delta}{1-\delta}\cdot \max_{x\in X}\min_{i\in N_m} \Delta_i(C^0, x^*, x) \\
&\le \tfrac{2\delta}{1-\delta} + \tfrac{1+\delta}{1-\delta}\cdot \min_{i\in N_m}\max_{x\in X} \Delta_i(C^0, x^*, x) \\
&= \tfrac{2\delta}{1-\delta} + \tfrac{1+\delta}{1-\delta}\cdot \min_{i\in N_m} A_i(C^0, x^*, 0). \quad \square
\end{aligned}
$$

Now it becomes clear that calculating the upper bound specified by Proposition 3 is as hard as calculating m times $A_i(C^0, x^*, 0)$, whose calculating according to Corollary 1 turns into solving the original single objective problem.

Observe that similar upper bound were obtained in the case of single objective combinatorial optimization problem in [20], and earlier in [32, 41] for linear programs. The following corollary is straightforward consequence from Propositions 1 and 3, and it specifies the upper bound for the accuracy function of the originally Pareto equilibrium $x^* \in P^m(C^0)$.

Corollary 2 *For $x^* \in P^m(C^0)$ and $\delta \in [0, 1)$,*

$$A_P(C^0, x^*, \delta) \le \frac{2\delta}{1-\delta}. \tag{22}$$

Corollary 3 *For $x^* \in N^m(C^0)$ and $\delta \in [0, 1)$, the equality $A_N(C^0, x^*, \delta) = 0$ holds.*

Now consider the case when x^* is an equilibrium in the original game with matrix C^0 implying $A_{P,N}(C^0, x^*, 0) = 0$. It is of special interest to know the extreme values of δ for which $A_{P,N}(C^0, x^*, \delta) = 0$, because these values determine maximum norms of perturbations which preserve the property of the given solution to be an equilibrium. These values are close analogues of the so-called stability radius introduced earlier for single and multiple objective combinatorial optimization problems (see e.g. [6]). Formally, the accuracy radii $R_{P,N}(C^0, x^*)$ are defined in the following way:

$$R_{P,N}(C^0, x^*) := \sup\Big\{\delta \in [0,1) : \; A_{P,N}(C^0, x^*, \delta) = 0\Big\}. \tag{23}$$

If these radii are equal to zero, then this means that there exist arbitrary small perturbations of the original game matrix C^0 such that the initial equilibrium x^* loses its property of being equilibrium under very small perturbations. Otherwise, the solution x^* remains equilibrium for any game with matrix $C \in \Theta_\delta(C^0)$, $\delta < R_{P,N}(C^0, x^*)$. The next theorem is a straightforward consequence of Theorem 1

Theorem 2 *The following statements are true.*

(i) For $x^* \in P^m(C^0)$, the Pareto accuracy radius can be expressed by the formula:

$$R_P(C^0, x^*) = \min\Big\{1, \min_{x \in X\setminus\{x^*\}} \max_{i \in N_m} \frac{C_i^0(x - x^*)}{\sum\limits_{j \in N_m} c_{ij}^0 |x_j - x_j^*|}\Big\}. \tag{24}$$

(ii) For $x^* \in N^m(C^0)$, the Nash accuracy radius can be expressed by the formula:

$$R_N(C^0, x^*) = \min\Big\{1, \min_{i \in N_m} \min_{x \in W_i(x^*) \cap X\setminus\{x^*\}} \frac{C_i^0(x - x^*)}{\sum\limits_{j \in N_m} c_{ij}^0 |x_j - x_j^*|}\Big\} = 1, \tag{25}$$

i.e. $x^* \in N^m(C^0)$ is accurate (i.e. $R_N(C^0, x^*) \geq 0$).

5 Robustness Tolerances

Robustness tolerances were first mentioned in [21] for single objective linear generic combinatorial optimization problem. Our approach develops the idea of [21] by extending it to the multiobjective case under game theoretic formulation. In this section we consider the case when only one column in matrix C^0 is uncertain, while all the other columns are kept unchanged. It corresponds to the situation in the game, when all players are uncertain about their own costs associated with the strategy choice of a given player. Assume j be the uncertain column in the original matrix C^0, so we denote the original matrix $C^0[j]$, where notation $[j]$ is used to indicate that column j is uncertain. Then for a fixed $\delta \in [0,1)$ we have

$$\Theta_\delta(C^0[j]) := \Big\{C \in \mathbf{R}_+^{m\times m} : \Big(|c_{ij}-c^0_{ij}| \le \delta\cdot c^0_{ij},\ i \in N_m\Big) \,\&\Big(c_{ik} = c^0_{ik},\ k \in N_m\backslash\{j\},\ i \in N_m\Big)\Big\}. \tag{26}$$

For $x^* \in X$ and $\delta \in [0,1)$, the definition of the accuracy function in this case transforms into the following:

$$A_{P,N}(C^0[j], x^*, \delta) := \max_{C[j]\in\Theta_\delta(C^0[j])} \varepsilon_{P,N}(C[j], x^*), \tag{27}$$

where $\varepsilon_{P,N}(C[j], x^*)$ are defined according to (6) and (7).

