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Effects of implementing quasiinvertion and optimization methods for nonlinear boundary inverse heat conduction problems

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Effects of implementing quasiinvertion and optimization methods for nonlinear boundary inverse heat conduction problems

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Abstract

In this paper we consider heat treatment by using the concentrated energy fluxes (laser). We estimate the solutions of the nonlinear boundary inverse heat conduction problems by using quasiinvertion and optimization methods. The results were obtained by using both exact and noisy input data. In order to increase result stability a smoothing filter was used for noisy input data. We analyse and compare results for different shapes of laser treatment such as: trapezoidal shape, sinusoidal shape and triangular one with low-angle anterior front.

Keywords: inverse problem, quasiinvertion method, optimization methods, laser treatment, nonlinear heat conduction problem

TUCS Laboratory
Turku Optimization Group

1. Introduction

Nowadays there exist a lot of construct materials with various physical parameters. In many cases it is possible to improve the physical parameters with the heat treatment and in most cases it is necessary to improve properties only at the work surface with a certain depth. These results can be achieved by using concentrated energy fluxes. The essence of the concentrated energy fluxes is that the relatively small amount of metal is treated with pulse energy flows of high intensity (with high speed), with simultaneous deformation and rapid cooling the metal by heat dissipation into the material depth.

At the moment there appear more and more requirements for the process of heating operation. Especially many requirements refer to the high-temperature processes, where it is important to take into account physical characteristics, depending on the temperature [1]. However, it is not possible to provide numerous experiments for obtaining required temperature field in a certain depth for improving its properties. The main obstacles are their cost and long period for every individual case. Hence there appears necessity for solving the inverse problems, where there is either known the heat flux or set the given temperature field at inside point. But firstly this type of problems are ill-posed and secondly the data of the temperature field are usually a discrete function. Furthermore the data have a limited amount of values, moreover these values are not exact (i.e. they have a small deviation from the exact value).

Many authors have paid attention to the solution process and obtained algorithms for inverse heat conduction problems such as Alifanov [2], Tikhonov [3], Cui [4], Samarskiy [5], Yang [6], Beck [7], Matsevity [8], Li [9], Panchenko [10], Shen [11], Terrola [12], Huang [13], et al. However, most of them have studied only linear problems.

There exist several principles for obtaining stable solutions of the inverse problems. One of them is the method using the principle of self-regularization (transfer from a parabolic equation to a hyperbolic one) [5], another uses Tikhonov's method [3], and the other one assumes usage of numerical optimization methods [2].

In this work we establish some new results solving the nonlinear inverse heat conduction problem by using two different approaches: first of them is the transformation of a parabolic equation to a hyperbolic one and the second one is the usage of numerical optimization methods. These approaches were analyzed and compared between each other with respect to both accuracy and time of obtaining the results.

2. Problem statement

Let us consider the inverse problem for one-dimensional case of the nonlinear heat conduction equation in the following way

$$\rho(T)c(T)\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right),$$

where $T(x,\tau)$ is the temperature field, $\rho(T)$, $c(T)$ and $\lambda(T)$ are material's density function, material's heat capacity function and material's heat conduction coefficient function, respectively. All these functions depend on temperature T. They are known functions and the initial temperature distribution is given. Our aim is to find the temperature field $T(x,\tau)$ in the area $D = \{0 \le x \le L;\ 0 \le \tau \le \tau_{\max}\}$ and the function of the heat-flux density $q(\tau)$ on the outer boundary. In the capacity of additional information we have temperature field at the internal point $x = M$ and the condition of the second type at the other boundary $(x = 0)$. Thus our problem can be represented in the way depicted in Figure 1.

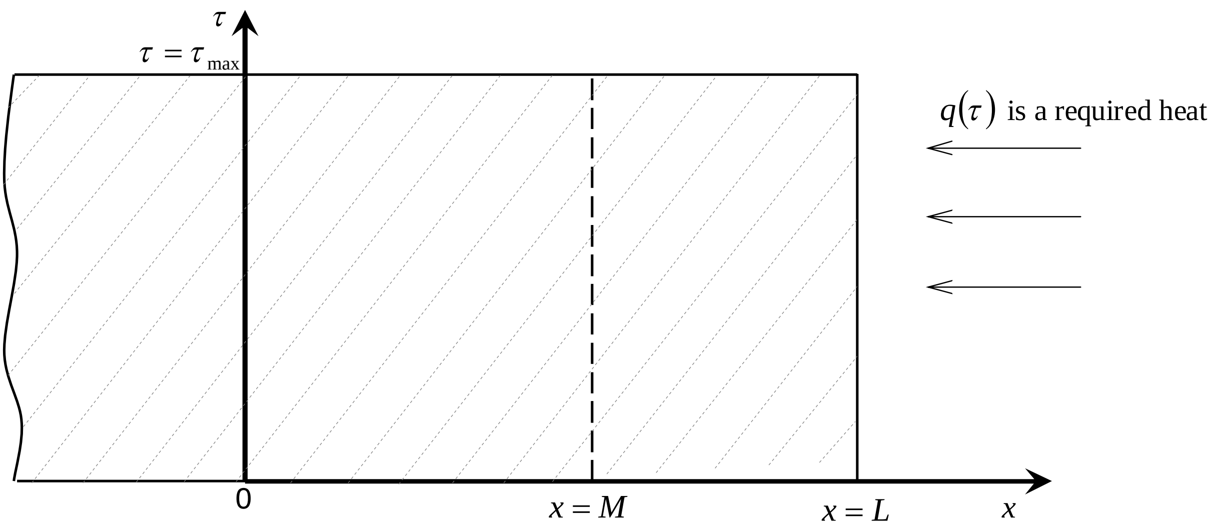

Figure 1. Problem statement

The problem can be written mathematically as follows

$$\rho(T)c(T)\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right),\ 0 \le x \le L,\ 0 \le \tau \le \tau_{\max}, \tag{1}$$

$$T(x,0) = \varphi(x),\ 0 \le x \le L, \tag{2}$$

$$T(M,\tau)=\xi(\tau),\ 0\le\tau\le\tau_{\max}, \tag{3}$$

$$-\lambda(T)\frac{\partial T(x,\tau)}{\partial x}\bigg|_{x=0}=q^{*}(\tau),\ 0\le\tau\le\tau_{\max}, \tag{4}$$

where x is a spatial coordinate, τ is a time variable**,** $q^{*}(\tau)$ is the heat flux at $x=0$, $\xi(\tau)$ is a temperature distribution at $x=M$, M is the inside point, L is a thickness and $\varphi(x)$ is an initial temperature distribution. If the heat flux (at $x=0$) $q^{*}(\tau)$ is equal to 0 it means the heat insulation (i.e. the neighboring nodes are equal). We assume that the initial and boundary conditions are harmonized between each other that is

$$T(L,0)=\varphi(L),\ \lambda(T)\cdot {}^{d\varphi(L)}\!/_{dx}=q^{*}(0).$$

The heat flux (at $x=L$) is an unknown value, thus if we can find the heat flux $q(\tau)$ we can also find the temperature field of the whole area from solving the direct problem for the heat conduction equation.

