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Abstract

Return words constitute a powerful tool for studying symbolic dynamical systems. They may be regarded as a discrete analogue of the first return map in dynamical systems. In this paper we investigate two abelian variants of the notion of return word, each of them gives rise to a new characterization of Sturmian words. We prove that a recurrent infinite word is Sturmian if and only if each of its factors has two or three abelian (or semi-abelian) returns. We study the structure of abelian returns in Sturmian words and give a characterization of those factors having exactly two abelian returns. Finally we discuss connections between abelian returns and periodicity in words.

Keywords: Sturmian word, return word, abelian equivalence

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Contents

1 Introduction

Let $w \in A^{\mathbb{N}}$ be an infinite word with values in a finite alphabet A. The *(factor) complexity function* $p : \mathbb{N} \to \mathbb{N}$ assigns to each n the number of distinct factors of w of length n. A fundamental result of Hedlund and Morse [15] states that a word w is ultimately periodic if and only if for some n the complexity $p(n) \leq n$. Infinite words of complexity $p(n) = n + 1$ are called *Sturmian words.* The most studied Sturmian word is the so-called Fibonacci word

$$0100101001001010010100100101001010\ldots$$

fixed by the morphism $0 \mapsto 01$ and $1 \mapsto 0$. In [16] Hedlund and Morse showed that each Sturmian word may be realized geometrically by an irrational rotation on the circle. More precisely, every Sturmian word is obtained by coding the symbolic orbit of a point x on the circle (of circumference one) under a rotation by an irrational angle α where the circle is partitioned into two complementary intervals, one of length α and the other of length $1 - \alpha$. And conversely each such coding gives rise to a Sturmian word. The irrational α is called the *slope* of the Sturmian word. An alternative characterization using continued fractions was given by Rauzy in [17] and [18], and later by Arnoux and Rauzy in [2]. Sturmian words admit various other types of characterizations of geometric and combinatorial nature (see for instance [6]). For example they are characterized by the following *balance* property: A word w is Sturmian if and only if w is a binary aperiodic (non-ultimately periodic) word and $||u|_i - |v|_i| \leq 1$ for all factors u and v of w of equal length, and for each letter i. Here $|u|_i$ denotes the number of occurrences of i in u.

In this paper we develop and study two abelian analogues of the notion of return word and apply it to characterize Sturmian words. Return words constitute a powerful tool for studying various problems in combinatorics on words, symbolic dynamical systems and number theory. Given a factor v of an infinite word w, by a *return word* to v (in w) we mean a factor u of w such that uv is a factor of w beginning and ending in v and having no other (internal) occurrence of v. In other words the set of all return words to v is the set of all distinct words beginning with an occurrence of v and ending just before the next occurrence of v. The notion of return words can be regarded as a discrete analogue of the first return map in dynamical systems. Many developments of the notion of return words have been given: For example, return words are used to characterize primitive substitutive sequences [8, 10]. Return words are used in studying the transcendence of Sturmian or morphic continued fractions [1]. Return words were fruitfully studied in the context of interval exchange transformations (see [21]). Words having a constant number of return words were considered in [5]. In [9] a generalization of the notion of balanced property for Sturmian words was introduced and the proof is based on return words. Return words are also used to characterize periodicity and Sturmian words. The following characterization was obtained by L. Vuillon in [20]:

Theorem 1.1. [20] *A binary recurrent infinite word w is Sturmian if and only if each factor u of w has two returns in w.*

In [12] the proofs were simplified and return words were studied in the context of episturmian words.

Two words are said to be *abelian equivalent* if they are permutations of each other. It is readily verified that this defines an equivalence relation on the set of all factors of an infinite word. Various abelian properties of words have been extensively investigated including abelian powers and their avoidance, abelian complexity and abelian periods [3, 4, 7, 13, 19]. Given a factor u of an infinite word w, let $n_1 < n_2 < n_3 < \dots$ be all integers n_i such that $w_{n_i} \dots w_{n_i+|u|-1}$ is abelian equivalent u. Then we call each $w_{n_i} \dots w_{n_i+|u|-1}$ a *semi-abelian return* to u. By an *abelian return* to u we mean an abelian class of $w_{n_i} \dots w_{n_{i+1}-1}$. We note that in both cases these definitions depend only on the abelian class of u. Each of these notions of abelian returns gives rise to a new characterization of Sturmian words:

Theorem 1.2. *A binary recurrent infinite word w is Sturmian if and only if each factor u of w has two or three abelian returns in w.*

Surprisingly, Sturmian words admit exactly the same characterization in terms of semi-abelian returns:

Theorem 1.3. *A binary recurrent infinite word w is Sturmian if and only if each factor u of w has two or three semi-abelian returns in w.*

Although the above characterizations of Sturmian words are similar to the one given in Theorem 1.1, our methods differ considerably from those used in [12, 20].

The paper is organized as follows: Section 2 is devoted to providing the necessary background and terminology relevant to the subsequent sections. In Section 3 we investigate connections between abelian returns and periodicity. In Section 4 we study the structure of abelian returns in Sturmian words. We prove that every factor of a Sturmian word has two or three abelian returns (Proposition 4.3) and moreover, a factor has two abelian returns if and only if it is singular (Theorem 4.8). In Section 5 we prove the sufficiency of the condition on the number of abelian returns for a word to be Sturmian (Corollary 5.7). In Section 6 we prove Theorem 1.3.

2 Preliminaries

2.1 Sturmian words and return words

We begin by presenting some background on Sturmian words and return words and terminology which will be used later in the paper.

Given a finite non-empty set Σ (called the alphabet), we denote by Σ^* and Σ^ω, respectively, the set of finite words and the set of (right) infinite words over the alphabet Σ. A word v is a *factor* (resp. a *prefix*, resp. a *suffix*) of a word w, if there exist words x, y such that $w = xvy$ (resp. $w = vy$, resp. $w = xv$). The set of factors of a finite or infinite word w is denoted by $F(w)$. Given a finite word $u = u_1u_2 \dots u_n$ with $n \geq 1$ and $u_i \in \Sigma$, we denote the length n of u by $|u|$. The empty word will be denoted by ε and we set $|\varepsilon| = 0$. For each $a \in \Sigma$, we let $|u|_a$ denote the number of occurrences of the letter a in u. An infinite word w is said to be *k-balanced* if and only if $||u|_a - |v|_a| \leq k$ for all factors u, v of w of equal length and all letters $a \in \Sigma$. If w is 1-balanced, then we say that w is *balanced*.

Two words u and v in Σ^* are said to be *abelian equivalent*, denoted $u \sim_{ab} v$, if and only if $|u|_a = |v|_a$ for all $a \in \Sigma$. It is easy to see that abelian equivalence is indeed an equivalence relation on Σ^*.

We say that a (finite or infinite) word w is *periodic*, if there exists T such that $w_{n+T} = w_n$ for every n. A right infinite word w is *ultimately periodic* if there exist T, n_0 such that $w_{n+T} = w_n$ for every $n \geq n_0$. A word w is *aperiodic*, if it is not (ultimately) periodic. A factor u of w is called *right special* if both ua and ub are factors of w for some pair of distinct letters $a, b \in \Sigma$. Similarly u is called *left special* if both au and bu are factors of w for some pair of distinct letters $a, b \in \Sigma$. The factor u is called *bispecial* if it is both right special and left special.

Sturmian words can be defined in many different ways. For example, they are infinite words having the smallest factor complexity among aperiodic words. By a celebrated result due to Hedlund and Morse [15], a word is ultimately periodic if and only if its factor complexity $p(n)$ is uniformly bounded. In particular, $p(n) <$

n for all n sufficiently large. Sturmian words are exactly words whose factor complexity $p(n) = n + 1$ for all $n \geq 0$. Thus, Sturmian words are those aperiodic words having the lowest complexity. Since $p(1) = 2$, it follows that Sturmian words are binary words. In what follows, we denote the letters of a Sturmian word by 0 and 1.

The condition $p(n) = n + 1$ implies the existence of exactly one right special and one left special factor of each length. The set of factors of a Sturmian word is closed under reversal, so for every length the right special factor is a reversed left special factor, and bispecial factors are palindromes. Bispecial factors play a crucial role in Sturmian words. *Standard factors* of a Stumian word w are letters and factors of the form Bab, where $a \neq b \in \{0, 1\}$ and B is a bispecial factor of w. A factor of a Sturmian word is called *singular* if it is the only factor in its abelian class. It is well known that singular factors have the form aBa, where a is a letter and B a bispecial factor. We will also use the notion of *Christoffel word*. One of the ways to define Christoffel words is the following: they are factors of a Sturmian word of the form aBb and letters.

In [16] it is shown that each Sturmian word may be realized measure-theoretically by an irrational rotation on the circle. That is, every Sturmian word is obtained by coding the symbolic orbit of a point x on the circle (of circumference one) under a rotation by an irrational angle α, $0 < \alpha < 1$, where the circle is partitioned into two complementary intervals, one of length α and the other of length $1-\alpha$. And conversely each such coding gives rise to a Sturmian word. The quantity α gives the frequency of letter 1 in the Sturmian word defined by such rotation. Other widely used characterizations are via mechanical words, cutting sequences, Sturmian morphisms etc., see [6] for further detail.

