Piecewise Directable Automata

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Abstract

In this paper a strong form of directability of automata is introduced and studied. A word w is called *piecewise directing* if any input word containing w as a piecewise subword takes the automaton to the same state from every starting state. The *piecewise directable automata* form a variety of automata situated properly between the varieties of definite automata and directable automata. We describe the sets of piecewise directing words of automata in terms of Haines' embedding order, study the structure of piecewise directable automata, and present criteria for an automaton to be piecewise directable. We also consider the congruences of an automaton which yield a piecewise directable quotient automaton and show that any n-state piecewise directable automaton has a piecewise directing word of length $\leq \binom{n}{2}$.

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1 Introduction

An automaton is said to be *directable* (or *synchronizable*) if it has a directing word which takes the automaton from every state to the same state. In the terminology of Moore experiments [11] a directing word is a special homing sequence which yields a known final state even if no outputs are observed. It seems that the notion of directability has occurred independently at different times to many authors. Some of the earliest sources are [10] and [5]. In [5] Černý states his famous conjecture claiming that any n-state ($n \geq 1$) directable automaton has a directing word of length $\leq (n-1)^2$. Although $(n-1)^2$ is shown to be a lower bound already in [5], the best upper bounds known for the length of the shortest directing words are still of order n^3. On the other hand, for some special classes considerably better, and even exact, bounds are known (cf. [12], [14] and [8], for example). For a general survey of the theory of directable automata and many references the reader is referred to [2].

In this paper we introduce and study a stronger form of directability. A word w is said to be *piecewise directing* (pwd) for an automaton $\mathcal{A}$ if any input word which contains w as a piecewise subword is a directing word of $\mathcal{A}$, that is to say, if the input word is of the form $v_0w_1v_1w_2\cdots v_{k-1}w_kv_k$, where $w_1w_2\cdots w_k = w$, then the final state of $\mathcal{A}$ is known even if the starting state is unknown. Of course, an automaton is called *piecewise directable* (pwd) if it has a pwd word.

In Section 2 we review the basic concepts to be used and fix some general notation. Piecewise directing words are introduced in the next section where we also consider the languages formed by the pwd words of a given automaton. Here the embedding order $\leq_e$ on the set X^* of all words over the input alphabet X plays a central role. This follows from the fact that if u is a pwd word for an automaton $\mathcal{A}$ and $u \leq_e v$, then also v is a pwd word of $\mathcal{A}$. Since by Haines' theorem [7] all antichains in $(X^*, \leq_e)$ are finite, this means that for any automaton $\mathcal{A}$, the set $PDW(\mathcal{A})$ of pwd words is a regular set and the union of finitely many languages of the form $X^*x_1X^*x_2X^*\cdots X^*x_kX^*$, where $k \geq 0$ and $x_1, x_2, \ldots, x_k \in X$. We also show that every language of this form is the set $PDW(\mathcal{A})$ for some automaton $\mathcal{A}$.

Section 4 forms the core of this paper. First we note that for any alphabet X containing at least two letters, the class $\mathbf{PDir}(X)$ of pwd X-automata lies properly between the classes of definite X-automata and directable X-automata. Likewise as these, $\mathbf{PDir}(X)$ is a variety of (finite) automata; it contains all subautomata, epimorphic images and finite direct products of its members. Next we show that each pwd automaton is the extension of a strongly connected pwd automaton by a pwd automaton with a trap state.

Two states a and b of an X-automaton $\mathcal{A}$ are said to be *piecewise mergeable* if there is a word $w \in X^*$ such that any word containing w as a piecewise subword takes a and b to the same state. An automaton is pwd if and only if all of its pairs of states are piecewise mergeable. To take advantage of this fact we associate two new automata $\mathcal{A}^{(2)}$ and $\mathcal{S}(\mathcal{A})$ with any given automaton $\mathcal{A}$. The *pair automaton* $\mathcal{A}^{(2)}$ has as its states all pairs of distinct states of $\mathcal{A}$ plus a state $\Diamond$ which represents all pairs of identical states. The automata $\mathcal{A}$ and $\mathcal{A}^{(2)}$ have the same pwd words but $\mathcal{A}^{(2)}$ has $\Diamond$ as a trap state. The states of $\mathcal{S}(\mathcal{A})$ are certain subautomata of $\mathcal{A}$. It is always cycle-free, has a trap state and is piecewise directable. If $\mathcal{A}$ itself is pwd, then $PDW(\mathcal{A})$ equals both $PDW(\mathcal{S}(\mathcal{A}))$ and the set of directing words of $\mathcal{S}(\mathcal{A})$. The automaton $\mathcal{S}(\mathcal{A}^{(2)})$ obtained by combining the two constructions can be used both for checking whether $\mathcal{A}$ is pwd and for recognizing the set $PDW(\mathcal{A})$.

In Section 5 we use results obtained in the previous section to show that any n-state ($n \geq 1$) pwd automaton has a pwd word of length $\leq \binom{n}{2}$. In Section 6 we consider congruences of automata which yield pwd quotient automata and show how the greatest such congruence of a given automaton can be found using the pair automaton. The last section contains a reasonably efficient algorithm for testing the piecewise directability of a given automaton.

2 Preliminaries

The cardinality and the power set of a set A are denoted by $|A|$ and $\wp(A)$, respectively. If $f : A \to B$ is a mapping, the value $f(a)$ of an element $a \in A$ is denoted also af, and for $H \subseteq A$, we may write Hf for $f(H) = \{af \,:\, a \in H\}$. The composition of two mappings $f : A \to B$ and $g : B \to C$ is the mapping $fg : A \to C$, $a \mapsto (af)g$. For a relation $\theta \subseteq A \times B$ and any elements $a \in A$, $b \in B$, we may express $(a, b) \in \theta$ also by writing $a\,\theta\,b$. The set of equivalence relations on a set A is denoted by $\mathrm{Eq}(A)$. If $\theta \in \mathrm{Eq}(A)$, then a/θ is the θ-class of an element a of A, and the set of all such θ-classes, the quotient set with respect to θ, is denoted A/θ. For any set A, let $\Delta_A = \{(a, a) \,:\, a \in A\}$ and $\nabla_A = A \times A$.

Let us also recall that a reflexive symmetric relation on a set is called a *tolerance*. Let $\mathrm{Tol}(A)$ denote the set of all tolerances on a set A. A closely related notion is that of a cover: a set $\mathcal{P}$ of nonempty subsets of A is a *cover* on A if $\bigcup \mathcal{P} = A$ and no member of $\mathcal{P}$ is a subset of another. The members of a cover are its *blocks*. Let $\mathrm{Cov}(A)$ denote the set of all covers on A. With any cover $\mathcal{P}$ on A we can associate the tolerance $r(\mathcal{P})$ which consists of all pairs $(a, b) \in A \times A$ such that $a, b \in H$ for some block H of $\mathcal{P}$.

In this paper X is always a finite nonempty alphabet. The set of all (finite) words over X, also called *X-words*, is denoted X^* and ε is the empty word. For the *length* of a word w we use the notation $\lg(w)$. For any integer $k \geq 0$, we set

$$X^{\leq k} = \{w \in X^* \,:\, \lg(w) \leq k\},$$

and $X^{\geq k}$ is defined similarly.