3. Solution of the inverse problem

The problem (1-4) was solved by using both the quasiinvertion method [5] and the parameter optimization method [2]. Let us next consider each case separately.

3.1. The quasiinvertion method

Before solving the inverse problem it is necessary to solve the direct problem in the area $\hat{D}=\{0\le x\le M;\ 0\le\tau\le\tau_{\max}\}$, with known boundaries and initial conditions. Then we can obtain the additional condition in the form of heat flux at the point $x=M$, thus the quasiinvertion method implies approximate solution of (1-4) from the perturbed equation

$$\rho(T)c(T)\frac{\partial T}{\partial\tau}-\alpha AA^{*}\frac{\partial T}{\partial x}=\frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right),$$

where $A={}^{\partial}\!/_{\partial\tau}$, A^{*} is a conjugated operator of A and α is a regularization parameter. Then the parabolic equation transformed to the hyperbolic equation is of the form

$$\rho(T)c(T)\frac{\partial T}{\partial\tau}-\alpha\frac{\partial^{3}T}{\partial\tau^{2}\partial x}=\frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right),\ M\le x\le L,\ 0\le\tau\le\tau_{\max},$$

$$T(x,0)=\varphi(x),\ M\le x\le L,$$

$$\frac{\partial T(x,0)}{\partial\tau}=0,\ M\le x\le L,$$

$$T(M,\tau)=\xi(\tau),\ 0\le\tau\le\tau_{\max},$$

$$-\lambda(T)\frac{\partial T(x,\tau)}{\partial x}\bigg|_{x=M}=q^{**}(\tau),\ 0\le\tau\le\tau_{\max},$$

where $q^{**}(\tau)$ is the heat flux at $x=M$ obtained from the solution of the direct problem (1-4) in the area $\hat{D}$. This problem was solved by using the finite difference method, and the terms of the heat conduction equation were approximated in the following way

$$\rho(T)c(T)\frac{\partial T}{\partial\tau}=\rho_i^{j-1}c_i^{j-1}\frac{T_i^j-T_i^{j-1}}{\Delta\tau},$$

$$\alpha\frac{\partial^3 T}{\partial\tau^2\partial x}=\alpha\frac{\frac{T_{i+1}^{j+1}-T_{i-1}^{j+1}}{2\Delta\tau}-2\cdot\frac{T_{i+1}^{j}-T_{i-1}^{j}}{2\Delta\tau}+\frac{T_{i+1}^{j-1}-T_{i-1}^{j-1}}{2\Delta\tau}}{\Delta x^2},$$

$$\frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right)=\frac{1}{\Delta x}\left[\lambda_{i+1/2}^{j-1}\frac{T_{i+1}^j-T_i^j}{\Delta x}-\lambda_{i-1/2}^{j-1}\frac{T_i^j-T_{i-1}^j}{\Delta x}\right],\ i=n,\ n+1,\ ...,\ N,\ \ j=1,\ 2,\ ...,\ K,$$

where T_i^j, ρ_i^j, c_i^j and λ_i^j are the discrete values of the temperature field function, material's density function, material's heat capacity function and material's heat conduction coefficient function, respectively, N and K are the numbers of spatial steps and time variables, respectively, n is the number of spatial step depending on finding the point M and $\Delta\tau$, Δx are the sizes of the spatial step and time variable, respectively.

The terms of the initial and boundary conditions were approximated as follows

$$T(x,0)=\varphi(x)=\varphi_i,\ \frac{\partial T(x,0)}{\partial\tau}=\frac{T_i^2-T_i^1}{\Delta\tau}=0,\ i=n,\ n+1,\ ...,\ N,$$

$$T(M,\tau)=\xi(\tau)=\xi^j,\ -\lambda(T)\frac{\partial T(x,\tau)}{\partial x}\bigg|_{x=M}=-\lambda^{j-1}\frac{T_2^j-T_1^j}{\Delta x},\ \ j=1,\ 2,\ ...,\ K.$$

Hence on each spatial step we have a system of equations which coefficients have only three nonzero diagonals and can be solved by using Thomas algorithm [14]. After solution we obtain the temperature field in the area $\overline{D}=\{M\le x\le L;\ 0\le\tau\le\tau_{\max}\}$. Hence we can find the heat flux on the outer boundary. The key idea is that we change the value of the regularization parameter α. Thus we look for the minimal residual between the exact value of the heat flux (temperature fields) and the obtained results in consequence of our calculations. The value of the residual is calculated by

$$\min_{\alpha>0}J(\alpha)=\int\left[T_{calc}(M,\tau,\alpha)-T_{input\ data}(M,\tau)\right]^2d\tau,$$

where $T_{input\ data}(M,\tau)$ is the given temperature distribution at the point $x=M$ and $T_{calc}(M,\tau,\alpha)$ is the calculated temperature values at the same point $x=M$.

In order to calculate this integral, replace $J(\alpha)$ by $J^l(\alpha)$ and we can represent a finite sum by

$$J^l(\alpha)=\Delta\tau\sum_{j=1}^{K}\left[T_{calc}(M,\tau_j,\alpha)-T_{input\ data}(M,\tau_j)\right]^2,$$

where l is an iteration number.

Typically the residual depends on value of α (regularization parameter) in the way depicted in Figure 2.

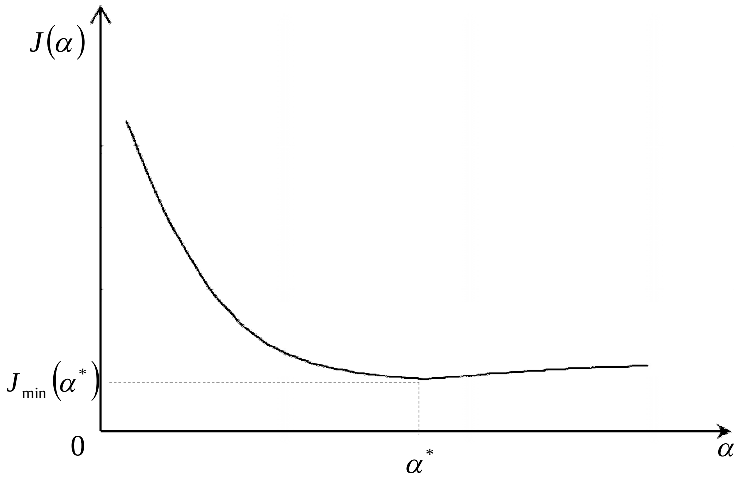

Figure 2. The residual $J(\alpha)$

Thus we can find the value of the regularization parameter α and obtain best values of the required heat flux.