Let $w = w_1 w_2 \dots$ be an infinite word. The word w is *recurrent* if each of its factors occurs infinitely many times in w. In this case, for $u \in F(w)$, let $n_1 < n_2 < \dots$ be all integers n_i such that $u = w_{n_i} \dots w_{n_i+|u|-1}$. Then the word $w_{n_i} \dots w_{n_{i+1}-1}$ is a *return word* (or briefly *return*) of u in w. An infinite word *has k returns*, if each of its factors has k returns. The following characterization of Sturmian words via return words was established in [20]: A word is Sturmian if and only if each of its factors has two returns (Theorem 1.1 in the Introduction).

Also there exists a simple characterization of periodicity via return words:

Proposition 2.1. [20] *A recurrent infinite word is ultimately periodic if and only if there exists a factor having exactly one return word.*

2.2 Abelian and semi-abelian returns

In this subsection we define the basic notions for the abelian case. In particular, we introduce two abelian versions of the notion of return word, abelian return and semi-abelian return.

For an infinite recurrent word w and for $u \in F(w)$, let $n_1 < n_2 < n_3 < \dots$ be all integers n_i such that $w_{n_i} \dots w_{n_i+|u|-1} \sim_{ab} u$. Then each $w_{n_i} \dots w_{n_i+|u|-1}$ is called a *semi-abelian return* to the abelian class of u. By an *abelian return* to the abelian class of u we mean an abelian class of $w_{n_i} \dots w_{n_{i+1}-1}$. So the number of abelian returns is the number of distinct abelian classes of semi-abelian returns. Hence for every factor u in an infinite word w the number of abelian returns to the abelian class of u is less or equal to the number of semi-abelian returns to the abelian class of u. For brevity in the further text we often say (semi-)abelian return to factor u meaning the abelian class of u. We will often denote abelian returns by an element from the abelian equivalence class, that is by a semi-abelian return from the class.

Example 2.2. *Consider the Thue-Morse word*

$$t = 0110100110010110\dots$$

fixed by the morphism μ: $\mu(0) = 01$, $\mu(1) = 10$. The abelian class of 01 consists of two words 01 and 10. Consider an occurrence of 01 starting at position i, i.e., $t_i = 0$, $t_{i+1} = 1$. It can be followed by either 0 or 10, i.e. we have either $t_{i+2} = 0$ or $t_{i+2} = 1$, $t_{i+3} = 0$. In the first case we have $t_{i+1}t_{i+2} = 10$, which is abelian equivalent to 01, and hence we have the semi-abelian return $t_i = 0$. In the second case $t_{i+1}t_{i+2} = 11$, which is not abelian equivalent to 01, so we consider the next factor $t_{i+2}t_{i+3} = 10 \sim_{ab} 01$, which gives the semi-abelian return $t_it_{i+1} = 01$. Symmetrically, 10 gives semi-abelian returns 1 and 10. So the abelian class of 01 has four semi-abelian returns: $\{0, 1, 01, 10\}$ and three abelian returns since $01 \sim_{ab} 10$.

For our considerations we will use the following definitions. We say that a letter a is *isolated* in a word $w \in \Sigma^\omega$, if aa is not a factor of w. A letter $a \in \Sigma$ *appears in w in a block of length $k > 0$*, if a word ba^kc is factor of w for some letters $b \neq a$, $c \neq a$.

In this paper we establish a new characterization of Sturmian words analogous to Theorem 1.1. Namely, we prove that a recurrent infinite word is Sturmian if and only if each of its factors has two or three abelian returns (see Theorem 1.2 in the Introduction). On the other hand, contrary to property of being Sturmian, abelian returns do not give a simple characterization of periodicity analogous to Proposition 2.1. In terms of semi-abelian returns Sturmian words have exactly the same characterization as in terms of abelian returns (see Theorem 1.3 in Introduction).

3 Abelian returns and periodicity

In this section we discuss relations between periodicity and numbers of abelian and semi-abelian returns. We begin by proving a simple sufficient condition for periodicity:

Lemma 3.1. *Let $|\Sigma| = k$. If each factor of a recurrent infinite word over the alphabet Σ has at most k abelian returns, then the word is periodic.*

Proof. Let w be a recurrent word over a k-letter alphabet, and let v be a factor of w containing all letters from the alphabet. Consider two occurrences of v in w, say in positions m and n (with $m < n$). Then the abelian class of $w_m \dots w_{n-1}$ has all letters as abelian returns, and hence no more, because every factor of w must have at most k abelian returns. Thus w is periodic with period $n - m$. □

Remark. Actually, this proves something stronger: Let w be any aperiodic word over an alphabet Σ, $|\Sigma| = k$, and let u be any factor of w containing k distinct letters, and let vu be any factor of w distinct from u beginning in u. Then the abelian class of v must have at least k abelian returns. It follows that if a word is not periodic, then for every positive integer N there exists an abelian factor of length $> N$ having at least $k + 1$ abelian returns. In other words, the value $k + 1$ must be assumed infinitely often.

Remark. Notice that the condition given by Lemma 3.1 is not necessary for periodicity. It is not difficult to construct a periodic word such that some of its factors have more than k abelian returns.

Notice also that a characterization of periodicity similar to Proposition 2.1 in terms of abelian returns does not exist. Moreover, in the case of abelian returns it does not hold in both directions. Consider an infinite aperiodic word of the form $\{110010, 110100\}^\omega$. It is easy to see that the factor 11 has one abelian return $110010 \sim_{ab} 110100$. So, the existence of a factor having one abelian return does not guarantee periodicity. The converse is not true as well: there exist periodic words such that each factor has at least two abelian returns. An example is given by the following word with period 24:

$$w = (001101001011001100110011)^\omega. \tag{1}$$

To check that every factor of this word has at least two abelian returns, one can check the factors up to the length 12. If we denote the period of w by u, then every factor v of length $12 < l \leq 24$ has the same abelian returns as abelian class of words of length $24 - l$ obtained from u by deleting v. For a factor of length longer than 24 its abelian returns coincide with abelian returns of part of this factor obtained by shortening it by u.

Now we continue with relations between semi-abelian returns and periodicity. In this connection semi-abelian returns show intermediate properties between normal and abelian returns. E. g., normal returns admit the characterization of periodicity given by Proposition 2.1, for abelian returns the proposition does not hold in both directions, and in the case of semi-abelian returns the proposition holds in one direction giving a sufficiency condition for periodicity:

Proposition 3.2. *If a recurrent infinite word has a factor with one semi-abelian return, then the word is periodic.*

Proof. It is readily verified that this unique semi-abelian return word gives the period. □

We note that this condition is not necessary for periodicity. One can take the same example (1) of a periodic word as for abelian returns. Since each of its factors has at least two abelian returns, it has at least two semi-abelian returns.

Lemma 3.1 holds also for semi-abelian returns (exactly the same proof works):

Lemma 3.3. *Let $|\Sigma| = k$. If each factor of a recurrent infinite word over the alphabet Σ has at most k semi-abelian returns, then the word is periodic.*

4 The structure of abelian returns of Sturmian words

In this section we prove the "only if" part of Theorem 1.2, and in addition we establish some properties concerning the structure of abelian returns of Sturmian words.

The following proposition follows directly from definitions and basic properties of Sturmian words:

Proposition 4.1. *Semi-abelian returns of factors of a Sturmian word are Christoffel words.*

Proof. Consider semi-abelian return to a factor v of length n starting at position i of a Sturmian word w. We should prove that its semi-abelian return is either a letter or of the form aBb, where $a \neq b$ are letters, B is a bispecial factor of w. If $w_i = w_{i+n}$, then the letter w_i is semi-abelian return. If $w_i = a$, $w_{i+n} = b$, $a \neq b$, then there exists $k \geq 0$, such that $w_{i+1} \dots w_{i+k} = w_{i+1+n} \dots w_{i+k+n}$, and $w_{i+k+1} \neq w_{i+k+1+n}$. Since w is balanced, we have that $w_{i+k+1} = b$, $w_{i+k+1+n} = a$. So, $w_{i+k+2} \dots w_{i+k+n+1} \sim_{ab} v$, and $w_i \dots w_{i+k+1} \sim_{ab} w_{i+n} \dots w_{i+k+n+1}$ is semi-abelian return to v. By definition the factor $w_{i+1} \dots w_{i+k} = w_{i+1+n} \dots w_{i+k+n}$ is bispecial. □

Corollary 4.2. *Fix $l \geq 2$. Then each factor u of a Sturmian word has at most one abelian return of length l.*

Now we proceed to the "only if" part of Theorem 1.2:

Proposition 4.3. *Each factor of a Sturmian word has two or three abelian returns.*

The proof of this proposition is based on the characterization of balanced words presented in [11]. We will need some notation from the paper.

Suppose $1 \leq p < q$ are positive integers such that $\gcd(p,q) = 1$. Let $\mathscr{W}_{p,q}$ denote the set of all words $w \in \{0,1\}^q$ with $|w|_1 = p$. If $w \in \mathscr{W}_{p,q}$ then the symbol 1 occurs with frequency p/q in w. Define the *shift* $\sigma : \{0,1\}^\omega \to \{0,1\}^\omega$ by $\sigma(w)_i = w_{i+1}$. Similarly define $\sigma : \{0,1\}^q \to \{0,1\}^q$ by $\sigma(w_0 \dots w_{q-1}) = w_1 \dots w_{q-1} w_0$.