An X-*automaton*, or simply an *automaton*, is a system $\mathcal{A} = (A, X, \delta)$, where A is the finite nonempty set of *states*, X is the *input alphabet*, and $\delta : A \times X \to A$ is the *transition function*. Since each letter $x \in X$ induces a unary operation

$$x^{\mathcal{A}} : A \to A,\ a \mapsto \delta(a, x),$$

and these operations determine uniquely the transition function δ by the condition

$$\delta(a, x) = ax^{\mathcal{A}} \quad (a \in A, x \in X),$$

we may view X as a set of unary operation symbols and the automaton $\mathcal{A}$ as a finite X-algebra, and we shall also write simply $\mathcal{A} = (A, X)$. The transition function is extended to $A \times X^*$ in the usual way and then each word $w \in X^*$ also defines a unary operation $w^{\mathcal{A}} : A \to A,\ a \mapsto \delta(a, w)$. The state $\delta(a, w)$ into which the input word w takes $\mathcal{A}$ from state a is usually denoted $aw^{\mathcal{A}}$, or simply aw if there is no danger of confusion. This notation is extended also to subsets of A: if $H \subseteq A$, then $Hw^{\mathcal{A}} = Hw = \{aw : a \in H\}$. Finally, for any $H \subseteq A$ and $L \subseteq X^*$, let $HL^{\mathcal{A}} = HL = \{aw : a \in H, w \in L\}$.

When automata are regarded as unary algebras, the basic algebraic notions can be modeled after universal algebra (cf. [4], for example). Hence an X-automaton $\mathcal{B} = (B, X)$ is a *subautomaton* of the X-automaton $\mathcal{A} = (A, X)$ if $B \subseteq A$ and $bx^{\mathcal{B}} = bx^{\mathcal{A}}$ for all $b \in B$ and $x \in X$. A subset B of A is *closed* in $\mathcal{A} = (A, X)$ if $bx \in B$ for all $b \in B$ and $x \in X$. The set of all closed subsets of $\mathcal{A} = (A, X)$ is denoted $\mathrm{Sub}(\mathcal{A})$. There is an obvious correspondence between the subautomata and the non-empty closed subsets of an automaton, and hence non-empty closed sets are also called subautomata. For any $H \subseteq A$, the least closed subset of $\mathcal{A} = (A, X)$ containing H is denoted by $\langle H \rangle_{\mathcal{A}}$ or simply $\langle H \rangle$. It is easy to see that $\langle H \rangle = \{aw : a \in H, w \in X^*\}$. If $H \neq \varnothing$, this is the *subautomaton generated* by H. For any $a \in A$, we write just $\langle a \rangle$ for $\langle \{a\} \rangle$. The automaton is *connected* from state a if $\langle a \rangle = A$, and $\mathcal{A}$ is *strongly connected* if it is connected from every state. A state a such that $ax = a$ for every $x \in X$ is called a *trap*. Let $\mathrm{Tr}(\mathcal{A})$ denote the set of all traps of $\mathcal{A}$.

An equivalence $\theta \in \mathrm{Eq}(A)$ is a *congruence* on $\mathcal{A} = (A, X)$ if for all $a, b \in A$ and $x \in X$, $a\,\theta\,b$ implies $ax\,\theta\,bx$. The set $\mathrm{Con}(\mathcal{A})$ of congruences on $\mathcal{A}$ forms a sublattice of the lattice $(\mathrm{Eq}(A), \subseteq)$. Moreover, $\Delta_A, \nabla_A \in \mathrm{Con}(\mathcal{A})$. For any $\theta \in \mathrm{Con}(\mathcal{A})$, the *quotient automaton* $\mathcal{A}/\theta = (A/\theta, X)$ is defined so that $(a/\theta)x^{\mathcal{A}/\theta} = (ax^{\mathcal{A}})/\theta$ for all $a/\theta \in A/\theta$ and $x \in X$. More generally, a tolerance θ on A is a *tolerance* of the automaton $\mathcal{A} = (A, X)$ if for all $a, b \in A$ and $x \in X$, $a\,\theta\,b$ implies $ax\,\theta\,bx$. Let $\mathrm{Tol}(\mathcal{A})$ be the set of all tolerances of $\mathcal{A}$.

A *morphism* of X-automata from $\mathcal{A} = (A, X)$ to $\mathcal{B} = (B, X)$ is a mapping $\phi : A \to B$ such that for all $a \in A$ and $x \in X$, $(ax^{\mathcal{A}})\phi = (a\phi)x^{\mathcal{B}}$. We write $\phi : \mathcal{A} \to \mathcal{B}$ to indicate that ϕ is a morphism. An *epimorphism* is a surjective morphism. If there exists an epimorphism $\phi : \mathcal{A} \to \mathcal{B}$, then $\mathcal{B}$ is an *image* of $\mathcal{A}$. An *isomorphism* is a bijective morphism, and $\mathcal{A}$ and $\mathcal{B}$ are said to be *isomorphic*, $\mathcal{A} \cong \mathcal{B}$ in symbols, if there is an isomorphism $\phi : \mathcal{A} \to \mathcal{B}$. The *direct product* of $\mathcal{A}$ and $\mathcal{B}$ is the X-automaton $\mathcal{A} \times \mathcal{B} = (A \times B, X)$ defined so that $(a, b)^{\mathcal{A}\times\mathcal{B}} = (ax^{\mathcal{A}}, bx^{\mathcal{B}})$ for all $(a, b) \in A \times B$ and $x \in X$.

If $\mathbf{K}$ is a class of automata, then $\mathbf{K}(X)$ denotes the class of X-automata belonging to $\mathbf{K}$. A nonempty class of X-automata is called a *variety of X-automata* if it is closed under the operations of forming subautomata, images and (finite) direct products.

Sets of X-words are called *X-languages*, or simply *languages*. By an *X-recognizer* we mean a system $\mathbf{A} = (A, X, \delta, a_0, F)$ which consists of an X-automaton (A, X, δ), an *initial state* $a_0\ (\in A)$ and a set $F\ (\subseteq A)$ of *final states*. The *language recognized* by $\mathbf{A}$ is $L(\mathbf{A}) = \{w \in X^* : a_0 w \in F\}$. An X-language is *recognizable*, or *regular*, if it is recognized by an X-automaton. The set of recognizable X-languages is denoted by $Rec(X)$.

For directable automata we refer the reader to [2], [6], [8], [15], for example. However, we shall recall some basic notions and facts which will be needed here. A word $w \in X^*$ is a *directing word* of an X-automaton $\mathcal{A} = (A, X)$ if it takes $\mathcal{A}$ from every state to the same state, i.e., if $|Aw| = 1$, and we call $\mathcal{A}$ *directable* if it has a directing word. The set of directing words of $\mathcal{A}$ is denoted by $DW(\mathcal{A})$. The class of all directable automata is denoted by $\mathbf{Dir}$. For any X-automaton $\mathcal{A}$, $DW(\mathcal{A}) \in Rec(X)$. If $\mathcal{A}$ is a directable automaton, let $d(\mathcal{A}) = \min\{\lg(w) : w \in DW(\mathcal{A})\}$, and for any $n \geq 1$, we define the number

$$d(n) = \max\{d(\mathcal{A}) : \mathcal{A} \in \mathbf{Dir},\ |A| = n\}.$$

Černý's conjecture may now be formulated as the claim that $d(n) = (n-1)^2$ for all $n \geq 1$. For the general case upper bounds of the order $O(n^3)$ only are known (cf. [15], [6], [12], for example). On the other hand, it is known that $d(n) \geq (n-1)^2$ for all $n \geq 1$ (cf. [5], [15]).