3.2. The parameter optimization method

Another approach is to use optimization methods. We replace the continuous function of the heat flux by the discrete values on the chosen time mesh $q(\tau)=q_i,\ i=1,\ 2,\ \dots,\ K-1,\ K.$ Thus we can consider this problem as a problem of the parameter optimization [2]. For finding the required value of the heat flux on each time step we use the following minimization criterion

$$\min_{q^k_{initial\ approx}\geq 0} J^k\left(q^k_{initial\ approx}\right)=\left|T_{calc}\left(M,\tau_k,q^k_{initial\ approx}\right)-T_{input\ data}\left(M,\tau_k\right)\right|,$$

where J^k is the minimization criterion and $q^k_{initial\ approx}$ is an initial approximate value of the heat flux. The temperature $T_{calc}\left(M,\tau_k,q^k_{initial\ approx}\right)$ corresponds to the solution of the direct heat conduction problem

$$\rho(T)c(T)\frac{\partial T}{\partial \tau}=\frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right),\ 0\leq x\leq L;\ 0\leq\tau\leq\tau_{\max}, \tag{5}$$

$$T(x,0)=\varphi(x),\ 0\leq x\leq L, \tag{6}$$

$$\lambda(T)\left.\frac{\partial T(x,\tau)}{\partial x}\right|_{x=0}=0,\ 0\leq\tau\leq\tau_{\max}, \tag{7}$$

$$-\lambda(T)\left.\frac{\partial T(x,\tau)}{\partial x}\right|_{x=L}=q(\tau),\ 0\leq\tau\leq\tau_{\max}. \tag{8}$$

Thus the search of the required value of the heat flux is realized by using the iterative process within we change the value of the heat flux so that we obtain the following results

$$J^k\left(\left(q^k_{initial\ approx}\right)^{(l+1)}\right) < J^k\left(\left(q^k_{initial\ approx}\right)^{(l)}\right),$$

where l is the iteration number. For finding the objective function value $J^k\left(q^k_{initial\,approx}\right)$ on each time step we use method of golden section [15]. Thus, in this case for solving our problem (1-4) and using method of golden section it is necessary to provide the interval containing a minimum of the objective function $J(q)$.

4. Numerical experiments and results

In order to check the efficiency of our methods let us consider the following case. We have a one-dimensional steel plate with the following physical characteristics [16]

$$\rho(T) = 7842 - 0.097\cdot T,\ c(T) = 112.3\cdot T^{0.256},\ \lambda(T) = 5\cdot 10^{-5}\cdot T^2 - 0.135\cdot T + 116.4.$$

For the calculations we use the width of this plate as $1mm$, $\Delta x = 0.01mm$, $\tau_{\max} = 0.35\sec$, $\Delta\tau = 0.0035\sec$ and $T_{initial} = 293K$. Additionally we suppose that the temperature distribution is known at the point M being on the depth $0.25mm$. Shapes of the laser treatment are represented in Figure 3.

It is required to find the temperature distribution field and the heat flux on the surface. The results of the solution process are represented by using different input data. First we used exact data which was obtained by solving the direct problem with certain type of laser treatment. Second we used noisy data which was obtained by changing exact data on $\pm 1\%$ by using uniform distribution on each time step.

The results are presented in Figures 4-7 (for temperature field of the outer boundary) and 8-13 (for heat flux of the outer boundary) as well as in Tables 1-3 (for temperature field of the outer boundary)

- Tables 1-3 present the results (the temperature distribution on the outer boundary) of the solution by using exact input data with the quasiinversion method and parameter optimization. We can see that average relative deviation of received results is around 0.40% for quasiinversion method and only 0.13-0.20% (depending on the type of the flux) for the method of the parameter optimization.
- The next results were obtained by using noisy input data, that means we changed exact input data on $\pm 1\%$. In Figure 4 there are represented the results obtained with the quasiinversion method, and in Figure 6 the results obtained with method of the parameter optimization.

- In order to receive more stable results for noisy input data we were smoothing them by using the Savitzky–Golay smoothing filter [17-18]. Received results are represented in Figures 5 (the quasiinversion method) and 7 (the method of the parameter optimization).
- In Figures 8-13 are represented the received values for heat flux with quasiinversion method (Figures 8, 10, 12) and the parameter optimization method (Figures 9, 11, 13).

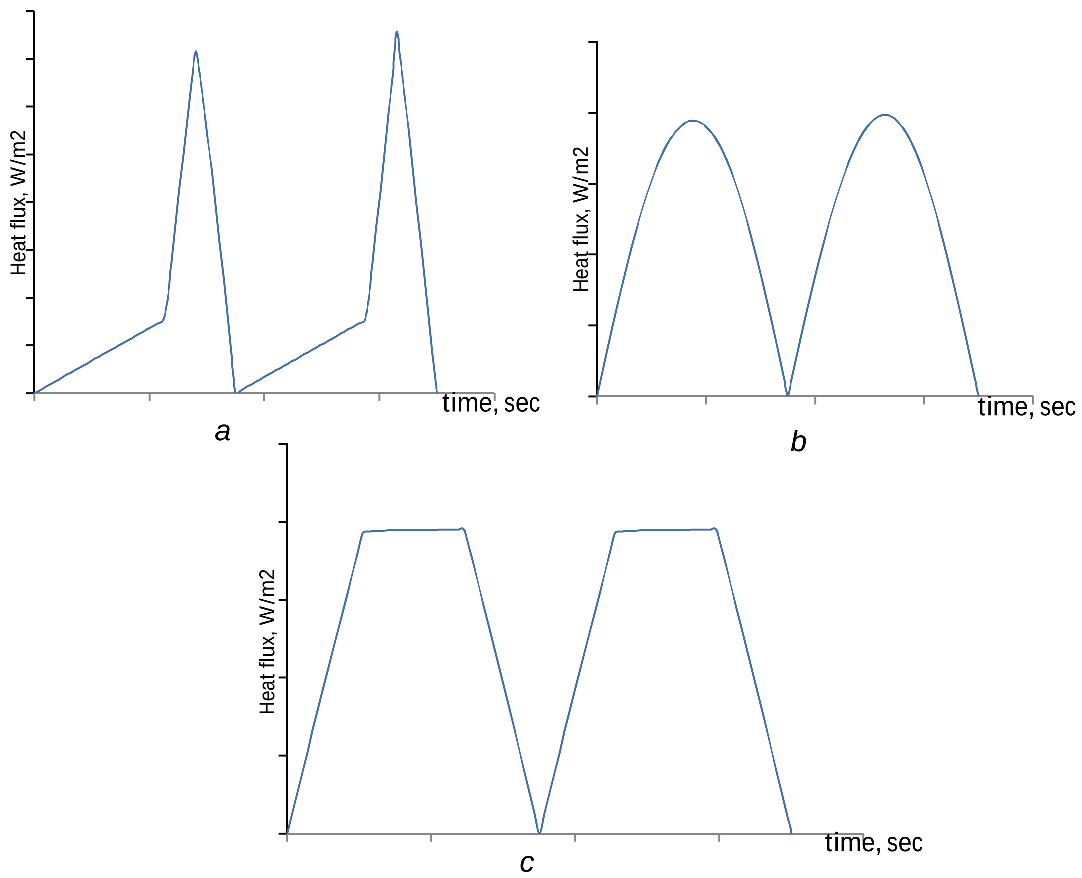

Figure 3. Shapes of the laser treatment: *a* - triangular shape with low-angle anterior front, *b* - sinusoidal shape, *c*- trapezoidal shape