Since $\gcd(p,q) = 1$, it follows that any element of $\mathscr{W}_{p,q}$ has the least period q under the shift map σ. We will write $w \sim w'$ if there exists $0 \leq k \leq q-1$ such that $w' = \sigma^k(w)$. In this case we say that w, w' are *cyclically conjugate*, or that w, w' are cyclic shifts of one another. The equivalence class $\{\sigma^i(w) : 0 \leq i < q\}$ of each $w \in \mathscr{W}_{p,q}$ contains exactly q elements. Let

$$\mathbb{W}_{p,q} = \mathscr{W}_{p,q} / \sim$$

denote the corresponding quotient. Elements of $\mathbb{W}_{p,q}$ are called orbits. It will usually be convenient to denote an equivalence class in $\mathbb{W}_{p,q}$ by one of its elements w.

Given an orbit $[w] \in \mathbb{W}_{p,q}$, let

$$w_{(0)} <_L w_{(1)} <_L \cdots <_L w_{(q-1)}$$

denote the lexicographic ordering of its elements. Define the lexicographic array $A[w]$ of the orbit $[w]$ to be the $q \times q$ matrix whose ith row is $w_{(i)}$. We will index this array by $0 \leq i,j \leq q-1$, so that $A[w] = (A[w]_{ij})_{i,j=0}^{q-1}$. For $0 \leq i,j \leq q-1$, let $w_{(i)}[j]$ denote the length-$(j+1)$ prefix of $w_{(i)}$; so the $w_{(i)}[j]$ are the length-$(j+1)$ factors of w, counted with multiplicity. For each j this induces the following lexicographic ordering:

$$w_{(0)}[j] \leq_L w_{(1)}[j] \leq_L \cdots \leq_L w_{(q-1)}[j].$$

Theorem 4.4. [11] *Suppose $w \in \{0,1\}^q$. The following are equivalent:*
(1) *w is a balanced word,*
(2) *$|w_{(i)}[j]|_1 \leq |w_{(i+1)}[j]|_1$ for all $0 \leq i \leq q-2$ and $0 \leq j \leq q-1$.*

The following proposition from [11] gives a very practical way of writing down the lexicographic array associated to a balanced word.

Proposition 4.5. [11] *Let $[w]$ be the unique balanced orbit in $\mathbb{W}_{p,q}$. Define $u \in \mathscr{W}_{p,q}$ by*

$$u = 0 \dots 0 \underbrace{1 \dots 1}_{p}$$

Then, for $0 \leq i,j \leq q-1$,
(1) *$A[w]_{ij} = (\sigma^{jp} u)_i$,*
(2) *The jth column of $A[w]$ is (the vector transpose of) the word $\sigma^{jp} u$*
(3) *$w_{(i)} = u_i (\sigma^p u)_i (\sigma^{2p} u)_i \dots (\sigma^{(q-1)p} u)_i$.*

Example 4.6. *Consider a balanced word* $w = 0101001 \in \mathscr{W}_{p,q}$*. The lexicographic ordering of* $[w]$ *is*

$$0010101 <_L 0100101 <_L 0101001 <_L 0101010 <_L 1001010 <_L 1010010 <_L 1010100,$$

so the corresponding lexicographic array is

$$A[w] = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

We now apply the above technique for studying abelian returns as follows:

Fix a Sturmian word s and a factor v. First notice that v cannot have only one abelian return, otherwise we immediately get a contradiction with the irrationality of letter frequencies in s. We consider a standard factor w of s of long enough length to contain v and all abelian returns to v. Let $|w| = q$, $|w|_1 = p$. Then all the conjugates of w are factors of s, they are pairwise distinct, and $\gcd(p, q) = 1$ (see, e. g. [14]). Without loss of generality we can assume that v is "poor" in 1-s, i.e., it contains fewer 1's than the unique other abelian class of the same length. Then if we consider in $A[w]$ the words $w_{(i)}[j]$, we have that there exists $n < q - 1$ such that $w_{(i)}[j] \sim_{ab} v$ for $0 \leq i \leq n$, and $w_{(i)}[j] \not\sim_{ab} v$ for $n < i \leq q - 1$. Note also that $A[w]_{im} = A[w]_{(i+q-p)(m+1)}$; from now on the indices are taken modulo q.

The lexicographic array allows us to find abelian returns to v as follows: For a word u denote by $u[m, l]$ the factor $u_m \dots u_l$. If for an i, $0 \leq i \leq n$, we have $w_{(i)}[k, k + j] \sim_{ab} v$, where v is as above and $k > 0$ is the minimal such length, then by definition $w_{(i)}[k - 1]$ is a semi-abelian return to v. Notice also that if $A[w]_{(i-1)k} = 1$ and $A[w]_{ik} = 0$, then $w_{(m)}[k, k + j] \sim_{ab} v$ for $m = i, \dots, i + n$. That is, we have exactly $n + 1$ words from the abelian class of v starting in every column, and these words are in consecutive $n + 1$ rows (the first and the last row are considered as consecutive).

Example 4.7. *Consider abelian returns to the abelian class of* 001 *in the Example 4.6.* $w_{(i)}[2] \sim_{ab} 001$ *for* $0 \leq i \leq 4$*;* $w_{(i)}[1, 3] \sim_{ab} 001$ *for* $i = 4, 5, 6, 0, 1$*,* $w_{(i)}[2, 4] \sim_{ab} 001$ *for* $i = 1, \dots, 5$*. So, the abelian returns are* $w_{(0)}[0] = w_{(1)}[0] = 0$*,* $w_{(4)}[0] = 1$*,* $w_{(2)}[1] = w_{(3)}[1] = 01$*.*

Proof of Proposition 4.3. Suppose that some factor v of length $j + 1$ has at least 4 abelian returns. Without loss of generality we may assume that v is poor in 1, and in the lexicographic array, rows $0 \dots n$ start with factors from the abelian class of v. By Corollary 4.2 there can be at most one abelian return of a fixed length

greater than 1 (length 1 will be considered separately), so in a lexicographic array we must have one of the following situations:

1) there exist $k_1 < k_2$ and $n_1 < n_2 < n$ such that $w_i[j]$ has semi-abelian returns of length k_1 for $i = 1, \ldots, n_1$, $w_i[j]$ has semi-abelian returns of length k_2 for $i = n_1 + 1, \ldots, n_2$, and $w_{n_2+1}[j]$ has semi-abelian returns of length greater than k_2;

2) symmetric case: there exist $k_1 < k_2$ and $n_1 < n_2 < n$ such that $w_i[j]$ has semi-abelian returns of length k_2 for $i = n_1 + 1, \ldots, n_2$, $w_i[j]$ has semi-abelian returns of length k_1 for $i = n_2 + 1, \ldots, n$, and $w_{n_1}[j]$ has semi-abelian returns of length greater than k_2.

We consider only case 1) as the proof of case 2) is similar. First, in case 1) one can notice that the words $w_{n_1}[k_1, k_1 + q]$ and $w_{n_2}[k_2, k_2 + q]$ coincide. So if we consider semi-abelian returns "to the left" of the words $w_{n_1}[k_1, k_1 + j]$ and $w_{n_2}[k_2, k_2 + j]$, they should be the same, but they are not: the first one is of length k_1, the second one is of length k_2.

It remains to consider the case when v has both letters as abelian returns. It can be seen directly from the lexicographic array, that the third and the last return is 01 (in this case after a word not from abelian class of v we will necessarily have a word from abelian class of v, i.e., the longest possible length of abelian return is 2). □

Theorem 4.8. *A factor of a Sturmian word has two abelian returns if and only if it is singular.*

Proof. The method of proof is similar to the proof of Proposition 4.3 and relies upon the characterization of balanced words from [11].

If a factor is singular, then it is the only word in its abelian class, so its semi-abelian returns coincide with usual returns. Since every factor of a Sturmian word has two returns [20], then a singular factor has two semi-abelian returns, and hence two abelian returns.

Now we will prove the converse, i.e., that if a factor v of a Sturmian word s of length $j + 1$ has two abelian returns, then it is singular.

As in the proof of Proposition 4.3, we consider a standard factor w of s of long enough length to contain v and all abelian returns to v, and denote $|w| = q$, $|w|_1 = p$. Without loss of generality we again assume that v is "poor" in 1-s, so that there exists $n < q - 1$ such that $w_{(i)}[j] \sim_{ab} v$ for $0 \leq i \leq n$, and $w_{(i)}[j] \not\sim_{ab} v$ for $n < i \leq q - 1$.