3 Piecewise directing words

The *embedding order* $\leq_e$ was defined by Haines [7] so that for any $u, v \in X^*$, $u \leq_e v$ iff u is a *piecewise subword* of v, that is to say, $u = u_1 u_2 \cdots u_k$ and $v = v_0 u_1 v_1 u_2 v_2 \cdots v_{k-1} u_k v_k$ for some $k \geq 1$ and $u_1, \ldots, u_k, v_0, \ldots, v_k \in X^*$.

Proposition 3.1 (Haines [7]) *For any alphabet X, $\leq_e$ is a partial order on X^*. Any set of pairwise incomparable words in the poset $(X^*, \leq_e)$ is finite.*

The *e-closure* of a word $u = x_1 x_2 \cdots x_k$, where $k \geq 0$, $x_1, x_2, \ldots, x_k \in X$, is defined as the set

$$e[u] = \{v \in X^* : u \leq_e v\} = X^* x_1 X^* x_2 X^* \cdots X^* x_k X^*.$$

The e-*closure* of a language $L \subseteq X^*$ is then the set $e[L] = \bigcup\{e[u] \,:\, u \in L\}$. It is clear that

$$e : \wp(X^*) \to \wp(X^*),\ L \mapsto e[L],$$

is an algebraic closure operator (cf. [4], for example). In particular, $e[e[w]] = e[w]$ for every word $w \in X^*$. Moreover, Haines' theorem yields the following useful description of the e-closed languages.

Corollary 3.1 *The e-closure of any language $L \subseteq X^*$ is a regular language; more precisely,*

$$e[L] = e[w_1] \cup e[w_2] \cup \ldots \cup e[w_k],$$

where $w_1, w_2, \ldots, w_k$ are the $\leq_e$-minimal words in L.

Hence the e-closed X-languages are precisely the unions $e[w_1] \cup \ldots \cup e[w_k]$, where $k \geq 0$ and $w_1, \ldots, w_k \in X^*$ are pairwise $\leq_e$-incomparable.

The following observations are also useful.

Lemma 3.1 *Let $\mathcal{A} = (A, X)$ be an automaton and $u, v, w \in X^*$ any words.*

(a) $u \leq_e v \Leftrightarrow e[u] \supseteq e[v]$.
(b) *If $u \leq_e v$, then $He[v]^{\mathcal{A}} \subseteq He[u]^{\mathcal{A}}$ for every $H \subseteq A$.*
(c) *If for some $H \subseteq A$, $He[u]^{\mathcal{A}} = He[v]^{\mathcal{A}}$, then $He[uw]^{\mathcal{A}} = He[vw]^{\mathcal{A}}$.*

Definition 3.1 We call $u \in X^*$ a *piecewise directing word* of an automaton $\mathcal{A} = (A, X)$ if for every $v \in e[u]$, $av = bv$ for all $a, b \in A$. The set of piecewise directing words of $\mathcal{A}$ is denoted by $PDW(\mathcal{A})$ and the automaton $\mathcal{A}$ is said to be *piecewise directable* if $PDW(\mathcal{A}) \neq \varnothing$. Let **PDir** be the class of piecewise directable automata.

We shall often abbreviate 'piecewise' to 'pw'. Furthermore, 'pwd' will mean either 'piecewise directing' or 'piecewise directable' depending on the context.

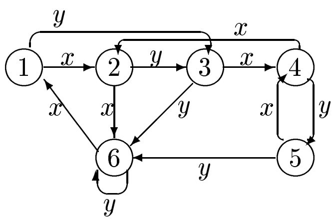
That a word $u = x_1x_2 \cdots x_k$ ($k \geq 0$, $x_1, x_2, \ldots, x_k \in X$) is pwd for an X-automaton $\mathcal{A}$ means that every word of the form $v_0x_1v_1x_2v_2 \cdots v_{k-1}x_kv_k$ is a directing word of $\mathcal{A}$. This observation and some further basic facts about piecewise directing words are presented in the following proposition.

Proposition 3.2 *Let $\mathcal{A}$ be any X-automaton.*

(a) *For any $u \in X^*$, $u \in PDW(\mathcal{A})$ iff $e[u] \subseteq DW(\mathcal{A})$.*
(b) *$PDW(\mathcal{A}) \subseteq DW(\mathcal{A})$ and the inclusion may be proper even if $\mathcal{A}$ is piecewise directable.*
(c) $PDW(\mathcal{A}) = e[PDW(\mathcal{A})]$.
(d) *$PDW(\mathcal{A})$ is a regular language which can be given as the union of finitely many languages of the form $X^*x_1X^*x_2X^* \cdots X^*x_kX^*$ ($k \geq 0$, $x_1, x_2, \ldots, x_k \in X$).*

Most of Proposition 3.2 follows directly from the definitions and Corollary 3.1. That the proper inclusion $PDW(\mathcal{A}) \subset DW(\mathcal{A})$ may hold even if $\mathcal{A} \in \mathbf{PDir}$, is shown by the following example.

Example 3.1 The automaton $\mathcal{A}$ defined by the state diagram below is pwd since $yyx \in PDW(\mathcal{A})$, but $xyy \in DW(\mathcal{A}) \setminus PDW(\mathcal{A})$ since $xyxy \notin DW(\mathcal{A})$.



Note, however, that for a one-letter alphabet, any directing word is also a pwd word.

Also the converse of Proposition 3.2 (c) holds and the languages $PDW(\mathcal{A})$ can be characterized as follows.

Proposition 3.3 *For a given language $L \subseteq X^*$ there is an X-automaton $\mathcal{A}$ such that $L = PDW(\mathcal{A})$ if and only if L is e-closed.*

Proof. By Proposition 3.2 (c) all languages $PDW(\mathcal{A})$ are e-closed, so we suppose that $L \subseteq X^*$ is an e-closed language.

If $L = \varnothing$, then $L = PDW(\mathcal{A})$ for any X-automaton $\mathcal{A}$ which is not pw directable. Assume now that $L = e[L] \neq \varnothing$. Let us consider a minimal recognizer $\mathbf{A} = (A, X, \delta, a_0, F)$ of L. Since $wx \in L$ whenever $w \in L$ and $x \in X$, F must consist of a single trap state, say c. We prove that $L = PDW(\mathcal{A})$, for $\mathcal{A} = (A, X, \delta)$.

If $u \in PDW(\mathcal{A})$, then $a_0u = cu = c$ and hence $u \in L$.

Let $u \in L$ and consider any $v \in e[u]$. For any $a \in A$ there is a word $w \in X^*$ such that $a_0w = a$, and therefore $av = a_0wv = c$. Hence $u \in PDW(\mathcal{A})$. $\square$

4 Piecewise directable automata

First we show that the class of pwd automata lies properly between the classes of definite automata and directable automata. Recall that an automaton is *definite* if there is a $k \geq 0$ such that $aw = bw$ for every $w \in X^{\geq k}$ and all $a, b \in A$. Let **Def** denote the class of definite automata.