Table 1. Solution with exact data (**triangular** shape of the laser treatment with low-angle anterior front)

Time, sec	Exact Solution	Inverse solution, quasiinversion method	relative deviation (%)	Inverse solution, parameter optimization	relative deviation (%)
0	293	293	0,00%	293	0,00%
0,0175	301	301	0,02%	299	0,40%
0,035	315	315	0,03%	314	0,34%
0,0525	335	334	0,05%	334	0,31%
0,07	360	360	0,07%	359	0,29%
0,0875	391	391	0,10%	390	0,26%
0,105	428	427	0,13%	427	0,24%
0,1225	521	519	0,23%	520	0,21%
0,14	785	772	1,73%	781	0,62%
0,1575	854	841	1,43%	852	0,13%
0,175	757	759	0,22%	756	0,14%
0,1925	729	729	0,01%	728	0,14%
0,21	738	737	0,03%	736	0,14%
0,2275	757	756	0,05%	756	0,14%
0,245	783	782	0,08%	782	0,13%
0,2625	814	813	0,12%	813	0,13%
0,28	850	849	0,16%	849	0,12%
0,2975	955	953	0,31%	954	0,12%
0,315	1268	1241	2,14%	1261	0,57%
0,3325	1318	1304	1,09%	1317	0,08%
0,35	1177	1185	0,71%	1176	0,09%
Average deviation			0,41%		0,22%

Table 2. Solution with exact data (**trapezoidal** shape of laser treatment)

Time, sec	Exact Solution	Inverse solution, quasiinversion method	relative deviation (%)	Inverse solution, parameter optimization	relative deviation (%)
0	293	293	0,00%	293	0,00%
0,0175	335	335	0,09%	334	0,35%
0,035	415	414	0,24%	414	0,27%
0,0525	530	527	0,57%	529	0,21%
0,07	633	629	0,69%	632	0,18%
0,0875	729	723	0,77%	728	0,16%
0,105	822	815	0,85%	821	0,15%
0,1225	915	906	1,02%	914	0,13%
0,14	947	941	0,67%	946	0,12%
0,1575	932	930	0,26%	931	0,12%
0,175	885	887	0,21%	884	0,12%
0,1925	920	919	0,09%	919	0,12%
0,21	1014	1010	0,33%	1013	0,11%
0,2275	1149	1140	0,80%	1148	0,10%
0,245	1256	1246	0,79%	1255	0,09%
0,2625	1344	1335	0,61%	1342	0,08%
0,28	1422	1417	0,35%	1421	0,08%
0,2975	1494	1491	0,17%	1493	0,07%
0,315	1502	1504	0,08%	1501	0,07%
0,3325	1466	1467	0,07%	1465	0,07%
0,35	1397	1402	0,36%	1396	0,07%
Average deviation			0,43%		0,13%

Table 3. Solution with exact data (**sinusoidal** shape of laser treatment)

Time, sec	Exact Solution	Inverse solution, quasiinversion method	relative deviation (%)	Inverse solution, parameter optimization	relative deviation (%)
0	293	293	0,00%	293	0,00%
0,0175	332	332	0,09%	331	0,36%
0,035	403	402	0,22%	401	0,27%
0,0525	493	492	0,39%	492	0,23%
0,07	596	592	0,58%	594	0,19%
0,0875	697	692	0,74%	696	0,17%
0,105	786	779	0,80%	785	0,15%
0,1225	849	843	0,72%	848	0,14%
0,14	879	874	0,50%	878	0,13%
0,1575	872	871	0,22%	871	0,13%
0,175	833	835	0,19%	832	0,13%
0,1925	866	866	0,09%	865	0,13%
0,21	947	944	0,29%	946	0,12%
0,2275	1050	1044	0,56%	1049	0,11%
0,245	1160	1151	0,78%	1159	0,10%
0,2625	1262	1252	0,79%	1261	0,09%
0,28	1340	1332	0,59%	1339	0,08%
0,2975	1386	1381	0,33%	1385	0,08%
0,315	1395	1393	0,15%	1394	0,08%
0,3325	1367	1367	0,04%	1366	0,08%
0,35	1306	1310	0,34%	1305	0,08%
Average deviation			0,40%		0,14%

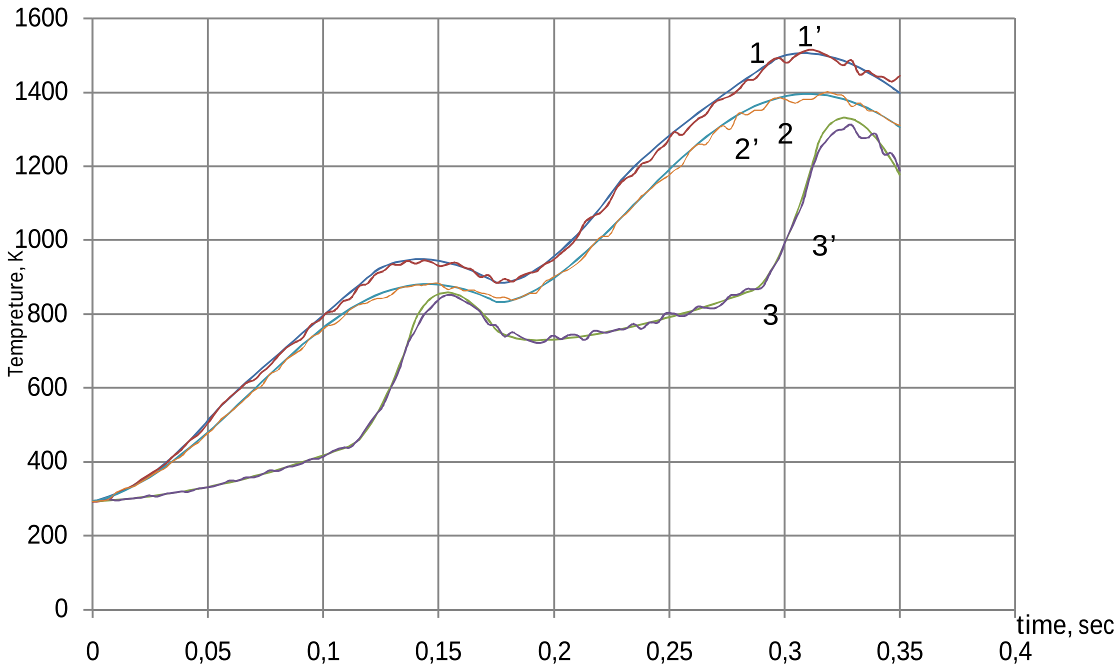

Figure 4. Temperature fields on the outer boundary. Curves 1, 2, 3 - exact data, 1’, 2’, 3’ - received results by quasiinversion method with noisy input data. Curves 1 and 1’ correspond trapezoidal shape, 2 and 2’ sinusoidal shape, 3 and 3’ - triangular shape