It is not difficult to see that two abelian returns are possible in one of the following cases:

Case 1) there exist $0 \leq m < n$, $0 < k_1, k_2 < q$ such that $w_{(i)}[k_1 - 1]$ is semi-abelian return for all $0 \leq i \leq m$, $w_{(i)}[k_2 - 1]$ is semi-abelian return for all $m + 1 \leq i \leq n$;

Case 2) there exist $0 \le m_1 < m_2 < n$, $0 < k_1 < k_2 < q$ such that $w_{(i)}[k_1 - 1]$ is semi-abelian return for all $0 \le i \le m_1$ and $m_2 + 1 \le i \le n$; $w_{(i)}[k_2 - 1]$ is semi-abelian return for all $m_1 + 1 \le i \le m_2$.

Case 1) In case 1) we will assume that $k_1 < k_2$, the proof in case $k_2 < k_1$ is symmetric. We will consider two subcases:

Case 1a) $A[w]_{mk_2} = 1$, $A[w]_{(m+1)k_2} = 0$. This means that $w_{(i)}[k_2, k_2 + j] \sim_{ab} v$ for $i = m+1, \ldots, m+n+1$, and $A[w]_{m(k_2-1)} = 0$, $A[w]_{(m+1)(k_2-1)} = 1$. So, the element $A[w]_{(m+1)k_2}$ is a left-upper element of a block of abelian class of v, and $A[w]_{m(k_2-1)}$ is a right-lower element of another such block. It is easy to see that the latter block starts in column k_1. Therefore, $|v| = j + 1 = k_2 - k_1 < k_2$.

In case 1a) we will prove that the abelian class of v consists of a single word, i.e., $w_{(i)}[j] = v$ for $i = 0, \ldots, n$. Suppose that $w_{(i)}[j] \ne w_{(i+1)}[j]$ for some $i \in \{0, \ldots, n-1\}$. Since the rows grow lexicogaphically, it means that there exists $0 \le l < j < k_2 - 1$ such that $A[w]_{il} = 0$, $A[w]_{(i+1)l} = 1$. Hence $A[w]_{i(l+1)} = 1$, $A[w]_{(i+1)(l+1)} = 0$, and so $w_{(i+1)}[l + 1, l + 1 + j] \sim_{ab} v$. If $m < i + 1 \le n$, then the word $w_{(i+1)}[j]$ has return $w_{(i+1)}[l]$, which is impossible, because it has return $w_{(i)}[k_2]$. Similarly we get that the case $0 \le i + 1 \le m$ and $l + 1 < k_1$ is impossible.

In case $0 \le i+1 \le m$ and $k_1 \le l+1 < k_2$ we get that the word $w_{(i+1)}[k_1, k_1 + j]$ has return $w_{(i+1)}[k_1, l]$ of length $l-k_1+1$. But in this case $w_{(t)}[l+1, l+1+j] \sim_{ab} v$ for $t = i+1, \ldots, i+1+n$. Contradiction with the condition that $w_{(t)}[k_2 - 1]$ is semi-abelian return to $w_{(t)}[j]$. So, the case $0 \le i + 1 \le m$ and $k_1 \le l + 1 < k_2$ is impossible. Hence $w_{(i)}[j] = w_{(i+1)}[j]$ for $i = 0, \ldots, n - 1$, i.e., the abelian class of v consists of a single word.

Case 1b) $A[w]_{mk_2} = 0$ or $A[w]_{(m+1)k_2} = 1$. This means that $w_{(m)}[k_2, k_2 + j] \sim_{ab} v$. Hence the word $w_{(n)}[j]$ has semi-abelian return $w_{(n)}[k_2]$ of length $k_2 + 1$, and the word $w_{(m)}[k_1, k_1 + j]$ has semi-abelian return $w_{(m)}[k_1, k_2]$ of length $k_2 - k_1 + 1$, so the returns are different. This is impossible since $w_{(n)} = w_{(m)}[k_1, k_1 + q - 1]$.

Case 2) In case 2) the fact that $w_{(i)}[k_1]$ is semi-abelian return for all $0 \le i \le m_1 - 1$ and $m_2 + 1 \le i \le n$ implies that $n > q/2$. So, $k_1 = 1$, i.e., we necessarily have return(s) of length 1. Since there are two abelian returns totally, we can have only one return of length 1, and this return is 0. It means that $A[w]_{i0} = 0$ for $0 \le i \le n$. Since $w_{(m_2)}[1, j + 1] \not\sim_{ab} v$ and $w_{(m_2+1)}[1, j + 1] \sim_{ab} v$, we have $A[w]_{m_2 1} = 1$, $A[w]_{(m_2+1)1} = 0$, and hence $A[w]_{m_2 0} = 0$, $A[w]_{(m_2+1)0} = 1$. We get a contradiction with $A[w]_{i0} = 0$ for $0 \le i \le n$.

So, the converse is proved, i.e., every factor of a Sturmian word having two abelian returns is singular. □

5 Proof of Theorem 1.2: the sufficiency

Here we prove the "if" part of Theorem 1.2, i.e., we establish the condition on the number of abelian returns forcing a word to be Sturmian, i.e., we prove that a binary recurrent word with each factor having two or three abelian returns is Sturmian.

Proposition 5.1. *If each factor of a binary recurrent infinite word has at most three abelian returns and at least two semi-abelian returns, then the word is balanced.*

Notice that we formulate and prove auxiliary lemmas and propositions in a bit stronger way than we need for sufficiency in Theorem 1.2: instead the condition "each factor has two or three abelian returns" we put a weaker condition "each factor has at most three abelian and at least two semi-abelian returns". Using this condition we will be able to prove the sufficiency in both Theorems 1.2 and 1.3: since both words with two or three abelian returns and words with two or three semi-abelian returns satisfy this condition, we solve two problems at once.

The proof of this proposition is rather technical, it is based on considering abelian returns to different possible factors of the infinite word and consecutively restricting the possible form of the word. Denote the binary word with at most three abelian returns by $w \in \{0,1\}^\omega$. In the rest of this section instead of abelian returns "to the left" we consider abelian returns "to the right": if vu is a factor having $v' \sim_{ab} v$ as its suffix, and vu does not contain as factors other words abelian equivalent to v besides suffix and prefix, then the abelian class of u is abelian return to the abelian class of v. It is easy to see that regardless of the definition, the set of abelian returns to each abelian factor is the same. We will refer to the word u as *right semi-abelian return* of the abelian class of v, so normal semi-abelian returns can be regarded as left semi-abelian returns. Right semi-abelian returns do not necessarily coincide with left semi-abelian returns, but their abelian classes also give the set of abelian returns. Though this does not make any essential difference, this modification of the definition is more convenient for our proof of this proposition.

We will make use of the following key lemma:

Lemma 5.2. *If each factor of a binary recurrent infinite word w has at most three abelian and at least two semi-abelian returns, then one of the letters is isolated.*

Proof. Considering abelian returns to letters, we get that every letter can appear in blocks of at most three different lengths. Denote these lengths for blocks of 0's by l_1, l_2, l_3, where $l_1 < l_2 < l_3$, for blocks of 1's by j_1, j_2, j_3, where $j_1 < j_2 < j_3$. Notice that a letter can appear in blocks of only two or one lengths, then the third length or the third and the second lengths are missing.

Consider right semi-abelian returns of the word 10^{l_1}: they are 1, $0^{l-l_1}1$ for $l = l_2, l_3$ (if 0 appears in blocks of corresponding lengths), $1^{j-1}0^{l_1}$ for $j = j_1 >$

$1, j_2, j_3$ (if 1 appears in blocks of corresponding lengths) and 0 for $j_1 = 1$. Some of these returns should be missing or abelian equivalent to others in order to have at most three abelian returns totally. So we have the following cases:

– j_2, j_3, l_3 are missing, i.e., $w \in \{0^{l_1}1^{j_1}, 0^{l_2}1^{j_1}\}^\omega$. In this case abelian returns are 1, $0^{l_2-l_1}1$, and $1^{j_1-1}0^{l_1}$ for $j_1 > 1$ or 0 for $j_1 = 1$.
– l_2, l_3, j_3 are missing, i.e., $w \in \{0^{l_1}1^{j_1}, 0^{l_1}1^{j_2}\}^\omega$. Abelian returns are 1, $1^{j_2-1}0^{l_1}$, and $1^{j_1-1}0^{l_1}$, if $j_1 > 1$, or 0, if $j_1 = 1$.
– j_2, j_3 are missing, $j_1 = 2$, $l_2 = 2l_1$ or $l_3 = 2l_1$, i.e., $w \in (\{0^{l_1}, 0^{2l_1}, 0^l\}1^{j_2})^\omega$. Abelian returns are 1, $0^{l_1}1$, $0^{l-l_1}1$.
– l_3, j_3 are missing, $l_2 = 2l_1$, $j_1 = 2$ or $j_2 = 2$, $w \in (\{0^{l_1}, 0^{2l_1}\}\{1^2, 1^j\})^\omega$. Abelian returns are 1, $0^{l_1}1$, $1^{j-1}0^{l_1}$ (if $j > 1$) or 0 (if $j = 1$).
– j_2, l_2, j_3, l_3 are missing, then $w = (0^{l_1}1^j)^\omega$ is periodic. This case is impossible since 0^{l_1} has only one semi-abelian return.