Moreover,

$$A_{P,N}(C^0[j], x^*, \delta) = REG_{P,N}(\Theta_\delta(C^0[j]), x^*),$$

It is easy to see that the analytic formulae (17) specified by Theorem 1 can be rewritten as follows For $x^* \in X$ and $\delta \in [0,1)$,

$$A_P(C^0[j], x^*, \delta) = \max_{x\in X}\min_{i\in N_m} \Xi_i(C^0[j], x^*, x, \delta) =$$

$$\max_{x\in X}\min_{i\in N_m} \frac{C^0_i(x^*-x) + \delta c^0_{ij}|x^*_j - x_j|}{C^0_i x - \delta c^0_{ij} x_j} =$$

$$\max\Big\{\max_{x\in X:x_j=0}\min_{i\in N_m}\frac{C^0_i(x^*-x)+\delta c^0_{ij}x^*_j}{C^0_i x},\ \max_{x\in X:x_j=1}\min_{i\in N_m}\frac{C^0_i(x^*-x)+\delta c^0_{ij}(1-x^*_j)}{C^0_i x-\delta c^0_{ij}}\Big\}.$$

It will be convenient now to state the last formula for calculating accuracy functions by splitting it into two cases. when $x^*_j = 0$ and $x^*_j = 1$. So, for $x^* \in X$ and $\delta \in [0,1)$, we have

$$A_P(C^0[j], x^*, \delta) = \begin{cases} \max\Big\{\max\limits_{x\in X:x_j=0}\min\limits_{i\in N_m}\frac{C^0_i(x^*-x)}{C^0_i x},\ \max\limits_{x\in X:x_j=1}\min\limits_{i\in N_m}\frac{C^0_i(x^*-x)+\delta c^0_{ij}}{C^0_i x-\delta c^0_{ij}}\Big\} & \text{if } x^*_j = 0, \\ \max\Big\{\max\limits_{x\in X:x_j=0}\min\limits_{i\in N_m}\frac{C^0_i(x^*-x)+\delta c^0_{ij}}{C^0_i x},\ \max\limits_{x\in X:x_j=1}\min\limits_{i\in N_m}\frac{C^0_i(x^*-x)}{C^0_i x-\delta c^0_{ij}}\Big\} & \text{if } x^*_j = 1. \end{cases} \tag{28}$$

Similarly, we get the expression for the accuracy function in the case of Nash optimality

$$A_N(C^0[j], x^*, \delta) = \begin{cases} \max\limits_{i\in N_m}\max\Big\{\max\limits_{x\in W_i(x^*)\cap X:x_j=0}\frac{C^0_i(x^*-x)}{C^0_i x},\ \max\limits_{x\in W_i(x^*)\cap X:x_j=1}\frac{C^0_i(x^*-x)+\delta c^0_{ij}}{C^0_i x-\delta c^0_{ij}}\Big\} & \text{if } x^*_j = 0, \\ \max\limits_{i\in N_m}\max\Big\{\max\limits_{x\in W_i(x^*)\cap X:x_j=0}\frac{C^0_i(x^*-x)+\delta c^0_{ij}}{C^0_i x},\ \max\limits_{x\in W_i(x^*)\cap X:x_j=1}\frac{C^0_i(x^*-x)}{C^0_i x-\delta c^0_{ij}}\Big\} & \text{if } x^*_j = 1. \end{cases} \tag{29}$$

Notice also that if $x^* \in P^m(C^0)$, then (28) transforms into

$$A_P(C^0[j], x^*, \delta) = \begin{cases} \max\left\{0, \max\limits_{x\in X: x_j=1} \min\limits_{i\in N_m} \frac{C_i^0(x^*-x)+\delta c_{ij}^0}{C_i^0 x - \delta c_{ij}^0}\right\} \text{ if } x_j^* = 0, \\ \max\left\{ \max\limits_{x\in X: x_j=0} \min\limits_{i\in N_m} \frac{C_i^0(x^*-x)+\delta c_{ij}^0}{C_i^0 x}, 0\right\} \text{ if } x_j^* = 1, \end{cases} \tag{30}$$

and if $x^* \in N^m(C^0)$, then (29) transforms into

$$A_N(C^0[j], x^*, \delta) = \begin{cases} \max\limits_{i\in N_m} \max\left\{0, \max\limits_{x\in W_i(x^*)\cap X: x_j=1} \frac{C_i^0(x^*-x)+\delta c_{ij}^0}{C_i^0 x - \delta c_{ij}^0}\right\} \text{ if } x_j^* = 0, \\ \max\limits_{i\in N_m} \max\left\{ \max\limits_{x\in W_i(x^*)\cap X: x_j=0} \frac{C_i^0(x^*-x)+\delta c_{ij}^0}{C_i^0 x}, 0\right\} \text{ if } x_j^* = 1. \end{cases} \tag{31}$$

Now we are interested in the maximal level of perturbation not violating robustness of a given optimal solution.