Proposition 4.1 *For any one-letter alphabet X, $\mathbf{Def}(X) = \mathbf{PDir}(X) = \mathbf{Dir}(X)$, but $\mathbf{Def}(X) \subset \mathbf{PDir}(X) \subset \mathbf{Dir}(X)$ if X contains at least two letters.*

Proof. By Proposition 3.2 (b) the inclusion $\mathbf{PDir}(X) \subseteq \mathbf{Dir}(X)$ holds for every X. It is also clear that $\mathbf{Def}(X) \subseteq \mathbf{PDir}(X)$. Indeed, if $\mathcal{A} \in \mathbf{Def}(X)$, then $X^{\geq k} \subseteq DW(\mathcal{A})$ for some $k \geq 0$. Moreover, equality obviously holds in both cases if $|X| = 1$.

If X contains at least two distinct letters x and y, then $\mathcal{A} \in \mathbf{Dir}(X) \setminus \mathbf{PDir}(X)$, when the X-automaton $\mathcal{A} = (\{1,2,3\}, X)$ is defined so that $1x = 3x = 1$, $2x = 2$, $1y = 2$, $2y = 3$, $3y = 1$ and $az = a$ for all $z \in X \setminus \{x,y\}$ and $a \in \{1,2,3\}$. Indeed, $xyyx$ is a directing word of $\mathcal{A}$ but there cannot be any pwd word since $\{1,2,3\}w = \{1,2\}$ for any word $w \in (yux)^k$, where $k \geq 1$ and $u \in (X \setminus \{x,y\})^*$. Finally, it is easy to see that $\mathcal{B} \in \mathbf{PDir}(X) \setminus \mathbf{Def}(X)$ when $\mathcal{B} = (\{1,2\}, X)$ is defined so that $1x = 1, 2x = 2, 1y = 2y = 2$ and $1z = 2z = 2$ for all $z \in X \setminus \{x,y\}$. $\square$

Proposition 4.2 *For any alphabet X, $\mathbf{PDir}(X)$ is a variety of X-automata.*

Proof. It is clear that $PDW(\mathcal{A}) \subseteq PDW(\mathcal{B})$ if $\mathcal{B}$ is a subautomaton or an image of $\mathcal{A}$. Hence $\mathbf{PDir}(X)$ is closed under the construction of subautomata and images. Finally, if $\mathcal{A}, \mathcal{B} \in \mathbf{PDir}(X)$, then also $\mathcal{A} \times \mathcal{B} \in \mathbf{PDir}(X)$ since for any $u \in PDW(\mathcal{A})$ and $v \in PDW(\mathcal{B})$, uv is clearly a pwd word of $\mathcal{A} \times \mathcal{B}$. $\square$

Any pwd automaton $\mathcal{A} = (A, X)$ has a unique strongly connected subautomaton which is accessible from all states. Indeed, if $u \in PDW(\mathcal{A})$, then au is the same state c for all $a \in A$. In particular, $cvu = c$ for every word $v \in X^*$, and hence $\langle c \rangle$ is strongly connected. Of course, $\langle c \rangle \in \mathbf{PDir}(X)$. At this point we have to recall some algebraic notions (cf. [3], for example).

The *Rees congruence* of an X-automaton $\mathcal{A} = (A, X)$ modulo a subautomaton $\mathcal{B} = (B, X)$ is the relation $\rho_{\mathcal{B}} = \Delta_A \cup (B \times B)$. The corresponding *Rees quotient* $\mathcal{A}/\rho_{\mathcal{B}}$ is written simply as $\mathcal{A}/\mathcal{B}$. Finally, $\mathcal{A}$ is said to be an *extension* of $\mathcal{B}$ by $\mathcal{C}$ if $\mathcal{B}$ is a subautomaton of $\mathcal{A}$ and $\mathcal{C} \cong \mathcal{A}/\mathcal{B}$.

In our present situation $\mathcal{A}/\langle c \rangle$ is a pwd automaton with a trap. It is also clear that if a pwd automaton $\mathcal{C}$ has a trap c, then $\mathrm{Tr}(\mathcal{C}) = \{c\}$ and every subautomaton of $\mathcal{C}$ contains c. We shall call a pwd automaton with a trap *piecewise trap-directable*. The above observations lead to the following description of the structure of pwd automata.

Proposition 4.3 *An automaton is piecewise directable if and only if it is the extension of a strongly connected piecewise directable automaton by a piecewise trap-directable automaton.*

Let us recall that an automaton is directable if and only if each pair a, b of distinct states has a *merging word* w such that $aw = bw$. This fact, noted already in [10], has been used for testing the directability of automata and for finding upper bounds for minimum-length directing words (cf. [15], [8], for example). We shall now consider a corresponding notion of piecewise mergeability.

For any $k \geq 0$, two states a and b of an automaton $\mathcal{A} = (A, X)$ are said to be *piecewise k-mergeable* if there exists a *piecewise k-merging word* $w \in X^{\leq k}$ such that $av = bv$ for every $v \in e[w]$. Let $PMW_k(a, b)$ denote the set of all piecewise k-merging words of a given pair of states $a, b \in A$. Then $PMW(a, b) = \bigcup_{k \geq 0} PMW_k(a, b)$ is the set of all *piecewise merging words* of the pair a, b and the pair is said to be *piecewise mergeable* if $PMW(a, b) \neq \varnothing$.

The following facts are quite obvious.

Lemma 4.1 *Let $\mathcal{A} = (A, X)$ be any automaton.*

(a) *For any pair of states $a, b \in A$, $e[PMW(a, b)] = PMW(a, b)$ holds and hence $PMW(a, b)$ is a regular language which can be represented as the union of finitely many languages of the form $X^*x_1X^*x_2X^* \cdots X^*x_kX^*$ $(k \geq 0,\ x_1, x_2, \ldots, x_k \in X)$.*

(b) *For any pair of states $a, b \in A$ and any $w \in X^*$, $PMW(a, b) \subseteq PMW(aw, bw)$.*

(c) $PDW(\mathcal{A}) = \bigcap\{PMW(a, b) \ :\ a, b \in A,\ a \neq b\}$.

The intersection $e[L_1] \cap e[L_2] \cap \ldots \cap e[L_n]$ of any nonempty languages $L_1, L_2, \ldots, L_n \subseteq X^*$ is also nonempty; it contains, for example, any word $w_1w_2 \cdots w_n$, where $w_1 \in L_1, w_2 \in L_2, \ldots, w_n \in L_n$. Hence Lemma 4.1 yields the following result.

Proposition 4.4 *An automaton is piecewise directable if and only if all pairs of distinct states are piecewise mergeable.*

Let us introduce the following symbols for the piecewise mergeability relations of an automaton $\mathcal{A}$:

$$\pi_{\mathcal{A}}(k) = \{(a, b) \ :\ a, b \in A, PMW_k(a, b) \neq \varnothing\}$$

for each $k \geq 0$, and let $\pi_{\mathcal{A}} = \bigcup_{k \geq 0} \pi_{\mathcal{A}}(k)$.