Notice that the first two cases are symmetric. Considering abelian returns to the word $1^{j_1}0$, we get symmetric cases (0 change places with 1, j_k change places with l_k, $k = 1, 2, 3$). Combining the cases obtained by considering abelian returns to 10^{l_1} with the cases obtained by considering abelian returns to $1^{j_1}0$, we finally get the following remaining cases (up to renaming letters):

1) j_2, j_3, l_3 are missing, i.e. w is of the form $w \in \{0^{l_1}1^{j_1}, 0^{l_2}1^{j_1}\}^\omega$.
2) l_3, j_3 are missing, $l_1 = 1$, $l_2 = 2$, $j_1 = 2$, $j_2 = 4$, i.e. $w \in (\{0, 0^2\}\{1^2, 1^4\})^\omega$.
3) l_3, j_3 are missing, $l_1 = 1$, $l_2 = 2$, $j_1 = 1$, $j_2 = 2$, i.e. $w \in (\{0, 0^2\}\{1, 1^2\})^\omega$.
4) l_3, j_3 are missing, $l_1 = 2$, $l_2 = 4$, $j_1 = 2$, $j_2 = 4$. i.e. $w \in (\{0^2, 0^4\}\{1^2, 1^4\})^\omega$.

Case 1): $w \in \{0^{l_1}1_1^j, 0^{l_2}1_1^j\}^\omega$.

In the first case we should prove that $j_1 = 1$. We omit the index 1 for brevity: $j = j_1$. Suppose that $j > 1$. Consider right abelian returns to the word 10^{l_2}. They are 1, $1^{j-1}(0^{l_1}1^j)^k0^{l_2}$ for all $k \geq 0$ such that the word $0^{l_2}1^j(0^{l_1}1^j)^k0^{l_2}$ is a factor of w. Therefore, we have at most two values of k (probably, including 0).

Right abelian returns to the word $1^j0^{l_1}1$ are 1, $(0^{l_2}1^j)^m0^{l_1}1$ for all $m \geq 0$ such that the word $10^{l_1}1^j(0^{l_2}1^j)^m0^{l_1}1$ is a factor of w. So, we have at most two values of m (probably, including 0).

Notice that we cannot have only one value of k and only one value of m simultaneously, since in this case we have periodic word $w = ((0^{l_1}1^j)^{k_1}(0^{l_2}1^j)^{m_1})^\omega$, and the word $(0^{l_2}1^{1^j})^{m_1-1}0^{l_2}$ has only one semi-abelian return.

Taking into account conditions for m and k, which we have just obtained from considering abelian returns to both 10^{l_2} and $1^j0^{l_1}1$, we find that there are two opportunities:

Case 1a) $w \in (\{(0^{l_1}1^j)^{k_1}, (0^{l_1}1^j)^{k_2}\}0^{l_2}1^j)^\omega$, $0 < k_1 < k_2$. The word $0^{l_2}1^j0^{l_1}1^{j-1}$ has returns 1, $0^{l_1}1$, $0^{l_2}(1^j0^{l_1})^{k-1}1$ for all k such that the word $0^{l_2}1^j(0^{l_1}1^j)^k0^{l_2}$ is a factor of w. To provide at most three abelian returns, w should admit only one value of k. In this case there is also only one value of m, so the case 1a) is impossible.

Case 1b) $w \in (0^{l_1}1^j, \{(0^{l_2}1^j)^{m_1}, (0^{l_2}1^j)^{m_2}\})^\omega$, $0 < m_1 < m_2$. The word $1^j0^{l_1}1^j0^{l_2}1$ has returns 1, 10^{l_2}, $10^{l_1}(1^j0^{l_2})^{m-1}$ for all m such that the word $10^{l_1}1^j(0^{l_2}1^j)^m0^{l_1}1$ is a factor of w. To provide at most three abelian returns, w should admit only one value of m. In this case there is also only one value of k, so the case 1b) is impossible.

Thus, in case 1) 1's are isolated.

In cases 2)–4) we need to consider words containing all four blocks, otherwise we get into conditions of case 1) in which we proved that 1-s are isolated. The proof is similar for the three cases, and is based on studying abelian returns of certain type. When we examine $w \in (\{0^{l_1}, 0^{l_2}\}, \{1^{j_1}, 1^{j_2}\})^\omega$, we consider abelian returns to the words $0^{l_1}1^{j_2}$ and $0^{l_2}1^{j_1}$, and with a technical case study obtain that if both words have at most three abelian returns, then w is periodic of a certain form, and then find its factor having one semi-abelian return.

Case 2): $w \in (\{0^2, 0^4\}\{1, 1^2\})^\omega$.

Consider abelian returns of the word 0^21^2. Factors of w from the abelian class of 0^21^2 are the following: 0^21^2, 1^20^2, 0110, 1001. Notice that each of these words is necessarily a factor of w. Consider right semi-abelian returns to each factor:

- 0^21^2, 01^20 have right semi-abelian return 0
- 1^20^2 has right semi-abelian returns of the form $\alpha_1 = (0^210^2)^{i_1}1$ and/or $\alpha_2 = (0^210^2)^{i_2}0^21^2$ for some $i_1, i_2 \geq 0$
- 10^21 has right semi-abelian returns of the form $\alpha_3 = (0^41)^{i_3}1$ and/or $\alpha_4 = (0^41)^{i_4}0^21$ for some $i_3, i_4 \geq 0$

We will also use abelian returns of the word 0^41:

- 0^41 could have right semi-abelian returns 0, returns of the forms $\alpha_1' = (10^21)^{j_1}0^2$ with $j_1 > 0$ and $\alpha_2' = (10^21)^{j_2}10^4$ for some $j_2 \geq 0$
- 0^310, 010^3 (not necessarily factors of w) have right semi-abelian return 0
- 0^210^2 could have right semi-abelian returns 0, returns of the forms $\alpha_3' = (1^20^2)^{j_3}0^2$ with $j_3 > 0$ and $\alpha_4' = (1^20^2)^{j_4}10^2$ for some $j_4 \geq 0$
- 10^4 has right semi-abelian return 1.

These are summarized in the table below:

abelian class	word	possible right semi-abelian returns
$0^2 1^2$	$0^2 1^2$, $01^2 0$	0
	$1^2 0^2$	$\alpha_1 = (0^2 10^2)^{i_1} 1$, $\alpha_2 = (0^2 10^2)^{i_2} 0^2 1^2$ for some $i_1, i_2 \geq 0$
	$10^2 1$	$\alpha_3 = (0^4 1)^{i_3} 1$, $\alpha_4 = (0^4 1)^{i_4} 0^2 1$ for some $i_3, i_4 \geq 0$
$0^4 1$	$0^4 1$	0, $\alpha'_1 = (10^2 1)^{j_1} 0^2$ with $j_1 > 0$, $\alpha'_2 = (10^2 1)^{j_2} 10^4$ for some $j_2 \geq 0$
	$0^3 10$, 010^3	0
	$0^2 10^2$	0, $\alpha'_3 = (1^2 0^2)^{j_3} 0^2$ with $j_3 > 0$, $\alpha'_4 = (1^2 0^2)^{j_4} 10^2$ for some $j_4 \geq 0$
	10^4	1

Notice that $\alpha_1 \sim_{ab} \alpha_3$ when $i_1 = i_3$, and $\alpha'_1 \sim_{ab} \alpha'_3$ when $j_1 = j_3$.

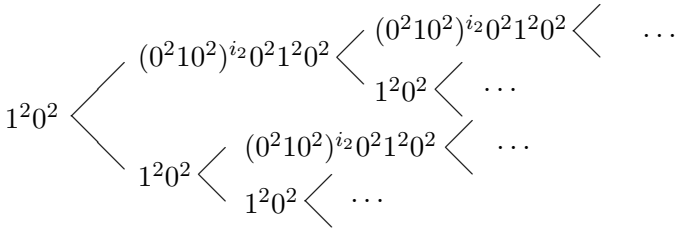
If factors from the abelian class of $0^2 1^2$ have only letters as abelian returns, then we obtain periodic word $w = (0^2 1^2)^\omega$, and this word does not contain all four blocks. So, a factor from the abelian class of $0^2 1^2$ should contain an abelian return of length longer than 1 (referred to as *long returns* in the further text), so we consider the four cases corresponding to returns α_1 through α_4.

Case 2a) let $1^2 0^2$ have a return α_1 with $i_1 > 0$. Then w contains a factor $u = 1^2 0^2 (0^2 10^2)^{i_1} 1$. Now consider right semi-abelian returns to the abelian class of $0^4 1$. One can find right semi-abelian returns 0 (in the factor $0^4 10$ of u) and 1 (in $10^4 1$). Since u has a prefix $1^2 0^4$, it means that there is a long right semi-abelian return ending in $1^2 0^4$, i.e., we have right semi-abelian return α'_2 or α'_3. A suffix $0^2 10^2 1$ of u implies that there is a long right semi-abelian return α'_3 or α'_4. So, the only possibility is that an abelian class of $0^4 1$ has abelian returns 0, 1 and $\alpha'_3 \sim_{ab} \alpha'_1$ with $j_1 = j_3 > 0$, and hence nothing else. The factor u has a suffix $0^2 10^2 1$, so the factor $0^2 10^2$ here has right semi-abelian return α'_3, and therefore u is continued in the unique way: $u' = 1^2 0^2 (0^2 10^2)^{i_1} (1^2 0^2)^{j_3} 0^2$. One can find here two right semi-abelian returns 0 and 1 to the abelian class of $0^2 1^2$, and we started with the first long right semi-abelian return α_1, so the three returns to $0^2 1^2$ are 0, 1 and $\alpha_1 \sim_{ab} \alpha_3$. The factor u' has a suffix $1^2 0^4$, so the factor $1^2 0^2$ here has right semi-abelian return α_1, therefore it is continued in the unique way: $u'' = 1^2 0^2 (0^2 10^2)^{i_1} (1^2 0^2)^{j_3} (0^2 10^2)^{i_1} 1$. Continuing this line of reasoning, we obtain a periodic word. One can find a factor having one semi-abelian return, e. g., $(1^2 0^2)^{j_3 - 1} 1^2$. Hence $1^2 0^2$ has no long right semi-abelian returns of the form α_1.