Definition 5 *For a given $x^* \in P^m(C^0)$ ($x^* \in N^m(C^0)$) the robustness tolerances in Pareto and Nash cases are defined as follows:*

$$t^r_{P,N}(C^0[j], x^*) := \sup\Big\{\delta \in [0,1) :\ A_{P,N}(C^0[j], x^*, \delta) \le A_{P,N}(C^0[j], x, \delta)\ \forall x \in X\Big\}. \tag{32}$$

Notice that the same definition can be formulated in terms of relative regrets as follows

$$t^r_{P,N}(C^0[j], x^*) := \sup\Big\{\delta \in [0,1) :\ REG_{P,N}(\Theta_\delta(C^0[j]), x^*) \le REG_{P,N}(\Theta_\delta(C^0[j]), x)\ \forall x \in X\Big\}.$$

The superscript r is used to emphasize that we are dealing with *robust* tolerances, which differ from usual tolerances (similar notation is also used in [21]).

Since for all $0 \le \delta \le R_{P,N}(C^0[j], x^*)$, we have that $A_{P,N}(C^0[j], x^*, \delta) = 0$, it implies that $t^r_{P,N}(C^0[j], x^*) \ge R_{P,N}(C^0[j], x^*)$. Finding analytical expressions for $t^r_{P,N}(C^0[j], x^*)$, which could be computationally tractable is a hard task. To compute robustness tolerances one should exploit some good search strategy such as branch and bound, or use some heuristic and ad-hoc strategies examining more carefully problem topology. We restrict our analysis by specifying $t^r_{P,N}(C^0[j], x^*)$ in the case when the original problem contains a single equilibrium only. So, let us assume that $P^m(C^0) = \{x^*\}$. Then it implies that for all $x \in X$ and all indices $i \in N_m$ the inequalities $C_i^0(x - x') \ge C_i^0(x^* - x')$, $C_i^0(x^* - x') \le 0$ hold for any $x' \in X$. Using this observation we deduce the following.

Consider now the case $x_j^* = 1$. For any fixed $x \in X$, assuming that $x_j = 1$, then we get

$$A_P(C^0[j], x, \delta) = \max\left\{ \max_{x'\in X: x'_j=0} \min_{i\in N_m} \frac{C_i^0(x-x')+\delta c_{ij}^0}{C_i^0 x'}, \max_{x'\in X: x'_j=1} \min_{i\in N_m} \frac{C_i^0(x-x')}{C_i^0 x' - \delta c_{ij}^0}\right\} \ge$$

$$\max\Big\{\max_{x'\in X: x'_j=0}\min_{i\in N_m}\frac{C_i^0(x^*-x')+\delta c_{ij}^0}{C_i^0x'}, \max_{x'\in X: x'_j=1}\min_{i\in N_m}\frac{C_i^0(x^*-x')}{C_i^0x'-\delta c_{ij}^0}\Big\} =$$

$$\max\Big\{\max_{x'\in X: x'_j=0}\min_{i\in N_m}\frac{C_i^0(x^*-x')+\delta c_{ij}^0}{C_i^0x'}, 0\Big\} = A_P(C^0[j], x^*, \delta).$$

For any fixed $x \in X$, assuming that $x_j = 0$, then we get

$$A_P(C^0[j], x, \delta) = \max\Big\{\max_{x'\in X: x'_j=0}\min_{i\in N_m}\frac{C_i^0(x-x')}{C_i^0x'}, \max_{x'\in X: x'_j=1}\min_{i\in N_m}\frac{C_i^0(x-x')+\delta c_{ij}^0}{C_i^0x'-\delta c_{ij}^0}\Big\} \geq$$

$$\max\Big\{0, \max_{x'\in X: x'_j=1}\min_{i\in N_m}\frac{C_i^0(x-x')+\delta c_{ij}^0}{C_i^0x'-\delta c_{ij}^0}\Big\} \geq \max_{x'\in X: x'_j=1}\min_{i\in N_m}\frac{C_i^0(x-x')+\delta c_{ij}^0}{C_i^0x'-\delta c_{ij}^0} \geq$$

$$\max_{x'\in X: x'_j=0}\min_{i\in N_m}\frac{C_i^0(x-x')+\delta c_{ij}^0}{C_i^0x'-\delta c_{ij}^0} \geq \max_{x'\in X: x'_j=0}\min_{i\in N_m}\frac{C_i^0(x-x')+\delta c_{ij}^0}{C_i^0x'} \geq A_P(C^0[j], x^*, \delta).$$

Thus, we will have $A_P(C^0[j], x^*, \delta) \leq A_P(C^0[j], x, \delta)$ for all $x \in X$ and $\delta \in [0, 1)$. Thus, we have just proven that if $x_j^* = 1$, then $t_P^r(C^0[j], x^*) = 1$.