Proposition 4.5 *Let $\mathcal{A}$ be any automaton.*

(a) $\Delta_A = \pi_{\mathcal{A}}(0) \subseteq \pi_{\mathcal{A}}(1) \subseteq \ldots \subseteq \pi_{\mathcal{A}}$.

(b) $\pi_{\mathcal{A}}(k) \in \mathrm{Tol}(\mathcal{A})$ *for all $k \geq 0$, and* $\pi_{\mathcal{A}} \in \mathrm{Con}(\mathcal{A})$.

(c) $\mathcal{A} \in \mathbf{PDir}(X)$ *iff* $\pi_{\mathcal{A}} = \nabla_A$.

Proof. Statement (a) and the fact that $\pi_{\mathcal{A}}(k), \pi_{\mathcal{A}} \in \mathrm{Tol}(\mathcal{A})$ follow directly from the definition of the relations $\pi_{\mathcal{A}}(k)$ and $\pi_{\mathcal{A}}$, and Lemma 4.1 (a) implies the transitivity of $\pi_{\mathcal{A}}$. Furthermore, $a \pi_{\mathcal{A}}(k) b$ implies that $ax\, \pi_{\mathcal{A}}(k)\, bx$ for every $x \in X$ by Lemma 4.1 (b). Statement (c) just restates Proposition 4.4 in symbols. $\square$

Let us define the *subuniverse automaton* $(\mathrm{Sub}(\mathcal{A}), X, \eta)$ of an automaton $\mathcal{A} = (A, X)$ so that

$$\eta(B, x) = \langle Bx^{\mathcal{A}} \rangle_{\mathcal{A}}$$

for all $B \in \mathrm{Sub}(A)$ and $x \in X$.

Lemma 4.2 *For any $B \in \mathrm{Sub}(A)$ and $w \in X^*$,*

(a) $\eta(B, w) = Be[w]^{\mathcal{A}}$ *and*
(b) $\eta(B, w) \subseteq B$.

Proof. To prove (a) we proceed by induction on $\lg(w)$. The case $w = \varepsilon$ is obvious, so let $w = ux$, where $x \in X$ and u is a word for which the claim holds. Then also

$$\begin{aligned} \eta(B, w) &= \eta(\eta(B, u), x) = \eta(Be[u]^{\mathcal{A}}, x) \\ &= \langle Be[u]^{\mathcal{A}} x^{\mathcal{A}} \rangle = B(e[u]xX^*)^{\mathcal{A}} \\ &= Be[w]^{\mathcal{A}}, \end{aligned}$$

where we used the fact that $\langle H \rangle_{\mathcal{A}} = HX^*$ for every $H \subseteq A$. Statement (b) follows from (a) since $Be[u]^{\mathcal{A}} \subseteq \langle B \rangle_{\mathcal{A}} = B$. $\square$

Let $\mathcal{S}(\mathcal{A}) = (S(\mathcal{A}), X)$ be the subautomaton of the subuniverse automaton generated by its state A. Hence

$$S(\mathcal{A}) = \{Ae[w]^{\mathcal{A}} \,:\, w \in X^*\}$$

and for any $B = Ae[w]^{\mathcal{A}}$ and $u \in X^*$,

$$Bu^{\mathcal{S}(\mathcal{A})} = Be[u]^{\mathcal{A}} = Ae[wu]^{\mathcal{A}}.$$

Let us recall that an automaton $\mathcal{A} = (A, X)$ is called *cycle-free* if there are no two distinct states $a, b \in A$ such that $au = b$ and $bv = a$ for some $u, v \in X^*$. An automaton is *trap-connected* if it has a trap which is reachable from every state. Clearly, if an automaton is trap-connected or piecewise trap-directable, it has just one trap. It is also clear that a pw trap-directable automaton must be trap-connected. Also the converse holds for ordinary directability (as noted in [13]), but not for pw directability as shown by the following example.

Example 4.1 If the transitions of the automaton $\mathcal{A} = (\{1, 2, 3\}, \{x, y\})$ are defined so that $1x = 1$, $1y = 2$, $2y = 1$, $2x = 3x = 3y = 3$, then $\mathcal{A}$ is trap-connected, but not pw directable. Indeed, for every $u \in \{x, y\}^*$ there is an $n \geq 0$ such that $u \leq_e (xyy)^n$, and the words $(xyy)^n$ are not directing.

Proposition 4.6 *For any $\mathcal{A} = (A, X)$, the automaton $\mathcal{S}(\mathcal{A})$ is cycle-free and piecewise trap-directable. Moreover, the trap of $\mathcal{S}(\mathcal{A})$ is the least element of the lattice $(S(\mathcal{A}), \subseteq)$.*

Proof. That $\mathcal{S}(\mathcal{A})$ is cycle-free follows from the fact that $Bu^{\mathcal{S}(\mathcal{A})} \subseteq \langle B \rangle_{\mathcal{A}} = B$ for all $B \in S(\mathcal{A})$ and $u \in X^*$. It is now also clear that $\mathcal{S}(\mathcal{A})$ has traps. Let $D = Aw^{\mathcal{S}(\mathcal{A})} \in \mathrm{Tr}(\mathcal{S}(\mathcal{A}))$. Consider any state $B = Au^{\mathcal{S}(\mathcal{A})}$. Since $De[u]^{\mathcal{A}} = D$, there exists for every $d \in D$ a state $c \in D$ and a word $v \in e[u]$ such that $d = cv^{\mathcal{A}}$, and hence $D \subseteq B$. This means that D is the least element of $S(\mathcal{A})$ and hence it must also be the only trap of $\mathcal{S}(\mathcal{A})$. Finally, it is clear that w is a pwd word: for any $B \in S(\mathcal{A})$, $D \subseteq Be[w]^{\mathcal{A}} \subseteq Ae[w]^{\mathcal{A}} = D$. $\square$

For pw trap-directable automata the following further fact can be noted.

Proposition 4.7 *If $\mathcal{A}$ is a piecewise trap-directable automaton, then*

$$PDW(\mathcal{A}) = PDW(\mathcal{S}(\mathcal{A})) = DW(\mathcal{S}(\mathcal{A})).$$

Proof. If $\mathcal{A} = (A, X)$ is pw trap-directable and $\mathrm{Tr}(\mathcal{A}) = \{c\}$, then for any $u \in X^*$,

$$\begin{aligned} u \in PDW(\mathcal{A}) &\Leftrightarrow Ae[u]^{\mathcal{A}} = \{c\} \\ &\Leftrightarrow Bu^{\mathcal{S}(\mathcal{A})} = \{c\} \text{ for every } B \in \mathcal{S}(\mathcal{A}) \\ &\Leftrightarrow u \in DW(\mathcal{S}(\mathcal{A})). \end{aligned}$$

Now $e(DW(\mathcal{S}(\mathcal{A})) = e[PDW(\mathcal{A})] = PDW(\mathcal{A}) = DW(\mathcal{S}(\mathcal{A}))$ shows that also $PDW(\mathcal{S}(\mathcal{A})) = DW(\mathcal{S}(\mathcal{A}))$ holds. $\square$

Note that a recognizer for the language $PDW(\mathcal{A})$ can be found by using the equality $PDW(\mathcal{A}) = DW(\mathcal{S}(\mathcal{A}))$. This fact could also be used for testing $\mathcal{A}$ for pw directability. However, in Proposition 4.10 a simpler recognizer is presented and in Section 7 a more efficient test for pw directability is given.