Case 2b) let $1^2 0^2$ have a return α_3 with $i_3 > 0$. Then w contains a factor $u = 10^2 1 (0^4 1)^{i_3} 1$. Now consider right semi-abelian returns to the abelian class of $0^4 1$. One can find right semi-abelian returns 0 (in the factor $0^4 10$ of u) and 1 (in $10^4 1$). Since u has a prefix $10^2 10^2$, it means that there is a long right semi-abelian return ending in $10^2 10^2$, i.e., we have right semi-abelian return α'_1 or α'_4. A suffix $0^4 1^2$ of u implies that there is a long right semi-abelian return α'_1 or α'_2.

So, the only possibility is that an abelian class of 10^4 has abelian returns 0, 1 and $\alpha'_1 \sim_{ab} \alpha'_3$ with $j_1 = j_3 > 0$. The factor u has a suffix $0^4 1^2$, so the factor $0^4 1$ here has right semi-abelian return α'_1, so u is continued in the unique way: $u' = 10^2 1(0^4 1)^{i_3}(10^2 1)^{j_1} 0^2$. This factor has a suffix $10^2 10^2$, so the factor $10^2 1$ here has right semi-abelian return α_3, and therefore it is continued in the unique way: $u'' = 10^2 1(0^4 1)^{i_3}(10^2 1)^{j_1}(0^4 1)^{i_3} 1$. Continuing this line of reasoning, we obtain a periodic word. One can find a factor having one semi-abelian return, e. g., $(0^4 1)^{i_3 - 1} 0^4$. Hence $10^2 1$ has no long right semi-abelian returns of the form α_3.

Case 2c) let $1^2 0^2$ have a return α_2 with $i_2 \geq 0$. Notice that if $1^2 0^2$ has only return α_2, then $w = (1^2 0^2 (0^2 10^2)^{i_2} 0^2)^\omega$, and w does not contain the block 0^2. We proved that there is no long right semi-returns of the forms α_1 and α_3, so the only possibility is that $1^2 0^2$ has two returns α_2 and 1, and $10^2 1$ always has return 1, otherwise this abelian class has more than 3 abelian returns. So, $1^2 0^2$ is followed by either $(0^2 10^2)^{i_2} 0^2 1^2$ or 1. In both cases we can determine several next letters: in the first case the next symbols are 00 (because w contains maximum two consecutive 1-s), in the second case the next symbols are 100 (since $10^2 1$ always has return 1, and 11 is always followed by 00). So, $1^2 0^2$ is followed by either $(0^2 10^2)^{i_2} 0^2 1^2 0^2$ or $1^2 0^2$. Both continuations have suffix $1^2 0^2$, which is followed by either 1 or α_2, etc:

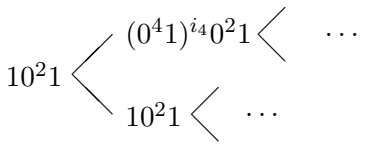


Thus $w \in \{(0^2 10^2)^{i_2} 0^2 1^2 0^2, 1^2 0^2\}^\omega$. We are interested in the case when all four blocks are contained in w, so we get $i_2 > 0$, otherwise w does not contain the block 1^1, and we get into case 1), which we proved is impossible.

So, w contains a factor $u = 1^2 0^2 (0^2 10^2)^{i_2} 0^2 1^2$ for some $i_2 > 0$. Now consider abelian returns to the abelian class of $0^4 1$. One can find right semi-abelian returns 0 (in the factor $0^4 10$ of u) and 1 (in $10^4 1$). Since u has a prefix $1^2 0^4$, it means that there is a long right semi-abelian return ending in $1^2 0^4$, i.e., we have right semi-abelian return α'_2 or α'_3. A suffix $0^4 1^2$ of u implies that there is a long right semi-abelian return α'_1 or α'_2. The only possibility is that an abelian class of $1^2 0^2$ has abelian returns 0, 1 and α'_2 with $j_2 \geq 0$, and nothing else. The set of abelian returns 0, 1 and $\alpha'_1 \sim_{ab} \alpha'_3$ is impossible since in this case the abelian class $1^2 0^2$ has other abelian returns than 0, 1, α_2. The factor u has a suffix $0^4 1^2$, so the factor $0^4 1$ here has right semi-abelian return α'_2, so u is continued in the unique way: $u = 1^2 0^2 (0^2 10^2)^{i_2} 0^2 1 (10^2 1)^{j_2} 0^2$. This factor has a suffix $10^2 10^2$, but we proved above that in the case 2c) the factor $10^2 1$ is always followed by 1, so we get a contradiction. Hence $1^2 0^2$ has no returns of the form α_2.

Case 2d) let $10^2 1$ have a return α_4 with $i_4 \geq 0$. Notice that if $10^2 1$ has only return

α_4, then $w = (0^2 1(0^4 1)^{i_4})^\omega$, and w does not contain the block 1^2. We proved that there is no long returns of the forms α_1, α_2 and α_3, so the only possibility is that $10^2 1$ has two returns α_4 and 1, and $1^2 0^2$ always has return 1. So, $10^2 1$ is followed by either $(0^4 1)^{i_4} 0^2 1$ or 1. In the second case we can determine several next letters to be 001 (because and 11 is always followed by 00, and $1^2 0^2$ always has return 1). So, $10^2 1$ is followed by either $(0^4 1)^{i_4} 0^2 1$ or $10^2 1$. Both continuations have suffix $10^2 1$, which is followed by either $(0^4 1)^{i_4} 0^2 1$ or 1:



Thus $w \in \{(0^4 1)^{i_4} 0^2 1, 10^2 1\}^\omega$. We are interested in the case when all four blocks are contained in w, so we get $i_4 > 0$, otherwise w does not contain the block 0^4.

Thus w contains a factor $u = 10^2 1(0^4 1)^{i_4} 0^2 1$. Now consider abelian returns to the abelian class of $0^4 1$. One can find right semi-abelian returns 0 (in a factor $0^2 10^3$ of u) and 1 (in $10^4 1$). Since u has a prefix $10^2 10^2$, we have a long right semi-abelian return ending in $10^2 10^2$, i.e., α'_1 or α'_4. A suffix $0^2 10^2 1$ of u implies that there is a long right semi-abelian return α'_3 or α'_4 with $j_4 \geq 0$. The only possibility is that an abelian class of $0^4 1$ has abelian returns 0, 1 and α'_4 with $j_4 \geq 0$. The set of abelian returns 0, 1 and $\alpha'_1 \sim_{ab} \alpha'_3$ is impossible since is this case the abelian class of $0^2 1^2$ has other abelian returns than 0, 1 and α_4. Considering the suffix $0^2 10^2 1$ of u, we get that the factor $0^2 10^2$ here has right semi-abelian return α'_4, so u is continued in the unique way: $u' = 10^2 1(0^4 1)^{i_4} 0^2 (1^2 0^2)^{j_4} 10^2$. The factor u' has a suffix $10^2 10^2$, so the factor $10^2 1$ here has right semi-abelian return α_4, so it is continued in the unique way: $u'' = 10^2 1(0^4 1)^{i_4} 0^2 (1^2 0^2)^{j_4} 1(0^4 1)^{i_4} 0^2 1$. Continuing this line of reasoning, we obtain a periodic word w. Its factor $(0^4 1)^{i_4 - 1} 0^4$ has only one semi-abelian return. Hence $10^2 1$ has no long returns α_4.

So, we are done with the case 2)

Case 3): $w \in (\{0, 0^2\}\{1, 1^2\})^\omega$.

Consider abelian returns for the word $0^2 1$. Factors of w from the abelian class of $0^2 1$ could be the following: 10^2, $0^2 1$, 010, and each of them necessarily appears in w.

- 10^2 has right semi-abelian return 1
- $0^2 1$ has right semi-abelian returns of the form $\alpha_1 = (101)^{i_1} 0$ and/or $\alpha_2 = (101)^{i_2} 10^2$ for some $i_1, i_2 \geq 0$.
- 010 has right semi-abelian returns of the form $\alpha_3 = (110)^{i_3} 0$ and/or $\alpha_4 = (110)^{i_4} 10$ for some $i_3, i_4 \geq 0$.