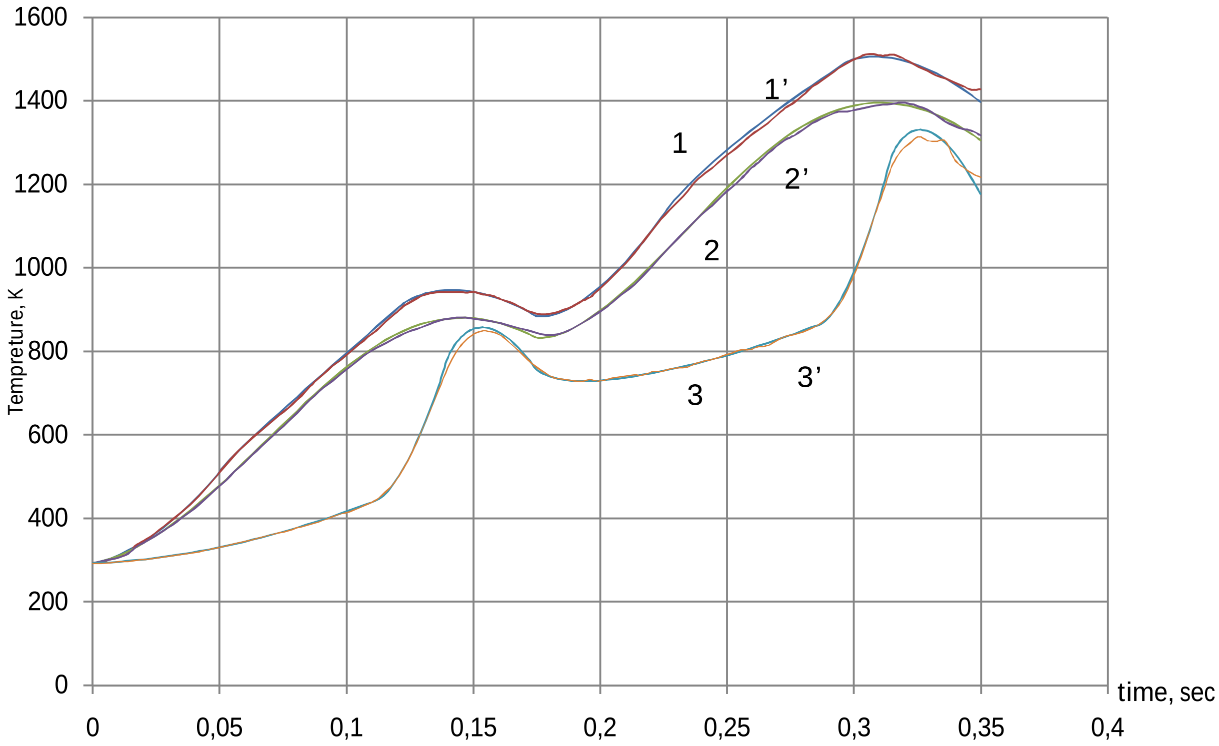

Figure 5. Quasiinversion method output using smoothed noisy input data

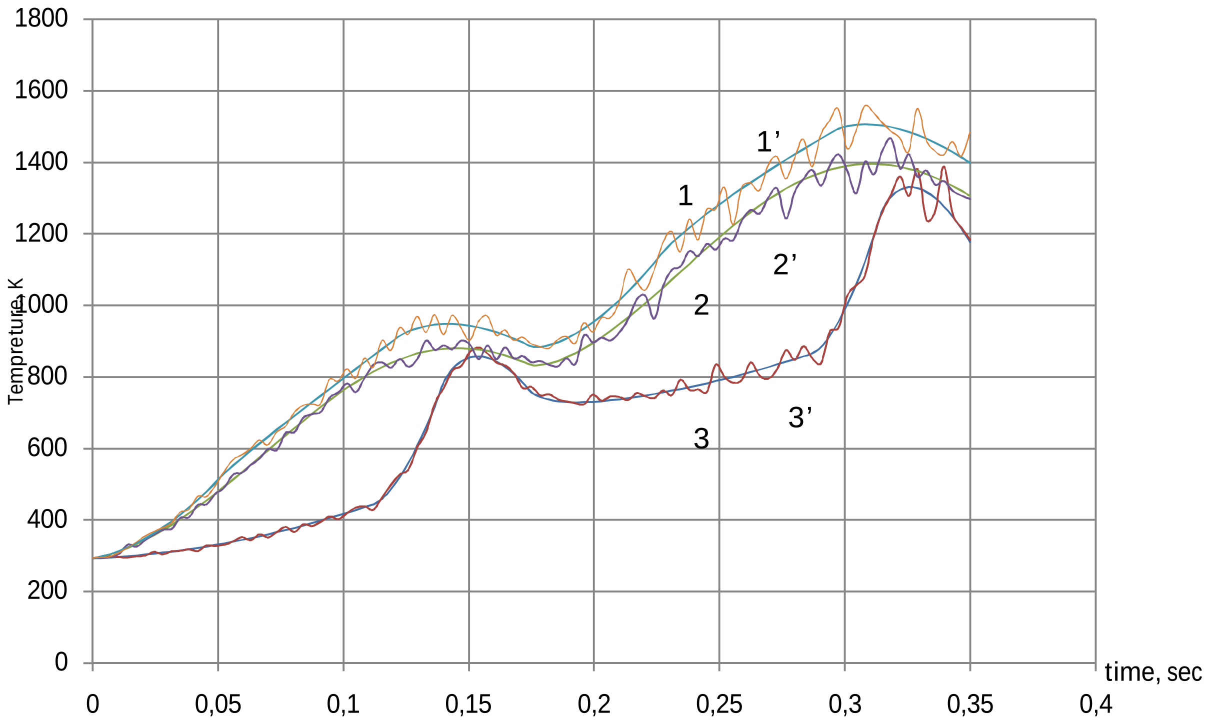

Figure 6. Parameter optimization method output using noisy input data