As a useful tool for the study of pw merging words and pw directability we introduce for each automaton $\mathcal{A} = (A, X)$ its *pair automaton* $\mathcal{A}^{(2)} = (A^{(2)}, X)$ which is defined as follows:

- $A^{(2)} = \{\{a, b\} : a \neq b,\ a, b \in A\} \cup \{\Diamond\}$;
- $\Diamond x = \Diamond$ for every $x \in X$;
- $\{a, b\}x = \begin{cases} \{ax, bx\}, & \text{if } ax \neq bx; \\ \Diamond, & \text{otherwise,} \end{cases}$

for every $x \in X$.

For each $k \geq 0$, let

$$PM_{\mathcal{A}}(k) = \{\Diamond\} \cup \{\{a, b\} \in A^{(2)} : a\pi_{\mathcal{A}}(k)b\}.$$

Moreover, let $PM_{\mathcal{A}} = \bigcup_{k \geq 0} PM_{\mathcal{A}}(k)$.

In some sense, $\Diamond$ represents all trivially pw mergeable pairs $\{a, a\}$. Hence $PM_{\mathcal{A}}(k)$ ($k \geq 0$) and $PM_{\mathcal{A}}$ can be regarded, respectively, as the sets of all pw k-mergeable and all pw mergeable pairs of states.

Lemma 4.3 *For any automaton* $\mathcal{A} = (A, X)$*,*

(a) $PM_{\mathcal{A}}(0) = \{\diamondsuit\} \subseteq PM_{\mathcal{A}}(1) \subseteq \ldots \subseteq PM_{\mathcal{A}}$*,*
(b) $PM_{\mathcal{A}}(k) \in \mathrm{Sub}(\mathcal{A}^{(2)})$ *for every* $k \geq 0$*, and*
(c) $PM_{\mathcal{A}} \in \mathrm{Sub}(\mathcal{A}^{(2)})$*.*

Proof. The first claim is completely obvious, and (b) and (c) hold by Proposition 4.5 (b). $\square$

Lemma 4.4 *Let* $\mathcal{A} = (A, X)$ *be an automaton and* $u \in X^*$ *any word.*

(a) *For any pair* $a, b \in A$*,* $a \neq b$*,*

$$u \in PMW(a, b) \ \Leftrightarrow\ \{a, b\}e[u]^{\mathcal{A}^{(2)}} = \{\diamondsuit\}.$$

(b) $u \in PDW(\mathcal{A}) \ \Leftrightarrow\ A^{(2)}e[u]^{\mathcal{A}^{(2)}} = \{\diamondsuit\}.$

Proof. Both claims follow from the obvious fact that

$$\{a, b\}v^{\mathcal{A}^{(2)}} = \begin{cases} \{av, bv\}, & \text{if } av \neq bv, \\ \diamondsuit, & \text{if } av = bv, \end{cases}$$

for any $v \in X^*$ and any $a, b \in A$ with $a \neq b$. $\square$

The pair automaton $\mathcal{A}^{(2)}$ of any automaton $\mathcal{A} = (A, X)$ has at least $\diamondsuit$ as a trap. Hence Lemma 4.4 (b) means that $\mathcal{A}$ and $\mathcal{A}^{(2)}$ have exactly the same pwd words. Moreover, if $\mathcal{A}$ is pw directable, then $\mathcal{A}^{(2)}$ can have no other trap than $\diamondsuit$. Hence, we get the following result.

Proposition 4.8 *For any automaton* $\mathcal{A} = (A, X)$*,*

(a) $PDW(\mathcal{A}) = PDW(\mathcal{A}^{(2)})$*,*

and hence

(b) $\mathcal{A}$ *is piecewise directable if and only if its pair automaton* $\mathcal{A}^{(2)}$ *is piecewise trap-directable.*

Let us now consider the automaton $\mathcal{S}(\mathcal{A}^{(2)}) = (S(\mathcal{A}^{(2)}), X)$ constructed from a given automaton $\mathcal{A} = (A, X)$. By Proposition 4.6 it is cycle-free and pw trap-directable. The crucial question is whether the unique trap is $\{\diamondsuit\}$ or a larger subautomaton of $\mathcal{A}^{(2)}$.

We say that a state $a \in A$ of an automaton $\mathcal{A} = (A, X)$ is on an *input-complete cycle* if there is a word $w \in X^*$ in which every letter of X appears and $aw = a$.

Proposition 4.9 *For any automaton $\mathcal{A} = (A, X)$ the following conditions are equivalent:*

(i) $\mathcal{A} \in \mathbf{PDir}$;

(ii) $\{\Diamond\}$ *is the trap of* $\mathcal{S}(\mathcal{A}^{(2)})$;

(iii) *for every pair* $\{a, b\} \in A^{(2)}$ *there is a word* $w \in X^*$ *such that* $\langle\{a, b\}\rangle w^{\mathcal{S}(\mathcal{A}^{(2)})} \neq \langle\{a, b\}\rangle$;

(iv) *no state* $\{a, b\} \in A^{(2)}$ *is on an input-complete cycle in* $\mathcal{A}^{(2)}$.

Proof. Let us write simply $\mathcal{S}$ for $\mathcal{S}(\mathcal{A}^{(2)})$.

(i)$\Leftrightarrow$(ii). For any $u \in X^*$,

$$\begin{aligned} u \in PDW(\mathcal{A}) \quad &\Leftrightarrow \quad \{a, b\}e[u]^{\mathcal{A}^{(2)}} = \{\Diamond\} \text{ for all } \{a, b\} \in A^{(2)} \\ &\Leftrightarrow \quad A^{(2)}u^{\mathcal{S}} = \{\{\Diamond\}\}, \end{aligned}$$

and hence $\mathcal{A}$ has a pwd word exactly in case $\{\Diamond\}$ is the trap of $\mathcal{S}$.

(ii)$\Rightarrow$(iii). If w is a word such that $A^{(2)}w^{\mathcal{S}} = \{\Diamond\}$, then by the proof of Proposition 4.6, for any $\{a, b\} \in A^{(2)}$, $\langle\{a, b\}\rangle w^{\mathcal{S}} = \{\Diamond\} \neq \langle\{a, b\}\rangle$.

(iii)$\Rightarrow$(iv). If (iv) does not hold, some $\{a, b\} \in A^{(2)}$ is on an input-complete cycle defined by some word $u \in X^*$. For any $w \in X^*$, there is a $k \geq 0$ such that $w \leq_e u^k$, and hence $\{a, b\} \in \langle\{a, b\}\rangle(u^k)^{\mathcal{S}} \subseteq \langle\{a, b\}\rangle w^{\mathcal{S}}$, which shows that (iii) does not hold either.