Now consider the case $x_j^* = 0$. For any fixed $x \in X$, assuming that $x_j = 0$, then we get

$$A_P(C^0[j], x, \delta) = \max\Big\{\max_{x'\in X: x'_j=0}\min_{i\in N_m}\frac{C_i^0(x-x')}{C_i^0x'}, \max_{x'\in X: x'_j=1}\min_{i\in N_m}\frac{C_i^0(x-x')+\delta c_{ij}^0}{C_i^0x'-\delta c_{ij}^0}\Big\} \geq$$

$$\max\Big\{0, \max_{x'\in X: x'_j=1}\min_{i\in N_m}\frac{C_i^0(x^*-x')+\delta c_{ij}^0}{C_i^0x'-\delta c_{ij}^0}\Big\} = A_P(C^0[j], x^*, \delta).$$

For any fixed $x \in X$, assuming that $x_j = 1$, then we get

$$A_P(C^0[j], x, \delta) = \max\Big\{\max_{x'\in X: x'_j=0}\min_{i\in N_m}\frac{C_i^0(x-x')+\delta c_{ij}^0}{C_i^0x'}, \max_{x'\in X: x'_j=1}\min_{i\in N_m}\frac{C_i^0(x-x')}{C_i^0x'-\delta c_{ij}^0}\Big\} \geq$$

$$\max\Big\{\max_{x'\in X: x'_j=0}\min_{i\in N_m}\frac{C_i^0(x^*-x')+\delta c_{ij}^0}{C_i^0x'}, 0\Big\} \geq \max\Big\{\max_{x'\in X: x'_j=1}\min_{i\in N_m}\frac{C_i^0(x^*-x')+\delta c_{ij}^0}{C_i^0x'}, 0\Big\}.$$

So, $A_P(C^0[j], x^*, \delta) \leq A_P(C^0[j], x, \delta)$ if and only if

$$\max\Big\{0, \max_{x'\in X: x'_j=1}\min_{i\in N_m}\frac{C_i^0(x^*-x')+\delta c_{ij}^0}{C_i^0x'-\delta c_{ij}^0}\Big\} \leq \max\Big\{\max_{x'\in X: x'_j=1}\min_{i\in N_m}\frac{C_i^0(x-x')+\delta c_{ij}^0}{C_i^0x'}, 0\Big\}$$

The last inequality holds for

$$\delta \leq \min\Big\{1, \frac{\sqrt{(C_{\hat{i}}^0\hat{x})^2 - (C_{\hat{i}}^0x^*)^2}}{c_{\hat{i}j}^0}\Big\},$$

i.e.

$$t^r_P(C^0[j], x^*) = \min\left\{1, \frac{\sqrt{(C^0_{\hat{i}}\hat{x})^2 - (C^0_{\hat{i}}x^*)^2}}{c^0_{\hat{i}j}}\right\},$$

where $\hat{x} := \arg \max_{x' \in X : x'_j = 1} \min_{i \in N_m} C^0_i x'$, and let $\hat{i} = \arg \min_{i \in N_m} C^0_i \hat{x}$. Summarizing, we can formulate the following main result in this section.

Theorem 3 *Assume that $P^m(C^0) = \{x^*\}$. Then the robustness tolerance can be computed according to the following expressions:*

if $x^*_j = 1$**, then**

$$t^r_P(C^0[j], x^*) = 1;$$

if $x^*_j = 0$ **then**

$$t^r_P(C^0[j], x^*) = \min\left\{1, \frac{\sqrt{(C^0_{\hat{i}}\hat{x})^2 - (C^0_{\hat{i}}x^*)^2}}{c^0_{\hat{i}j}}\right\},$$

where $\hat{x} := \arg \max_{x' \in X : x'_j = 1} \min_{i \in N_m} C^0_i x', \ \hat{i} = \arg \min_{i \in N_m} C^0_i \hat{x}.$