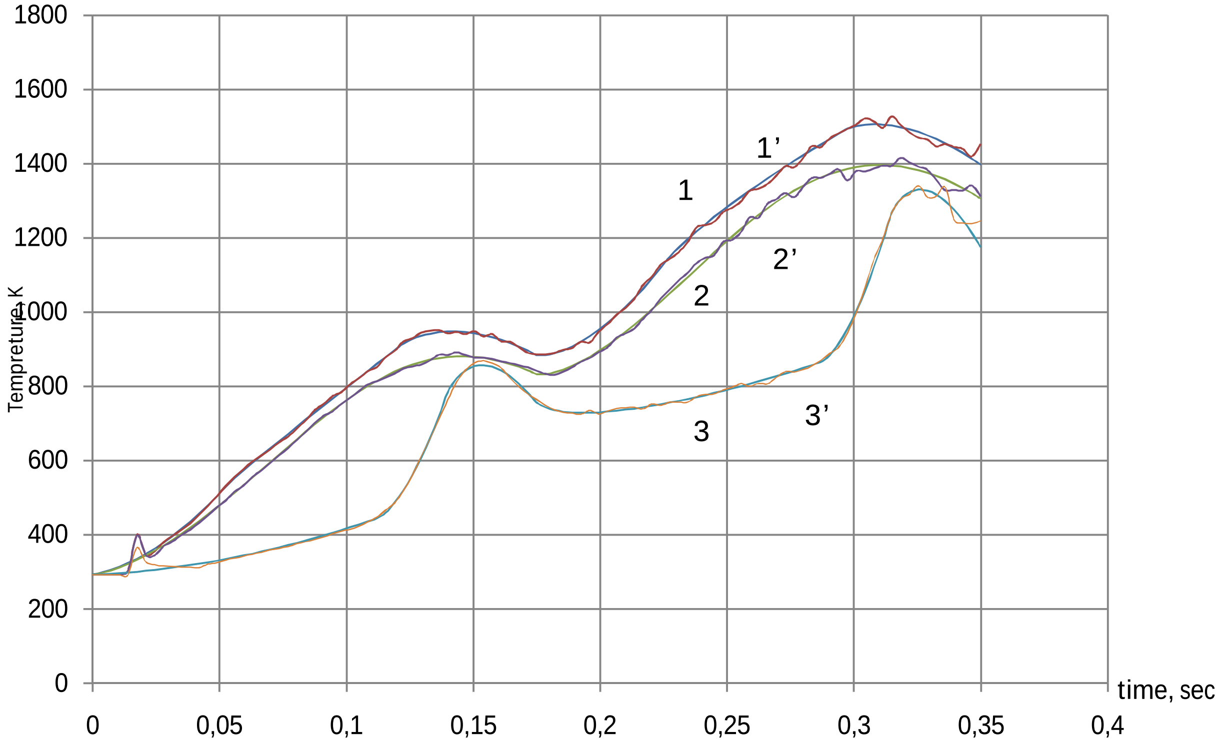

Figure 7. Parameter optimization method output using smoothed noisy input data

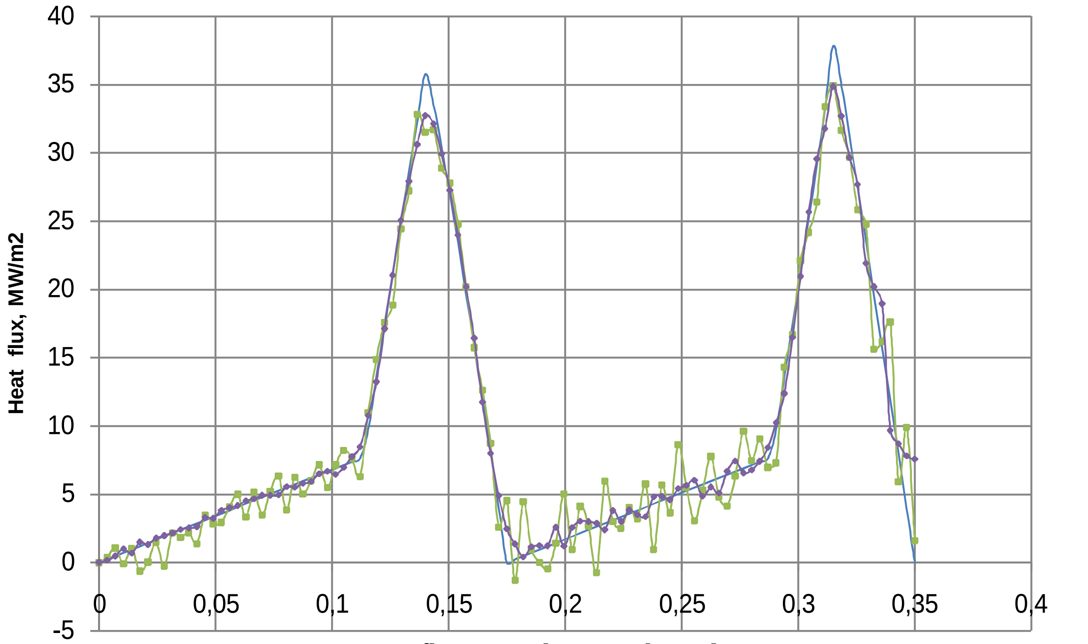

Figure 8. Heat fluxes on the outer boundary.
Quasiinversion method output using triangular shape