Symmetrically, we get possible abelian returns for $1^2 0$:

- 01^2 has right semi-abelian return 0

- 1^20 has right semi-abelian returns of the form $\alpha'_1 = (010)^{j_1}1$ and/or $\alpha'_2 = (010)^{j_2}01^2$ for some $j_1, j_2 \geq 0$.
- 101 has right semi-abelian returns of the form $\alpha'_3 = (001)^{j_3}1$ and/or $\alpha'_4 = (001)^{j_4}01$ for some $j_3, j_4 \geq 0$.

These are summarized in the table below:

abelian class	word	possible right semi-abelian returns
0^21	10^2	1
	0^21	$\alpha_1 = (101)^{i_1}0$, $\alpha_2 = (101)^{i_2}10^2$ for some $i_1, i_2 \geq 0$
	010	$\alpha_3 = (110)^{i_3}0$, $\alpha_4 = (110)^{i_4}10$ for some $i_3, i_4 \geq 0$
1^20	01^2	0
	1^20	$\alpha'_1 = (010)^{j_1}1$, $\alpha'_2 = (010)^{j_2}01^2$ for some $j_1, j_2 \geq 0$
	101	$\alpha'_3 = (001)^{j_3}1$, $\alpha'_4 = (001)^{j_4}01$ for some $j_3, j_4 \geq 0$

Notice that $\alpha_1 \sim_{ab} \alpha_3$ when $i_1 = i_3$, and $\alpha'_1 \sim_{ab} \alpha'_3$ when $j_1 = j_3$. In this case the lengths of blocks of 0's and 1's are the same, so we can use symmetry in the proofs.

If factors from the abelian class of 0^21 have only letters as abelian returns, then $w = (0^21)^\omega$, and this word does not contain all four blocks. So, a factor from the abelian class of 0^21 should contain a long abelian return (of length longer than 1), so we consider the four cases corresponding to long returns α_1–α_4.

Case 3a) let 0^21 have a return α_1 with $i_1 > 0$. Then w contains a factor $u = 0^21(101)^{i_1}0$. Now consider abelian returns to the abelian class of 1^20. One can find right semi-abelian returns 1 (in a factor 1101) and 0 (in 0110). Since u has a prefix 001^2, it means that there is a long right semi-abelian return ending in 0^21^2, i. e., α'_2 or α'_3. A suffix 1010 of u implies that there is a long right semi-abelian return α'_3 or α'_4. So, the only possibility is that an abelian class of 1^20 has abelian returns 0, 1 and $\alpha'_3 \sim_{ab} \alpha'_1$ with $j_1 = j_3 > 0$. Considering the suffix 1010 of u, we get that the factor 101 here has right semi-abelian return α'_3, so u is continued in the unique way: $u' = 0^21(101)^{i_1}(001)^{j_3}1$. One can find in u' abelian returns 0 and 1 to the abelian class of 0^21, and we started with the long return $\alpha_1 \sim_{ab} \alpha_3$. The factor u' has a suffix 0^21^2, so the factor 001 here has right semi-abelian return α_1, and hence u' is continued in the unique way: $u'' = 0^21(101)^{i_1}(001)^{j_3}(101)^{i_1}0$. Continuing this line of reasoning, we obtain a periodic word, in which the abelian class of $1(101)^{i_1}$ has one semi-abelian return. Hence 0^21 has no long returns α_1, and symmetrically 1^20 has no long returns α'_1.

Case 3b) let 010 have a return α_3 with $i_3 > 0$. Then w contains a factor $u = 010(110)^{i_3}0$. Now consider abelian returns to the abelian class of 1^20. One can find right semi-abelian returns 1 (in a factor 1011) and 0 (in 0110). Due to the prefix 0101 of u, there is a long right semi-abelian return ending in 0101, i.e., α'_1 or α'_4. The suffix 1100 of u implies that there is a long right semi-abelian return α'_1 or α'_2. We proved that there are no long returns of the form α'_1, so 1^20 has

right semi-abelian returns 0, 1, α'_4, α'_2. None of them are abelian equivalent, a contradiction. Hence $0^2 1$ has no returns of the form α_3, and symmetrically $1^2 0$ has no returns α'_3.

Case 3c) let $0^2 1$ have a return α_2. The abelian class of 001 always has abelian return 1. If $0^2 1$ has only return α_2, then $w = ((101)^{i_2} 10^2 1)^\omega$, and the factor 0^2 has only one abelian return. So, $0^2 1$ has also other abelian returns. Taking into account that there are no long returns of the forms α_1 and α_3, and α_2 is never abelian equivalent to α_4, we get that there should be abelian return 0. Hence, there is no abelian return α_4 and 010 is always followed by 0, $0^2 1$ is followed by either 0 or α_2. So, w contains a factor $u = 0^2 1(101)^{i_2} 10^2$, $i_2 \geq 0$. Now consider abelian returns to the abelian class of $1^2 0$. Since u has a prefix $0^2 1^2$, it means that there is a long right semi-abelian return ending in $0^2 1^2$, i.e., we have right semi-abelian return α'_2 or α'_3. A suffix $1^2 0^2$ of u implies that there is a long right semi-abelian return α'_1 or α'_2. We proved that we never have long return α'_1, so we have right semi-abelian return α'_2. Symmetrically to what we proved above, we get that 101 is always followed by 1, 110 is followed by either 1 or α'_2. So, the last occurrence of 110 in u is extended by α'_2, i.e. we get the unique extention of u: $u' = 0^2 1(101)^{i_2} 10(010)^{j_2} 01^2$. Considering the last occurrence of the factor 001 in u', we get that it should have right semi-abelian return α_2, i.e. we get the unique extention: $u'' = 0^2 1(101)^{i_2} 10(010)^{j_2} 01(101)^{i_2} 10^2$. Continuing this line of reasoning, we get a periodic word, in which the factor $0(010)^{j_2} 0$ has only one semi-abelian return. Hence we have no returns of the form α_2 and α'_2.

Case 3d) In the remaining case the word 010 has returns 0 and α_4 with $i_4 \geq 0$, and the word 101 has returns 1 and α'_4 with $j_4 \geq 0$. So, w contains a factor $u = 010(110)^{i_4} 10$. Considering the last occurrence of 101 in u, we see that it has return α'_4, so u is extended in the following way: $010(110)^{i_4} 1(001)^{j_4} 01$. The last occurrence of 010 in this word necessarily has right semi-abelian return α_4, so the word is extended uniquely as follows: $010(110)^{i_4} 1(001)^{j_4} 0(110)^{i_4} 10$. Continuing this line of reasoning, we get a periodic word. In this word $i_4 > 0$, otherwise we do not have occurrences of the block 1^2, and the abelian class of $(110)^{i_4} 1$ has only one semi-abelian return.

So, we are done with the case 3)

Case 4) $w \in (\{0^2, 0^4\}\{1^2, 1^4\})^\omega$

This case is considered in exactly the same way as the case 3) by considering abelian returns to $0^4 1^2$ and $0^2 1^4$. The only changes which should be done are doubling 0's and 1's everywhere except returns of length 1 (letters). □

Lemma 5.3. *If $w \in \{0^{l_1} 1, 0^{l_2} 1\}^\omega$, $0 < l_1 < l_2$ is a recurrent word such that each of its factors has at most three abelian returns and at least two semi-abelian returns, then $l_2 = l_1 + 1$.*

Proof. Suppose that $l_2 > l_1 + 1$. Consider abelian returns to the word 0^{l_1+1}: it has right abelian returns 0 and $1(0^{l_1} 1)^k 10^{l_1+1}$ for all $k \geq 0$ such that $0^l_2 1(0^{l_1} 1)^k 0^{l_2}$

is a factor of w, thus there could be at most two different values of k (probably, including 0). Consider abelian returns to the word $10^{l_1}10$: it has right abelian returns 0 and $(0^{l_2-1}10)^j0^{l_1-1}1$ for all $j \geq 0$ such that $10^{l_1}1(0^{l_2}1)^j0^{l_1}1$ is a factor of w, thus there could be at most two different values of j (probably, including 0). If we have only one value of k and one value of j simultaneously, then w is periodic, $w = ((0^{l_1}1)^{k_1}(0^{l_2}1)^{j_1})^\omega$. In this periodic word if $k_1 = 0$, then the factor 0_2^l has one semi-abelian return, if $k_1 > 0$, then the abelian class of $1(0^{l_1}1)^{k_1}$ has only one semi-abelian return. So, we have two cases:

Case I: $w \in (0^{l_2}1\{(0^{l_1}1)^{k_1}, (0^{l_1}1)^{k_2}\})^\omega$, $0 < k_1 < k_2$. In this case one can find four abelian returns to $0^{l_2}10^{l_1-1}$: 0, 10^{l_1-1}, $(10^{l_1})^{k_1-1}10^{l_2-1}$, $(10^{l_1})^{k_2-1}10^{l_2-1}$.