(iv)$\Rightarrow$(ii). Assume that the trap D of $\mathcal{S}$ is not $\{\Diamond\}$. Let $\langle\{a, b\}\rangle$ be a maximal monogenic subautomaton of $\mathcal{A}^{(2)}$ contained in D and consider any word u in which all letters of X appear. Since $Du^{\mathcal{S}} = D$, there exists a state $\{c, d\} \in D$ and a word $v \in e[u]$ for which $\{a, b\} = \{c, d\}v^{\mathcal{A}^{(2)}}$. Then $\langle\{a, b\}\rangle \subseteq \langle\{c, d\}\rangle$, but by the maximality of $\langle\{a, b\}\rangle$ equality must hold. Therefore, $\{a, b\}w^{\mathcal{A}^{(2)}} = \{c, d\}$ for some $w \in X^*$. Hence $\{a, b\}$ is on an input-complete cycle defined by wv. $\square$

Finally, we note that a recognizer for the set of pwd words of a pwd automaton $\mathcal{A}$ can be constructed from the automaton $\mathcal{S}(\mathcal{A}^{(2)})$.

Proposition 4.10 *If $\mathcal{A} = (A, X)$ is a piecewise directable automaton, then* $(\mathcal{S}(\mathcal{A}^{(2)}), A^{(2)}, \{\{\Diamond\}\})$ *recognizes* $PDW(\mathcal{A})$.

Proof. By Proposition 4.8 the pair automaton $\mathcal{A}^{(2)}$ is trap-directable and $PDW(\mathcal{A}) = PDW(\mathcal{A}^{(2)})$. Now $PDW(\mathcal{A}) = DW(\mathcal{S}(\mathcal{A}^{(2)}))$ by Proposition 4.7, and obviously $DW(\mathcal{S}(\mathcal{A}^{(2)})) = L(\mathbf{A})$ for the X-recognizer $\mathbf{A} = (\mathcal{S}(\mathcal{A}^{(2)}), A^{(2)}, \{\{\Diamond\}\})$. $\square$

5 Minimum-length piecewise directing words

For any pwd automaton $\mathcal{A}$, let

$$pd(\mathcal{A}) = \min\{\lg(u) \,:\, u \in PDW(\mathcal{A})\}.$$

Furthermore, for each $n \geq 1$ let

$$pd(n) = \max\{pd(\mathcal{A}) \,:\, \mathcal{A} \in \mathbf{PDir},\, |A| = n\}.$$

For any pwd automaton $\mathcal{A}$, $d(\mathcal{A}) \leq pd(\mathcal{A})$ as $PDW(\mathcal{A}) \subseteq DW(\mathcal{A})$. However, since all directable automata are not pw directable, this fact does not imply anything about the relationship between the functions $d(n)$ and $pd(n)$. In particular, Černý's automata (cf. [5], [15]) which yield the lower bound $(n-1)^2$ for $d(n)$ are not pwd for any $n \geq 3$.

Lemma 5.1 *If $\mathcal{A} = (A, X)$ is an n-state $(n \geq 1)$ piecewise trap-directable automaton, then $pd(\mathcal{A}) \leq n - 1$.*

Proof. If $u = x_1 x_2 \cdots x_k$ is a minimum-length pwd word of $\mathcal{A}$ and c is the trap of $\mathcal{A}$, then by Lemma 3.1

$$A \supset Ae[x_1]^{\mathcal{A}} \supset Ae[x_1x_2]^{\mathcal{A}} \supset \ldots \supset Ae[x_1x_2\cdots x_{k-1}]^{\mathcal{A}} \supset Au^{\mathcal{A}} = \{c\},$$

and hence $k \leq n - 1$. □

Clearly, $n - 1$ is an exact upper bound for pw trap-directable automata and hence a lower bound for $pd(n)$. Moreover, we get now the following upper bound.

Proposition 5.1 $pd(n) \leq \binom{n}{2}$ *for every $n \geq 1$.*

Proof. Let $\mathcal{A}$ be an n-state $(n \geq 1)$ pwd automaton. By Proposition 4.8 $\mathcal{A}^{(2)}$ is pw trap-directable and $PDW(\mathcal{A}) = PDW(\mathcal{A}^{(2)})$. Since $\mathcal{A}^{(2)}$ has $\binom{n}{2} + 1$ states, this means by Lemma 5.1 that $pd(\mathcal{A}) = pd(\mathcal{A}^{(2)}) \leq \binom{n}{2}$. □

6 Piecewise directing congruences

Congruences which yield directable quotient automata were introduced in [9] and studied further in [8]. Analogously, we call a congruence ρ on an automaton $\mathcal{A} = (A, X)$ *piecewise directing* if $\mathcal{A}/\rho \in \mathbf{PDir}$. Let $\mathrm{Con}_{pd}(\mathcal{A})$ denote the set of all pwd congruences on $\mathcal{A}$.

That $\mathcal{A}$ is pwd means that $\Delta_A \in \mathrm{Con}_{pd}(\mathcal{A})$. Of course, $\mathrm{Con}_{pd}(\mathcal{A})$ always contains the congruence ∇_A. Proposition 4.2 implies now that $\mathrm{Con}_{pd}(\mathcal{A})$ is a filter of the congruence lattice $\mathrm{Con}(\mathcal{A})$. Indeed, if $\rho \in \mathrm{Con}_{pd}(\mathcal{A})$ and $\rho \subseteq \theta \in \mathrm{Con}(\mathcal{A})$, then also $\theta \in \mathrm{Con}_{pd}(\mathcal{A})$ since $\mathcal{A}/\theta$ is an image of $\mathcal{A}/\rho$, and $\theta \cap \rho \in \mathrm{Con}_{pd}(\mathcal{A})$ for any $\theta, \rho \in \mathrm{Con}_{pd}(\mathcal{A})$ since $\mathcal{A}/\rho \cap \theta$ is isomorphic to a subautomaton of $\mathcal{A}/\rho \times \mathcal{A}/\theta$. As $\mathrm{Con}(\mathcal{A})$ is finite, the filter $\mathrm{Con}_{pd}(\mathcal{A})$ is principal. If $\mu_{\mathcal{A}}$ denotes the least pwd congruence $\bigcap \mathrm{Con}_{pd}(\mathcal{A})$ on $\mathcal{A}$, the above observations can be summed up as follows:

Proposition 6.1 *For any automaton $\mathcal{A}$, the piecewise directing congruences form a principal filter of the lattice* $\mathrm{Con}(\mathcal{A})$ *generated by* $\mu_{\mathcal{A}}$.

We shall now show how $\mu_{\mathcal{A}}$ can be found from the subuniverse automaton $\mathcal{S}(\mathcal{A}^{(2)})$ considered in Section 4.

Proposition 6.2 *Let $\mathcal{A} = (A, X)$ be an automaton. If D is the trap of $\mathcal{S}(\mathcal{A}^{(2)})$, then $\mu_{\mathcal{A}}$ is the transitive closure ρ^+ of the relation ρ on A given by*

$$(a, b) \in \rho \ \Leftrightarrow \ \{a, b\} \in D \text{ or } a = b.$$

Proof. That $\mu_{\mathcal{A}} = \rho^+$ follows from (1)–(4):

(1) $\rho^+ \in \mathrm{Eq}(A)$ since ρ is clearly reflexive and symmetric.

(2) $\rho^+ \in \mathrm{Con}(\mathcal{A})$ since ρ is compatible with the state transitions: if $a \,\rho\, b$, then for every $x \in X$, either $ax = bx$ or $\{ax, bx\} = \{a, b\}x^{\mathcal{A}^{(2)}} \in D$.