Similar result can be obtained in the case of Nash optimality

Corollary 4 *Assume that $x^* \in N^m(C^0)$. Then*

$$t^r_N(C^0[j], x^*) = 1.$$

Indeed, the robustness tolerance in this case can be computed in a similar way: if $x^*_j = 1$, then $t^r_N(C^0[j], x^*) = 1$, and if $x^*_j = 0$ then

$$t^r_N(C^0[j], x^*) = \begin{cases} 1, & \text{if } \tilde{x} \notin X; \\ \min\left\{1, \frac{\sqrt{(C^0_{\tilde{i}}\tilde{x})^2 - (C^0_{\tilde{i}}x^*)^2}}{c^0_{\tilde{i}j}}\right\}, & \text{otherwise,} \end{cases}$$

where $\tilde{x} := (x^*_1, x^*_2, ..., \bar{x}^*_j = 1, ..., x^*_n)$, $\tilde{i} = \arg \max_{i \in N_m} C^0_i \tilde{x}$. Since when $x^*_j = 0$, we have that

$$\frac{\sqrt{(C^0_{\tilde{i}}\tilde{x})^2 - (C^0_{\tilde{i}}x^*)^2}}{c^0_{\tilde{i}j}} \geq 1,$$

the result which says that the Nash equilibrium is always robustness tolerant is true. Notice also that here we do not need assumption about optimum uniqueness, which was crucial in the case of Pareto optimality.

6 Numerical examples

In this section we would like to illustrate how the accuracy function calculation can be used to rank alternatives from the robustness point of view in Pareto case. Consider the following examples.

Example 1 *Let* $m = 4$, *and*

$$C^0 = \begin{pmatrix} 2 & 2 & 1.5 & 2 \\ 0.5 & 1 & 2 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 3 & 3 & 3 \end{pmatrix}.$$

Assume also that $X = \{x^1, x^2, x^3, x^4\}$, $x^1 = (0,1,1,0)^T$, $x^2 = (1,0,1,0)^T$, $x^3 = (0,1,0,1)^T$, $x^4 = (0,0,1,1)^T$. *Then* $C^0x^1 = (3.5,3,2,6)^T$, $C^0x^2 = (3.5,2.5,3,4)^T$, $C^0x^3 = (4,2,3,6)^T$, $C^0x^4 = (3.5,3,3,6)^T$ *and* $P^3(C^0) = \{x^1, x^2, x^3\}$. *Using formula (24), we calculate* $R^4(C^0, x^1) = \frac{1}{3}$, $R^4(C^0, x^2) = \frac{1}{5}$ *and* $R^4(C^0, x^3) = \frac{1}{9}$. *The accuracy functions of all feasible solutions are depicted in Fig. 1, and zoomed in Fig. 2. It happens that for example both* x^1 *and* x^2, *behaves better under small* δ, *since they have higher accuracy radius values than* x^3. *However while* δ *is increased,* $A_P(C^0, x^2, \delta)$ *is dominated not only by* $A_P(C^0, x^1, \delta)$ *but also by* $A_P(C^0, x^4, \delta)$ *despite on the fact that* $x^4 \notin P^4(C^0)$.

Example 2 *Let* $m = 4$, *and original matrix* C^0 *be the same as in the previous example. Assume also that* $X = \{x^1, x^2, x^3, x^4\}$, $x^1 = (0,1,1,0)^T$, $x^2 = (1,1,1,1)^T$, $x^3 = (0,1,1,1)^T$, $x^4 = (0,0,1,1)^T$. *Then* $C^0x^1 = (3.5,3,2,6)^T$, $C^0x^2 = (7.5,4.5,6,10)^T$, $C^0x^3 = (5.5,4,4,9)^T$, $C^0x^4 = (3.5,3,3,6)^T$ *and* $P^3(C^0) = \{x^1\}$, *and* $R^4(C^0, x^1) = \frac{1}{3}$. *The accuracy functions together with corresponding upper bounds specified by (21) and (22) are depicted in Fig. 3 In this example, it happens that* $A_P(C^0, x^1, \delta)$ *dominates all others for all* $\delta \in [0, 1)$, *thus* x^1 *can be considered as a very robust Pareto optimum.*

7 Conclusions

The examples in previous section suggest that small changes or inaccuracies in estimating payoff function coefficients may have significant influence on the set of Pareto equilibria. Moreover, some situations being initially equilibria, cannot be considered 'robust', because very small changes of data destroy their properties of being equilibria.

The simplest measure of the 'robustness' of the equilibrium is its accuracy radius. But frequently these radii are not sufficient to rank the equilibria. Therefore, calculating accuracy radii only cannot be sufficient to make a conclusion about robustness, so it is necessary to calculate complementary more