Case II: $w \in (0^{l_1}1\{(0^{l_2}1)^{j_1}, (0^{l_2}1)^{j_2}\})^\omega$, $0 < j_1 < j_2$. In this case one can find four abelian returns to $10^{l_2}10^{l_1}10$: 0, $0^{l_2-1}1$, $(0^{l_2-1}10)^{j_1-1}0^{l_1-1}1$, $(0^{l_2-1}10)^{j_2-1}0^{l_1-1}1$. □

The proofs of Lemma 5.2 and Lemma 5.3 immediately imply

Corollary 5.4. *If each factor of an infinite binary recurrent word w has at most three abelian returns and at least two semi-abelian returns, then $w \in \{0^{l_1}1, 0^{l_1+1}1\}^\omega$.*

Lemma 5.5. *If each factor of a recurrent infinite binary word w has at most three abelian returns and at least two semi-abelian returns, then w is 2-balanced.*

Proof. For a length n, consider abelian classes of factors of length n of w. Denote by A the abelian class of factors containing the smallest number of 1-s: $A = \{u \in F_n(w) : |u|_1 = \min_{v \in F_n(w)} |v|_1\}$. The next class we denote by B: $B = \{u \in F_n(w) : |u|_1 = \min_{v \in F_n(w)} |v|_1 + 1\}$, the next one by C. If w has only two abelian classes, then it is Sturmian, so we are interested in the case when w has at least three abelian classes. For a length n, we associate to a word w a word $\xi^{(n)}$ over the alphabet of abelian classes of w of length n as follows: for an abelian class M of words of length n, $\xi_k^{(n)} = M$ iff $w_k \dots w_{k+n-1} \in M$. In other words, $(\xi_k^{(n)})_{k \geq 0}$ is the sequence of abelian classes of consecutive factors of length n in w.

It is easy to see that $\xi^{(n)}$ contains the following sequence of classes: $CB^{j_1}A^{j_2}B$ for some $j_1, j_2 \geq 1$, i.e. for some i we have $\xi_i^{(n)} \dots \xi_{i+j_1+j_2+1}^{(n)} = CB^{j_1}A^{j_2}B$. Then we have

$$
\begin{aligned}
&w_i = 1, w_{i+n} = 0,\\
&w_k = w_{k+n} \text{ for } k = i+1, \dots, i+j_1-1,\\
&w_{i+j_1} = 1, w_{i+j_1+n} = 0,\\
&w_k = w_{k+n} \text{ for } k = i+j_1+1, \dots, i+j_1+j_2,\\
&w_{i+j_1+j_2} = 0, w_{i+j_1+j_2+n} = 1.
\end{aligned}
$$

I. e., $w_i \dots w_{i+j_1+j_2} = 1u1v0$, $w_{i+n} \dots w_{i+j_1+j_2+n} = 0u0v1$.

By Corollary 5.4 we have $w \in \{0^{l_1}1, 0^{l_1+1}1\}^\omega$, so $|u| \geq 2l_1 + 1$; u contains both letters 0 and 1 and has a suffix 0^{l_1}. It follows that $j_2 = 1$. So, the class B has the following 3 abelian returns: $0, 1, 01$. All the returns are of length at most 2, so if after an occurrence of B we have C, then the next class is B again, otherwise we will get a longer return. So there are no other classes than these. In addition, we proved that if for length n there are three abelian classes, then in $\xi^{(n)}$ letters A and C are isolated. □

Proof of Proposition 5.1. By Corollary 5.4 and Lemma 5.5, we have that w is 2-balanced and it is of the form $\{0^{l_1}1, 0^{l_1+1}1\}^\omega$ for some integer l_1. Suppose that w is not balanced. Then there exists n for which there exist three classes of abelian equivalence in $F_n(w)$; as above, denote these classes by A, B and C. Arguing as in the proof of Lemma 5.5, consider a sequence of classes BCB^jAB which we necessarily have in $\xi^{(n)}$ for some integer j, denote its starting position by $i - 1$. Corresponding factor in w is

$$\begin{aligned}
&w_{i-1} = 0, w_{i-1+n} = 1,\\
&w_i = 1, w_{i+n} = 0,\\
&w_k = w_{k+n} \text{ for } k = i+1, \dots i+j-1,\\
&w_{i+j} = 1, w_{i+j+n} = 0,\\
&w_{i+j+1} = 0, w_{i+j+1+n} = 1.
\end{aligned}$$

I. e., $w_i \dots w_{i+j+1} = 1u10$, $w_{i+n} \dots w_{i+j+1+n} = 0u01$. Remark that $u = w_{i+1} \dots w_{i+j}$ has prefix $0^{l_1}10$.

Now consider abelian returns to an abelian class $B0 = A1$ of length $n + 1$. The factor starting from the position $i + 1$ is of the form $B0$ so it belongs to this class, and has an abelian return 0. The word starting from the position $i + j$ is of the form $B0$ and has an abelian return 1. The word starting from the position $i + l_1 - 1$ belongs to this class, and has an abelian return 10. So we have at least three returns 0, 1 and 10. Now consider the occurrence of class $B0 = A1$ to the left from the position $i + 1$. One can see that the positions i and $i - 1$ are from the class $B1 = C0$, so the preceding occurrence of $B0 = A1$ has an abelian return of length greater than 2, which is a fourth return, though there should be at most three. So we cannot have more than two classes of abelian equivalence in a binary word having two or three abelian returns, i.e., such word should be balanced. Proposition 5.1 is proved. □

Lemma 5.6. *Let $w \in \{0, 1\}^\omega$ be a recurrent balanced word. Then w is either Sturmian or periodic. In the latter case there exists a (possibly empty) bispecial factor B of ω and a letter $a \in \{0, 1\}$ such that aBa is a factor of w having exactly one first return in w. Since aBa is the unique element in its abelian class, it follows that if w is periodic then w contains a factor having only one semi-abelian return.*

Proof. Since w is assumed balanced, w contains at most one right special factor for each length n. If w is not Sturmian, then w is ultimately periodic, and hence periodic since it is recurrent. From here on we shall assume that w is periodic. Thus w has only a finite number of right special factors. As w is recurrent, the longest right special factor of w is also a bispecial factor of w. Let $\varepsilon = B_0, B_1, \ldots, B_n$ denote the bispecial factors of w in order of increasing length. Thus B_n is also the longest right special factor of w. Set $B = B_{n-1}$. Then there exists a unique letter $a \in \{0, 1\}$ such that aB is a right special factor. In particular both aBa and bBa are factors of w where $a \neq b \in \{0, 1\}$. We claim that the only right special factor of w which begins in Ba is B_n. Clearly, B_n is a right special factor beginning in Ba (since Ba is left special and hence must coincide with the prefix of B_n of its same length). To see that no other right special factor of w begins in Ba, let R denote the shortest right special factor of w beginning in Ba. Then R is also left special and hence bispecial. It follows that $R = B_n$. Since B_n is also the longest right special factor of w the claim is established. Having established the claim, it follows that aBa has a unique first return in w. If not, there would exist a right special factor beginning in aBa. From the previous claim it would follow that aB_n is right special contradicting that B_n is the longest right special factor. □

We are now ready to prove the sufficiency condition:

Corollary 5.7. *If each factor of a binary recurrent infinite word has two or three abelian returns, then the word is Sturmian.*

Proof. Follows from Proposition 5.1 and Lemma 5.6. □

Corollary 5.8. *An aperiodic recurrent infinite word w is Sturmian if and only if each factor u of w has two or three abelian returns in w.*

Proof. Lemma 3.1 implies that an aperiodic word with 2 or 3 abelian returns must necessarily be binary. □

6 Proof of Theorem 1.3

In this section we prove the characterization of Sturmian words in terms of semi-abelian returns.

Proof of Theorem 1.3. We have that for every factor in an infinite word the number of its semi-abelian returns is not less than the number of abelian returns. So, Proposition 5.1 and Lemma 5.6 imply that if each factor of an infinite binary recurrent word has two or three semi-abelian returns, then the word is Sturmian.

Now we prove that each factor of a Sturmian word has at most three semi-abelian returns. Suppose that a factor v of a Sturmian word has more than three semi-abelian returns. By Proposition 4.3 this factor has at most three abelian returns, so there are at least two semi-abelian returns which are abelian equivalent.

Due to Proposition 4.1, semi-abelian returns to factors of Sturmian words are Christoffel words, i.e., letters or words of the form aBb, so if we have more than three semi-abelian returns to v, then there should be both returns $0B1$ and $1B0$.

In the case $|v| \geq |0B1|$ the return $0B1$ is given by a factor $0B1x1B0$ for some $x \in \{0,1\}^*$, where $0B1x$ is abelian equivalent to v. The return $1B0$ is given by a factor $1B0y0B1$ for some $y \in \{0,1\}^*$, where $1B0y$ is abelian equivalent to v. So, we have factors $1x1$ and $0y0$, where x and y are abelian equivalent, a contradiction with balance.

In the case $1 < |v| < |0B1|$ we have a factor z whose (intersecting) prefix and suffix are $0B1$ and $1B0$, resp., and another factor z' of the same length whose prefix and suffix are $1B0$ and $0B1$, resp. So B should have 1 and 0 at the same position.

If $|v| = 1$, i.e., v is a letter, it is easy to see that v has two semi-abelian returns.

Thus, two different semi-abelian returns of the same length greater than 1 are impossible. This concludes the proof. □

Similarly to Corollary 5.8, we get

Corollary 6.1. *An aperiodic recurrent infinite word w is Sturmian if and only if each factor u of w has two or three semi-abelian returns in w.*

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