(3) $\mathcal{A}/\rho^+ \in \mathbf{PDir}$ is seen as follows. Let $u \in X^*$ be a word such that $D = A^{(2)}u^{\mathcal{S}}$, where $\mathcal{S} = \mathcal{S}(\mathcal{A}^{(2)})$. Then for any $a, b \in A$ and $v \in e[u]$, either $av = bv$ or $\{av, bv\} = \{a, b\}v^{\mathcal{A}^{(2)}} \in A^{(2)}u^{\mathcal{S}} = D$. In any case $(av, bv) \in \rho$, and hence $(a/\rho^+)v^{\mathcal{A}/\rho^+} = av/\rho^+ = bv/\rho^+ = (b/\rho^+)v^{\mathcal{A}/\rho^+}$ which shows that $u \in PDW(\mathcal{A}/\rho^+)$.

(4) Let $\theta \in \mathrm{Con}_{pd}(\mathcal{A})$. To prove $\rho^+ \subseteq \theta$, it suffices to show that $\rho \subseteq \theta$. If $D = \{\Diamond\}$, then $\rho = \Delta_A \subseteq \theta$. Let $\{a, b\} \in D$. If $\langle\{c, d\}\rangle$ is maximal such that $\{a, b\} \in \langle\{c, d\}\rangle \subseteq D$, then by the last part of the proof of Proposition 4.9 the pair $\{c, d\}$ is on an input-complete cycle in $\mathcal{A}^{(2)}$ defined by some word u. Let v be a word for which $\{c, d\}v = \{a, b\}$. If now $w \in PDW(\mathcal{A}/\theta)$, there is a $k \geq 0$ such that $w \leq_e u^k$, and then $(cu^k v, du^k v) \in \theta$. On the other hand, $\{cu^k v, du^k v\} = \{a, b\}$ and hence $(a, b) \in \theta$. □

7 Testing for piecewise directability

Our algorithm for testing whether a given automaton $\mathcal{A}$ is pw directable is based on Proposition 4.9; we just check whether the incomplete automaton obtained by removing the trap state $\Diamond$ from $\mathcal{A}^{(2)}$ has any input-complete cycles. Since no special properties of pair automata are used in the search for such cycles, the algorithm is presented for an arbitrary incomplete automaton $\mathcal{B} = (B, X)$ whose states are denoted by $\alpha, \beta, \gamma, \ldots$. The incompleteness of $\mathcal{B}$ means simply that the next state $\alpha x^{\mathcal{B}}$ may be undefined for some pairs $\alpha \in B,\ x \in X$. All definitions needed are easily adapted for such automata. In particular, for any $\alpha \in B$, $\langle\alpha\rangle$ is the set of states reachable from α.

First we define a Boolean $B \times B$ -matrix M so that for each pair $\alpha, \beta \in B$,

$$M(\alpha, \beta) = \begin{cases} 1 & \text{if } \beta \in \langle \alpha \rangle, \\ 0 & \text{if } \beta \notin \langle \alpha \rangle. \end{cases}$$

Hence, the 1-labeled entries in row α represent the subautomaton $\langle \alpha \rangle$. The subautomata $\langle \alpha \rangle$ could be found by using some well-known graph algorithm. However, the method described in Steps 1 and 2 also takes advantage of the fact that usually $|X| < |B|$.

For deciding whether some state is on an input-complete cycle in $\mathcal{B}$ it suffices to look for a state α such that $\alpha \in \langle \langle \alpha \rangle x \rangle$ for every $x \in X$. Indeed, $\alpha \in \langle \langle \alpha \rangle x \rangle$ if and only if there are words $u, v \in X^*$ such that $\alpha = \alpha u x v$. Since $\alpha \in \langle \langle \alpha \rangle x \rangle$ exactly in case there is a state $\beta \in \langle \alpha \rangle$ such that $\alpha \in \langle \beta x \rangle$, state α is on an input-complete cycle if and only if

$$\bigwedge_{x \in X} \bigvee_{M(\alpha,\beta)=1} M(\beta x, \alpha) = 1.$$

This is tested in Step 3.

In the algorithm $:=$ is the usual assignment operator, L is a list of states, λ denotes the empty list, $\alpha \to L$ means that state α is appended to L, and $\alpha \leftarrow L$ that α is taken from L. At any moment L contains the states which are already known to be in $\langle \alpha \rangle$, but for which the successors have not yet been tested for membership in $\langle \alpha \rangle$. At the end the variable P has the value YES or NO depending on whether or not an input-complete cycle exists.

Algorithm for checking the existence of an input-complete cycle in an incomplete automaton

Input: An incomplete automaton $\mathcal{B} = (B, X)$.
Output: $P =$ YES if $\mathcal{B}$ has an input-complete cycle, and $P =$ NO otherwise.

Step 1: For all $\alpha, \beta \in B$
    if $\alpha = \beta$ then $M(\alpha, \beta) = 1$ else $M(\alpha, \beta) = 0$.
Step 2: For every $\alpha \in B$ do
    $L := \lambda$;
    $\alpha \to L$;
    while $L \neq \lambda$ do
      $\beta \leftarrow L$;
      for every $x \in X$
        if βx is defined then
          if $M(\alpha, \beta x) = 0$ then $M(\alpha, \beta x) := 1$ and $\beta x \to L$;
Step 3: For every $\alpha \in B$ do
    if $\bigwedge_{x \in X} \bigvee_{M(\alpha,\beta)=1} M(\beta x, \alpha) = 1$ then $P :=$ YES and STOP.
Step 4: $P :=$ NO.

If $|B| = k$ and $|X| = m$, Step 1 takes $O(k^2)$ time, but Steps 2 and 3 both require $O(k^2 m)$ time. Hence, testing whether an n-state automaton $\mathcal{A}$ with m input letters is pw directable requires $O(n^4 m)$ steps when this algorithm is applied to the incomplete automaton obtained from $\mathcal{A}^{(2)}$ by omitting $\diamondsuit$.

The algorithm for testing the existence of an input-complete cycle can be used also for determining the least pwd congruence of an automaton $\mathcal{A}$. By Proposition 6.2 we have to determine the trap D of the automaton $\mathcal{S}(\mathcal{A}^{(2)})$, and from the proof of Proposition 4.9 we know that D consists of $\diamondsuit$ and all pairs of states of $\mathcal{A}$ which are in the subautomata generated by the pairs that are on an input-complete cycle in $\mathcal{A}^{(2)}$. Step 3 can easily be modified so that it forms the set C of all states which are on an input-complete cycle. Then D is a subautomaton of $\mathcal{A}^{(2)}$ generated by C. When $\mathcal{B}$ is the incomplete automaton obtained from $\mathcal{A}^{(2)}$ by omitting the state $\diamondsuit$, this set D yields the relation $\rho = \Delta_A \cup D$ which has the least pwd congruence $\mu_{\mathcal{A}}$ as its transitive closure (Proposition 6.2). The transitive closure can be computed in $O(n^3)$ time by Warshall's algorithm (cf. [1]), for example, and hence the congruence $\mu_{\mathcal{A}}$ is obtained in $O(n^6 + n^4 m)$ time.

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