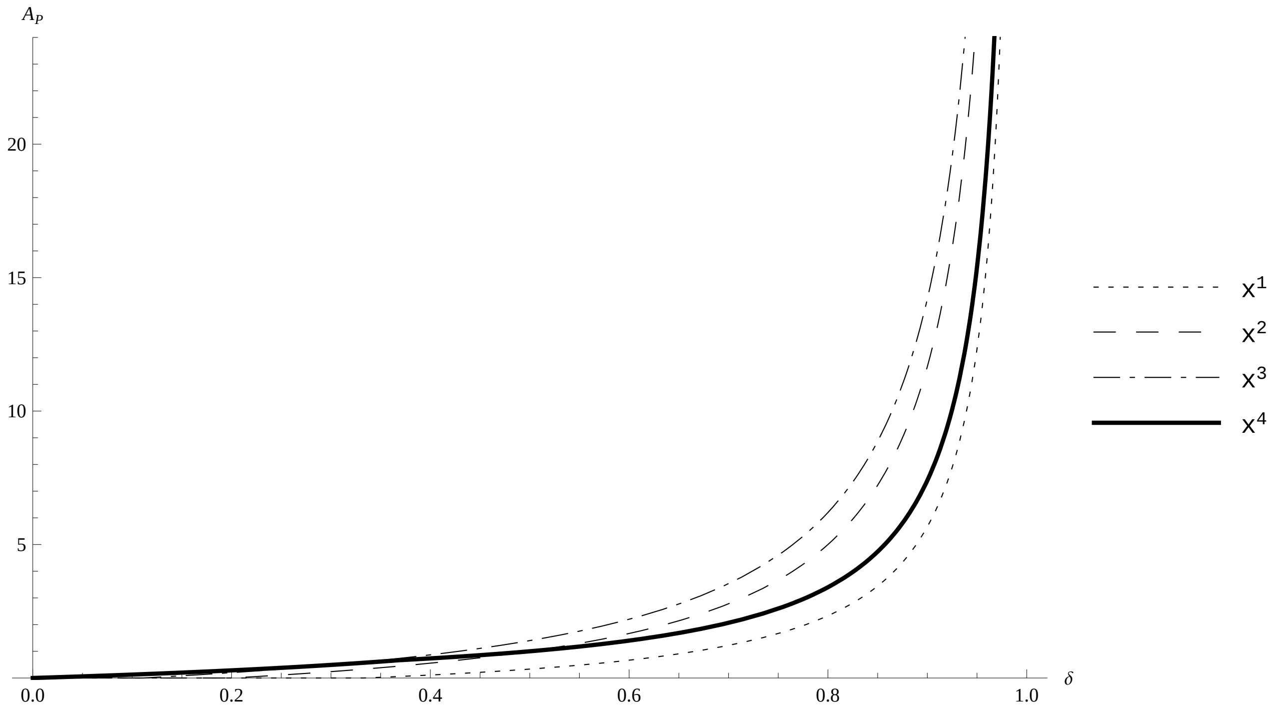

Figure 1: Accuracy functions; $\rho \in [0, 1)$.

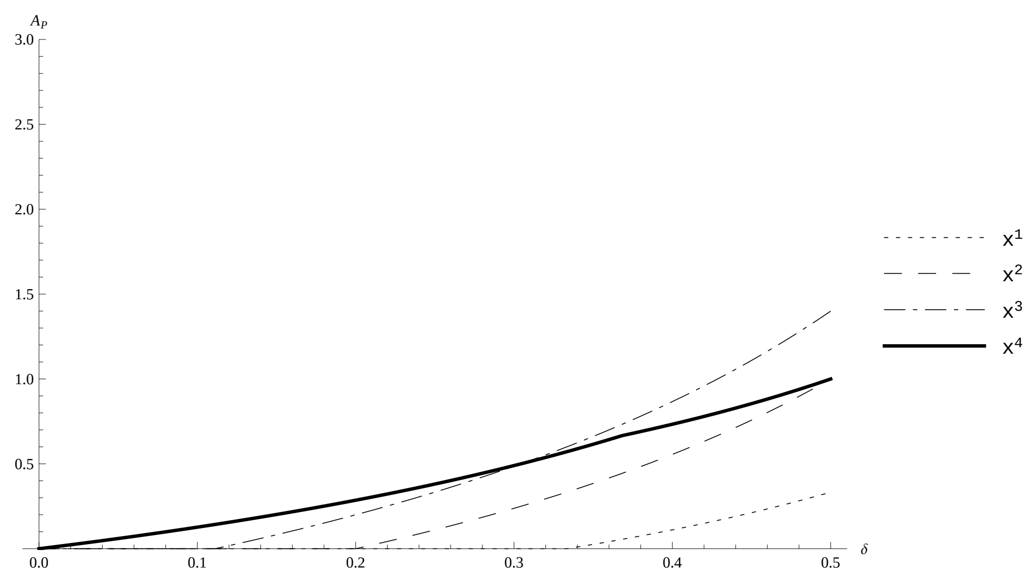

Figure 2: Accuracy functions; $\rho \in [0, 0.5)$.