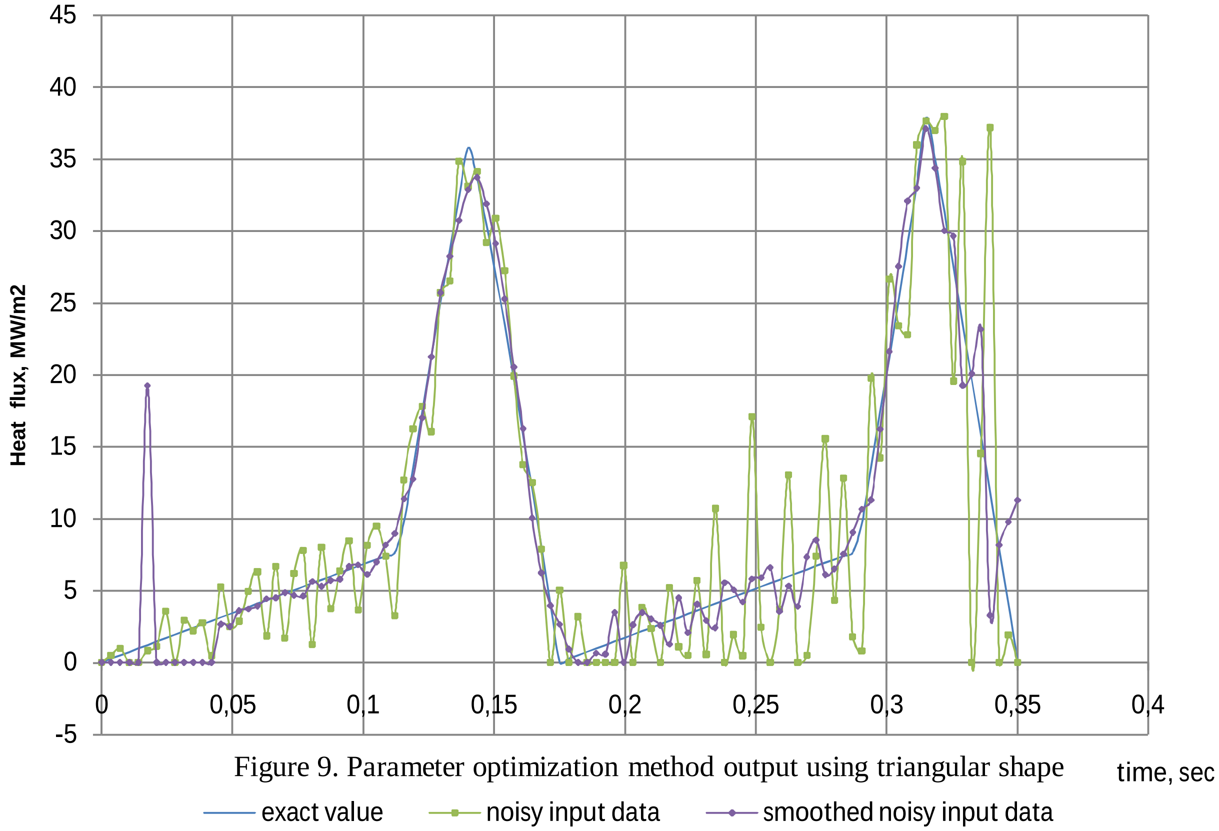

Figure 9. Parameter optimization method output using triangular shape

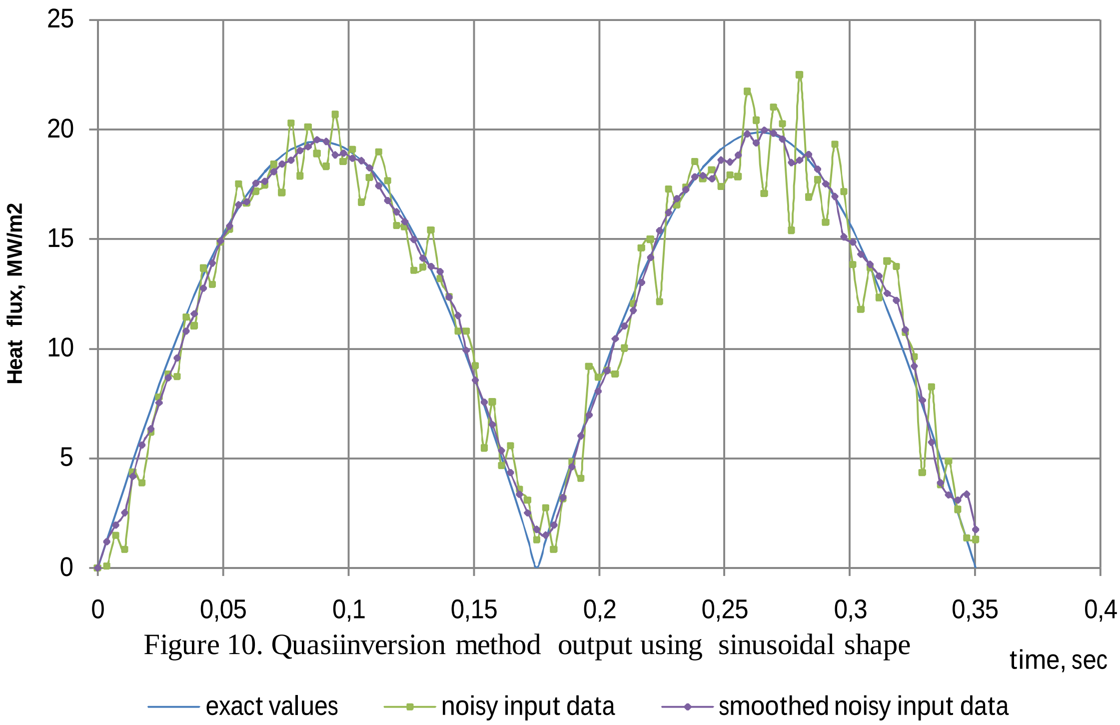

Figure 10. Quasiinversion method output using sinusoidal shape

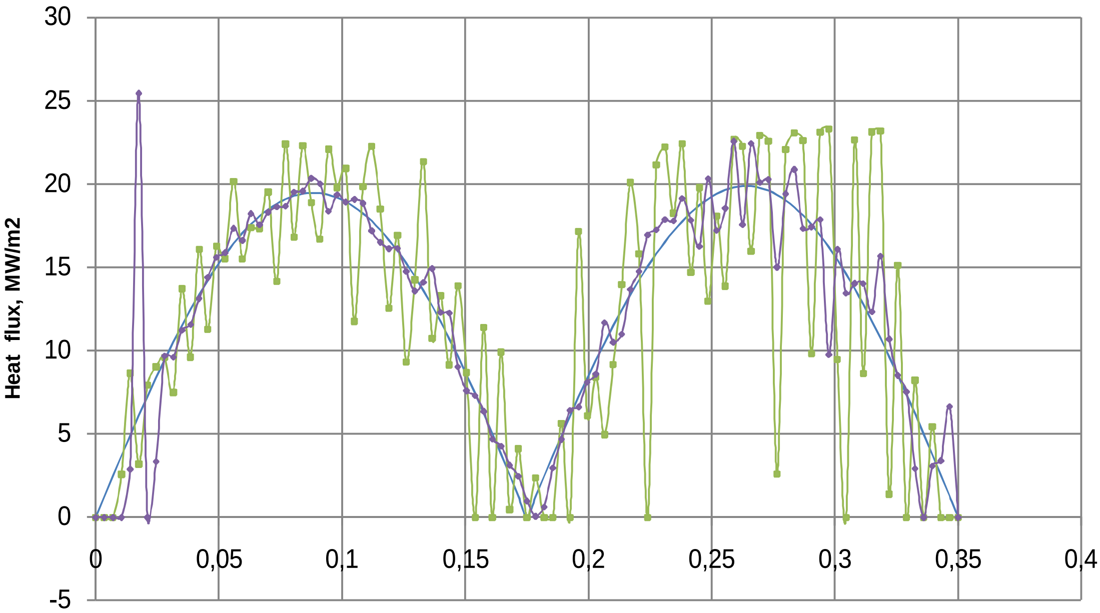

Figure 11. Parameter optimization method output using sinusoidal shape

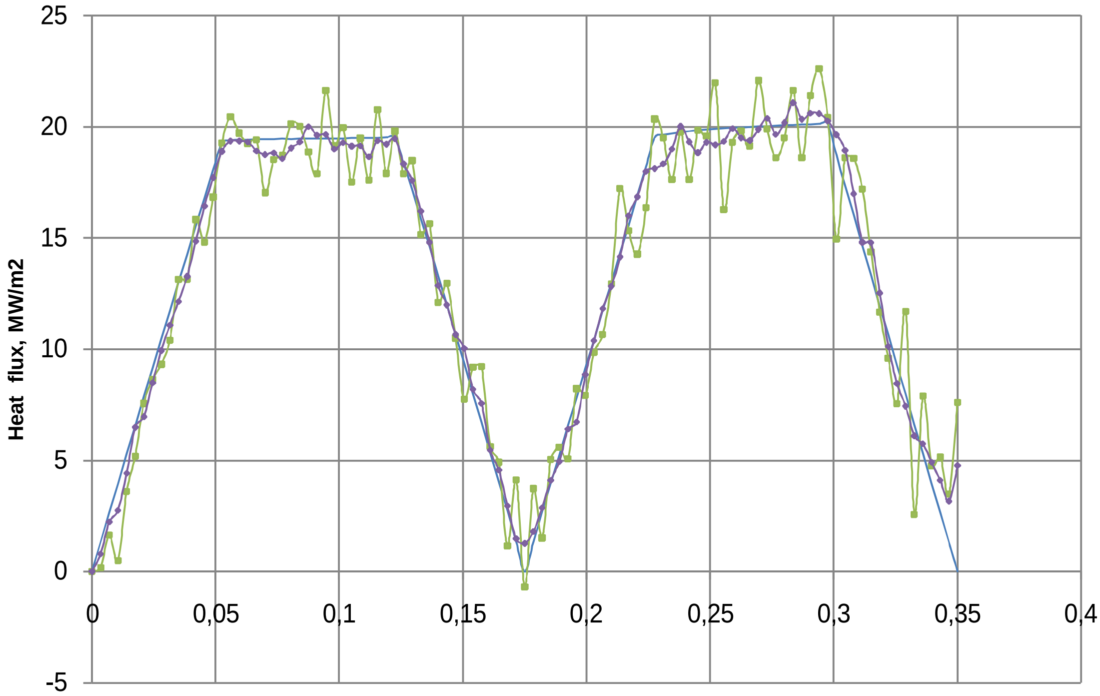

Figure 12. Quasiinversion method output using trapezoidal shape

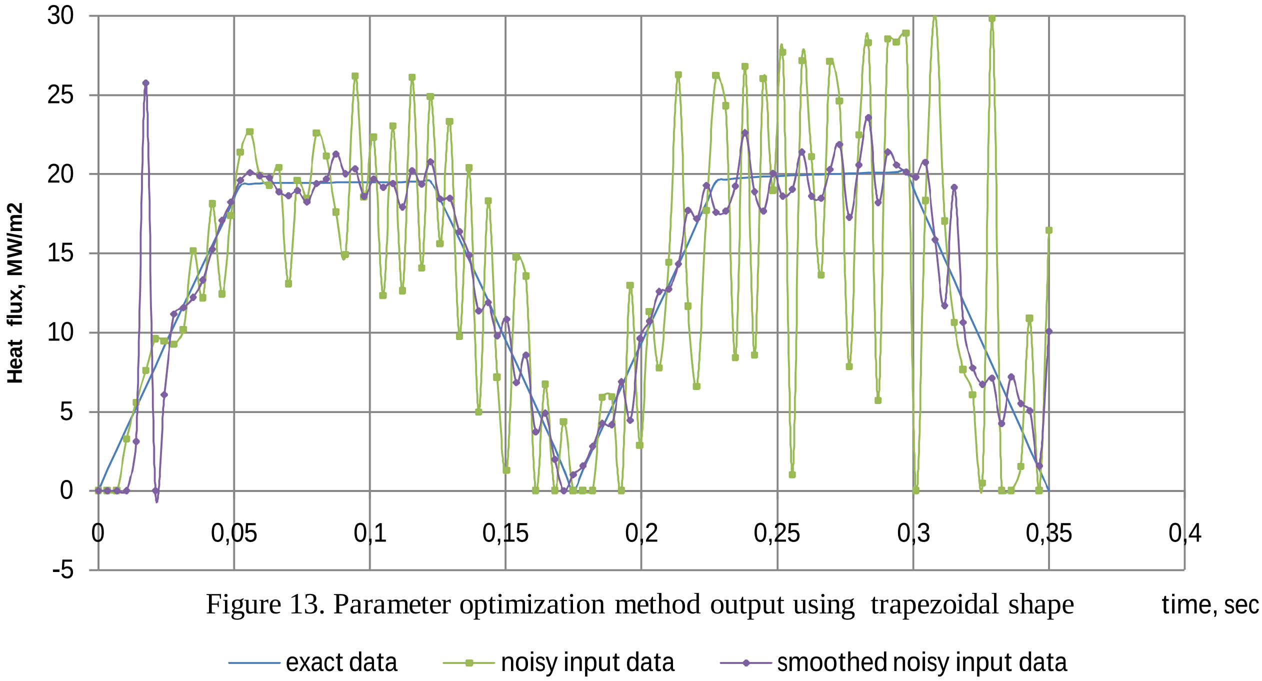

Figure 13. Parameter optimization method output using trapezoidal shape

5. Conclusions

By analyzing the obtained results we can come to the following observations and conclusions.

- Within results presented in Tables 1-3 one can see that both parameter optimization method and quasiinversion method permit to calculate perfect values of boundary conditions (both heat flux and temperature field) which are almost familiar to the exact solution. But according to the parameter optimization method, average relative deviation is several times less than the relative deviation obtained by the quasiinversion method. Thus, in case of having exact enough input data, the parameter optimization method is more reasonable to be used.
- Studying deviations on each time step in Tables 1-3 for the results obtained by the parameter optimization method we can notice in all cases (trapezoidal, sinusoidal and triangular shape), that there exist deviations of the first several values exceeding the average one. The explanation to this observation is that while starting process of heating, there is no visible heat in the internal points, which means that the temperature changes with light rate. According to this fact there appeared hardship with exact values. Also in the case with triangular shape there occurred deviation exceeding the average one at the top point of triangle shape, which can be explained by missing derivative at this point.
- Studying deviations on each time step in Tables 1-3 we can notice that when using quasiinversion method in all cases (trapezoidal, sinusoidal and triangular shape) there occurred two segments of values where there exist deviations