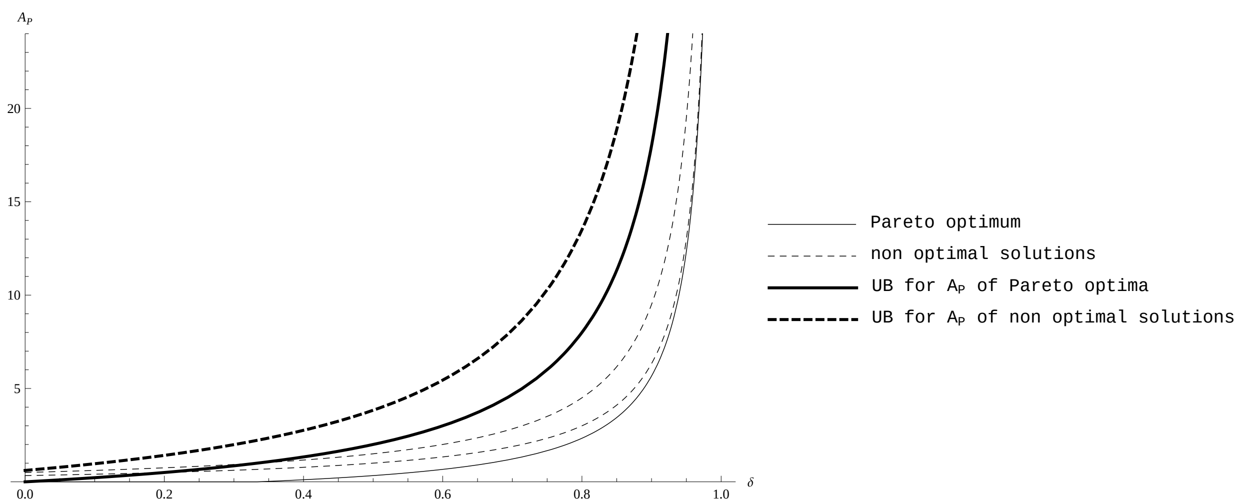

Figure 3: Accuracy functions with corresponding upper bounds; $\rho \in [0, 1)$.

general characteristics of situations like accuracy functions whose definitions are directly connected with given optimality principle.

The other big challenge in robust and sensitivity analysis is to construct efficient algorithms to calculate the analytical expressions. To the best of our knowledge there are not so many results (see e.g. [8], [17]) known in that area, and moreover some of those results which have been already known, put more questions than answers. It seems that calculating exact values is an extremely difficult task in general, so one could concentrates either on finding "easy" computable classes of problems or developing general metaheuristic approaches.

References

[1] **I. Averbakh**, (2001). "On the complexity of a class of combinatorial optimization problems with uncertainty". *Mathematical Programming 90*, 263 – 272.

[2] **A. Ben-Tal, A. Nemirovski**, (1999). "Robust solutions to uncertain programs". *Operations Research Letters 25*, 1 – 13.

[3] **D. Bertsimas, M. Sim**, (2003). "Robust discrete optimization and network flows". *Mathematical Programming 98*, 49 – 71.

[4] **S. Bukhtoyarov1, V. Emelichev and Y. Stepanishina**, (2006). "Stability of Discrete Vector Problems with the Parametric Principle of Optimality". *Cybernetics and systems analysis 39*, 604 – 614.

[5] **N. Chakravarti, A.P.M. Wagelmans**, (1998). "Calculation of stability radii for combinatorial optimization problem". *Operations Research Letters 23*, 1 – 7.

[6] **V. Emelichev, E. Girlich, Y. Nikulin, D. Podkopaev**, (2002). "Stability and regularization of vector problems of integer linear programming". *Optimization 51*, 645 – 676.

[7] **V. Emelichev, O. Karelkina, Y. Nikulin**, (2010). "Stability analysis of a multicriteria combinatorial median location problem", *TUCS Technical Report 966*.

[8] **V. Emelichev, D. Podkopaev**, (2010). "Quantitative stability analysis for vector problems of 0-1 programming". *Discrete Optimization 7*, 48 – 63.

[9] **M. Ehrgott**, *Multicriteria Optimization*, Springer, Berlin, 2000.

[10] **M. Ehrgott, X. Gandibleux**, (2000). "A survey and annotated bibliography of multiobjective combinatorial optimization". *OR Spectrum 22*, 425 – 460.

[11] **A.V. Fiacco**, *Mathematical Programming with Data Perturbations*, Marcel Dekker, New York, 1988.

[12] **H. Greenberg**, (1998). "An annotated bibliography for post-solution analysis in mixed integer and combinatorial optimization." In D. Woodruff (ed.), Advances in computational and stochastic optimization, Logic programming and heuristic search, 97 – 148.

[13] **S. van Hoesel, A. Wagelmans**, (1999). "On the complexity of postoptimality analysis of 0–1 programs". *Discrete Applied Mathematics 91*, 251 – 263.

[14] **A. Kasperski**, *Discrete Optimization with Interval Data. Minmax Regret and Fuzzy Approach*, Springer, Berlin, 2008.

[15] **A. Kasperski, P. Zielinski**, (2010). "Minmax regret approach and optimality evaluation in combinatorial optimization problems with interval and fuzzy weights". *European Journal of Operational Research 200*, 680 – 687.

[16] **P. Kouvelis, G. Yu**, *Robust discrete optimization and its applications*, Kluwer Academic Publishers, Norwell, 1997.

[17] **M. Libura, E.S. van der Poort, G. Sierksma, J.A.A van der Veen**, (1998). "Stability aspects of the traveling salesman problem based on k-best solutions", *Discrete Applied Mathematics 87*, 159 – 185.

[18] **M. Libura**, (1999). "On accuracy of solution for combinatorial optimization problems with perturbed coefficients of the objective function". *Annals of Operation Research 86*, 53 – 62.

[19] **M. Libura**, (2000). "Quality of solutions for perturbed combinatorial optimization problems". *Control and Cybernetics 29*, 199 – 219.

[20] **M. Libura**, (2009). "On the robustness of optimal solutions for combinatorial optimization problems". *Control and Cybernetics 38*, 671 – 685.

[21] **M. Libura**, (2010). "A note on robustness tolerances for combinatorial optimization problems". *Information Processing Letters 110*, 725 – 729.

[22] **M. Libura, Y. Nikulin**, (2004). "Stability and accuracy functions in multicriteria combinatorial optimization problem with Σ-MINMAX and Σ-MINMIN partial criteria". *Control and Cybernetics 33*, 511 – 524.

[23] **M. Libura, Y. Nikulin**, (2006). "Stability and accuracy functions in multicriteria linear combinatorial optimization problems". *Annals of Operations Research 147*, 255 – 267.

[24] **K. Miettinen**, *Nonlinear multiobjective optimization*, Kluwer Academic Publishers, Boston, 1999.

[25] **R. Montemanni, L. Gambardella**, (2004). "An exact algorithm for the robust shortest path problem with interval data". *Computers and Operations Research 31*, 1667 – 1680.

[26] **R. Montemanni, L. Gambardella**, (2005). "A branch and bound algorithm for the robust spanning tree problem with interval data". *European Journal of Operational Research 161*, 771 – 779.

[27] **J. Mulvey, R. Vanderbei, S. Zenios**, (1995). "Robust optimization of large-scale systems". *Operations Research 43*, 264 – 281.

[28] **M.M. Mäkelä, Y. Nikulin**, (2010). "Stability and accuracy functions for a multicriteria Boolean linear programming problem with parameterized principle of optimality "from Condorcet to Pareto". *European Journal of Operational Research 207*, 1497 – 1505.

[29] **Y. Nikulin**, (2009). "Stability and accuracy functions for a coalition game with linear payoffs, antagonistic strategies and bans". *Annals of Operations Research 172*, 25 – 35.

[30] **J. Nash**, (1950). "Equilibrium points in n-person games". *Proceedings of the National Academy of Sciences 36*, 48 – 49.

[31] **J. Nash**, (1951). "Non-cooperative games". *Annals of Mathematics 54*, 286 – 295.

[32] **O. Oguz**, (2000). "Bounds on the opportunity costs of neglecting reoptimization in mathematical programming". *Management Science 46*, 1009 – 1012.

[33] **V. Pareto**, *Manuel d'ecoonomie politique*, Qiard, Paris, 1909.

[34] **Y. Sawaragi, H. Nakayama, T. Tanino**, *Theory of Multi-Objective Optimization*. Academic Press, Orlando, 1985.

[35] **Y.N. Sotskov, V.K. Leontev, E.N. Gordeev**, (1995). "Some concepts of stability analysis in combinatorial optimization". *Discrete Applied Mathematics 58*, 169 – 190.

[36] **Y.N. Sotskov, N.Y. Sotskova, T.-C. Lai, F. Werner**, *Scheduling under uncertainty. Theory and algorithms.* Belorusskaya nauka, Minsk, 2010

[37] **Y.N. Sotskov, V.S. Tanaev, F. Werner**, (1998). "Stability radius of an optimal schedule: A survey and recent developments". *Industrial Applications of Combinatorial Optimization*, 72 – 108.

[38] **R. Steuer**, *Multiple Criteria Optimization: Theory, Computation and Application*. John Wiley & Sons, New York, 1986.

[39] **T. Tanino, Y. Sawaragi**, (1980). "Stability of nondominated solutions in multicriteria decision-making". *Journal of Optimization Theory and Applications 30*, 229 – 253.

[40] **T. Tanino**, (1988). "Sensitivity analysis in multiobjective optimization". *Journal of Optimization Theory and Applications 56*, 479 – 499.

[41] **R.E. Wendell**, (2005). "Tolerances sensitivity and optimality bounds in linear programming". *Management Science 50*, 797 – 803.

[42] **H. Yaman, O. Karasan, M. Pinar**, (2001). "The robust spanning tree problem with interval data". *Operations Research Letters 29*, 31 – 40.

[43] **P. Zielinski**, (2004). "The computational complexity of the relative robust shortest path problem with interval data". *European Journal of Operational Research 158*, 570 – 576.

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