exceeding the average one. Thus there occurred a maximal increase of the speed temperature, causing in these segments a maximal change of the physical characteristics depending on the temperature. In order to calculate the physical characteristics on each time step we used those obtained by using the temperature values of previous time step. According to this fact there appears influence on the solution accuracy.

- Studying deviations in Tables 1-3 one can see that the shape of the laser treatment influences on the solution accuracy especially when using the parameter optimization method. This can be explained by the fact that we calculate the temperature values on each time step independently. Thus in the case of triangle shape (where there exist worst accuracy) one can see that the accuracy deterioration occurs during the process starts and on the top of triangle (reason to that was explained earlier).
- Within analyzed results obtained by noisy data and represented in Figures 4, 6, 8-13, one can see that quasiinversion method permits to determine good values for both heat flux and temperature field. The usage of the parameter optimization method gave us good enough values of the temperature field, while values of heat flux are unstable enough. Thus the quasiinversion method is more stable when there exist noisy input data.
- While analyzing the obtained results of the temperature fields (Figures 4-7) for both inverse method and for all shapes, it can be observed the increase in the deviation value. This is due to the nonlinearity of the problem. In our case there exists not only error in input data but also in calculated physical characteristics obtained by using noisy data.
- With noisy data we utilized Savitzky–Golay smoothing filter in order to stabilize the results. As a result of using this filter we obtained good enough values for the temperature field and the heat flux. These results are represented in Figures 5,7-11 for both methods.

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